Discrete-time Indefinite Stochastic LQ Optimal Control: Infinite Horizon Case

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Abstract The asymptotic analysis method is applied to the infinite horizon discrete-time indefinite stochastic LQ problem. This paper carries out the research on the basis of the results of its finite horizon counterpart and the mean-square stabilizing assumption. Properties of the solutions to the generalized algebraic Riccati equation (GARE) are also considered. Finally, two examples are provided to justify the developed theory.

Key words Mean-square stabilization, generalized Lyapunov inequality (GLI), generalized algebraic Riccati equation (GARE)

In recent decades, both the LQ problem for Itô stochastic systems and the LQ problem for discrete-time systems with multiplicative noise have been extensively investigated by researchers; see e.g., [1−7] and references therein. Moreover, the study revealed that for Itô stochastic systems, even if the control weighting matrix is indefinite, the corresponding LQ problem can still be well-posed. It was also justified in mathematical finance that these theoretical results enable us to have a deeper understanding of the mean-variance portfolio theory[8].

From a technical point of view, it is not difficult to observe that the related Riccati equations have played an important role within the framework of the investigation concerning LQ problems. For Itô stochastic systems, the relationship between the generalized differential Riccati equation and the generalized algebraic Riccati equation (GARE) was studied in [9] to obtain the optimal control of the infinite horizon indefinite LQ problem. For discrete-time systems with multiplicative noise, the generalized difference Riccati equation (GDRE) was derived in [1] to deduce the optimal control of the finite horizon indefinite LQ problem.

In [4], the authors devoted to discussing the infinite horizon discrete-time stochastic LQ problem by investigating the relationship with its finite horizon counterpart. However, this research was carried out on the assumption that the control weighting matrix is positive definite. As a matter of fact, practical problems may require us to consider the case, in which the control weighting matrix is not positive definite. Therefore, it seems worthwhile to relax the coercive assumption proposed in [4]. Based on this viewpoint, in this paper we use the asymptotic analysis method to explore the limit behavior of the solution to the GDRE and discuss the infinite horizon discrete-time indefinite stochastic LQ problem.

1 Problem formulation

Consider a system which is formulated as

$$\begin{cases} \boldsymbol{x}(t+1)=[A(t)\boldsymbol{x}(t)+B(t)\boldsymbol{u}(t)]+[C(t)\boldsymbol{x}(t)+D(t)\boldsymbol{u}(t)]w(t) \\ \boldsymbol{x}(t_0)=\boldsymbol{x}_0, \quad (t_0,\boldsymbol{x}_0)\in N_{T-1}\times \mathbf{R}^n \end{cases} \tag{1}$$

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where $\boldsymbol{x}(t)\in \mathbf{R}^n$ is the state, $\boldsymbol{u}(t)\in\mathbf{R}^p$ is the control input, $w(t)\in\mathbf{R}$ is the noise, $t\in\{t_0,t_0+1,\cdots,T-1\}$, and $N_T=\{0,1,2,\cdots,T\}$. The process $\{w(t),t=t_0,t_0+1,\cdots,T-1\}$ is a sequence of second-order stationary random variables. Without loss of generality, we assume that $w(\cdot)$ satisfies $\mathrm{E}\{w(t)\}=0, \mathrm{E}\{w(s)w(t)\}=\delta_{st}$, where $\mathrm{E}\{\cdot\}$ is the expectation operator, δ_{st} is the Kronecker function, $s,t\in\{t_0,t_0+1,\cdots,T-1\}$.

The cost functional associated with system (1) is given by

$$J(t_0,\boldsymbol{x}_0;\boldsymbol{u}(t_0),\boldsymbol{u}(t_0+1),\cdots,\boldsymbol{u}(T-1))= \\ \mathrm{E}\left\{\sum_{t=t_0}^{T-1}\begin{bmatrix}\boldsymbol{x}(t)\\ \boldsymbol{u}(t)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}Q(t) & L(t)\\ L^{\mathrm{T}}(t) & R(t)\end{bmatrix}\begin{bmatrix}\boldsymbol{x}(t)\\ \boldsymbol{u}(t)\end{bmatrix}+\boldsymbol{x}^{\mathrm{T}}(T)M\boldsymbol{x}(T)\right\} \tag{2}$$

where $\begin{bmatrix}Q(t) & L(t)\\ L^{\mathrm{T}}(t) & R(t)\end{bmatrix}$, $t\in\{t_0,t_0+1,\cdots,T-1\}$ and M are all symmetric matrices. Notice that no restriction is imposed on the definiteness of the weighting matrices.

The value function is defined by

$$V(t_0,\boldsymbol{x}_0)=\inf_{\boldsymbol{u}(t_0),\boldsymbol{u}(t_0+1),\cdots,\boldsymbol{u}(T-1)} J(t_0,\boldsymbol{x}_0;\boldsymbol{u}(\cdot)) \tag{3}$$

Definition 1. Indefinite stochastic LQ problem (1) ∼ (3) is well-posed, if for any $(t_0,\boldsymbol{x}_0)\in N_{T-1}\times\mathbf{R}^n$, $V(t_0,\boldsymbol{x}_0)>-\infty$.

Definition 2. Indefinite stochastic LQ problem (1) ∼ (3) is attainable, if for any $(t_0,\boldsymbol{x}_0)\in N_{T-1}\times\mathbf{R}^n$, there exists a sequence $\{\boldsymbol{u}(t),t=t_0,t_0+1,\cdots,T-1\}$, such that

$$V(t_0,\boldsymbol{x}_0)=J(t_0,\boldsymbol{x}_0;\boldsymbol{u}(t_0),\boldsymbol{u}(t_0+1),\cdots,\boldsymbol{u}(T-1))$$

holds. In this case, $\{\boldsymbol{u}(t),\ t=t_0,t_0+1,\cdots,T-1\}$ is called an optimal control.

To faciliate the upcoming discussions, we recall some results associated with problem (1) ∼ (3), which were proved in [1].

Lemma 1. Suppose indefinite stochastic LQ problem (1) ∼ (3) is well-posed. The following statements are equivalent:

1) Indefinite stochastic LQ problem (1) ∼ (3) has a unique optimal control.

2) There exists a unique solution sequence $\{P(t),t=t_0,t_0+1,\cdots,T\}$ to the GDRE

$$\begin{cases} P(t)=A^{\mathrm{T}}(t)P(t+1)A(t)+C^{\mathrm{T}}(t)P(t+1)C(t)+ \\ \qquad Q(t)-H(t)G^{-1}(t)H^{\mathrm{T}}(t) \\ H(t)=A^{\mathrm{T}}(t)P(t+1)B(t)+C^{\mathrm{T}}(t)P(t+1)D(t)+L(t) \\ G(t)=B^{\mathrm{T}}(t)P(t+1)B(t)+D^{\mathrm{T}}(t)P(t+1)D(t)+R(t)>0 \\ P(T)=M \\ t=t_0,t_0+1,\cdots,T-1 \end{cases} \tag{4}$$

3) There exists a matrix sequence $\{P(t)=P^{\mathrm{T}}(t),t=t_0,t_0+1,\cdots,T\}$ satisfying the linear matrix inequality

(LMI) $\begin{bmatrix} \Sigma(t) & H(t) \\ H^{\mathrm{T}}(t) & G(t) \end{bmatrix} \geq 0$, where

$$\begin{cases} \Sigma(t) = A^{\mathrm{T}}(t)P(t+1)A(t) + C^{\mathrm{T}}(t)P(t+1)C(t) - P(t) + \\ \qquad Q(t) \\ H(t) = A^{\mathrm{T}}(t)P(t+1)B(t) + C^{\mathrm{T}}(t)P(t+1)D(t) + L(t) \\ G(t) = B^{\mathrm{T}}(t)P(t+1)B(t) + D^{\mathrm{T}}(t)P(t+1)D(t) + R(t) > 0 \\ P(T) \leq M \end{cases}$$

Moreover, the value function $V(t_0, \boldsymbol{x}_0) = \boldsymbol{x}_0^{\mathrm{T}} P(t_0) \boldsymbol{x}_0$, and the optimal control is given by $\boldsymbol{u}(t) = -[R(t) + B^{\mathrm{T}}(t)P(t+1)B(t) + D^{\mathrm{T}}(t)P(t+1)D(t)]^{-1}[L^{\mathrm{T}}(t) + B^{\mathrm{T}}(t)P(t+1)A(t) + D^{\mathrm{T}}(t)P(t+1)C(t)]\boldsymbol{x}(t)$.

Lemma 2 (Comparison theorem). If $\{P_1(\cdot)\}$ and $\{P_2(\cdot)\}$ are two solution sequences to GDRE (4) with terminal values satisfying $P_1(T) \leq P_2(T)$, then $P_1(t) \leq P_2(t)$ holds for any $t \in \{t_0, t_0+1, \cdots, T-1\}$.

2 Solvability of GARE

For simplicity, assume henceforth that A, B, C, D and L, Q, R are all time-invariant matrices, and introduce the notation $P(\cdot, T)$ to represent the solution to GDRE (4) with terminal time T.

Definition 3 (Mean-square stabilization). System (1) is said to be mean-square stabilizable, if there is a state feedback control law $\boldsymbol{u}(t) = K\boldsymbol{x}(t)$, $K \in \mathbf{R}^{p \times n}$, such that the state of the closed-loop system

$$\begin{cases} \boldsymbol{x}(t+1) = (A+BK)\boldsymbol{x}(t) + (C+DK)\boldsymbol{x}(t)w(t) \\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0, \quad (t_0, \boldsymbol{x}_0) \in \mathbf{N} \times \mathbf{R}^n \end{cases} \tag{5}$$

satisfies $\lim_{t\to\infty} \mathrm{E}\{\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{x}(t)\} = 0$. In this case, $\boldsymbol{u}(t) = K\boldsymbol{x}(t)$ is called a mean-square stabilizing control.

Remark 1. System (1) is mean-square stabilizable if and only if there is a matrix $K \in \mathbf{R}^{p \times n}$, such that when $\boldsymbol{u}(t) = K\boldsymbol{x}(t)$, the state of the closed-loop system (5) satisfies $\sum_{t=t_0}^{\infty} \mathrm{E}\{\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{x}(t)\} < \infty$.

In the sequel, to assure the well-posedness of the infinite horizon indefinite LQ problem, we shall make the following assumption.

Assumption 1. System (1) is mean-square stabilizable.

Remark 2. According to Proposition 2.2 presented in [10], system (1) is mean-square stabilizable if and only if there is a matrix $K \in \mathbf{R}^{p \times n}$, such that the generalized Lyapunov inequality (GLI)

$$-P + (A+BK)^{\mathrm{T}}P(A+BK) + (C+DK)^{\mathrm{T}}P(C+DK) < 0$$

admits a solution $P > 0$.

Using Schur's Lemma, it seems direct to get Remark 3.

Remark 3. System (1) is mean-square stabilizable if and only if there exist matrices $P > 0, U$, such that the LMI

$$\begin{bmatrix} -P & AP + BU & CP + DU \\ PA^{\mathrm{T}} + U^{\mathrm{T}}B^{\mathrm{T}} & -P & 0 \\ PC^{\mathrm{T}} + U^{\mathrm{T}}D^{\mathrm{T}} & 0 & -P \end{bmatrix} < 0 \tag{6}$$

holds. Moreover, a mean-square stabilizing control for system (1) can be expressed as $\boldsymbol{u}(t) = UP^{-1}\boldsymbol{x}(t)$.

The following theorem considers the asymptotic behavior of the solution to GDRE (4) with terminal value chosen from a specified matrix set, which is formulated as

$$\mathcal{P}_{L,Q,R} = \left\{ P = P^{\mathrm{T}} \,\middle|\, R + B^{\mathrm{T}}PB + D^{\mathrm{T}}PD > 0 \right.$$
$$\left. \begin{bmatrix} -P + A^{\mathrm{T}}PA + C^{\mathrm{T}}PC + Q & L + A^{\mathrm{T}}PB + C^{\mathrm{T}}PD \\ L^{\mathrm{T}} + B^{\mathrm{T}}PA + D^{\mathrm{T}}PC & R + B^{\mathrm{T}}PB + D^{\mathrm{T}}PD \end{bmatrix} \geq 0 \right\}$$

Theorem 1. If $\mathcal{P}_{L,Q,R} \neq \emptyset$, then for any $\tilde{P} \in \mathcal{P}_{L,Q,R}$, the solution $P(t)$ to GDRE (4) with terminal value $P(T) = \tilde{P}$ exists and is bounded when $t \in Z_T$, where $Z_T = \{\cdots, -2, -1, 0, 1, 2, \cdots, T\}$. Moreover, $P(t)$ converges increasingly to a solution of the GARE

$$\mathcal{R}(P, L, Q, R) = 0, \quad R + B^{\mathrm{T}}PB + D^{\mathrm{T}}PD > 0 \tag{7}$$

as t decreases to $-\infty$, where $\mathcal{R}(P, L, Q, R) = -P + A^{\mathrm{T}}PA + C^{\mathrm{T}}PC + Q - (L + A^{\mathrm{T}}PB + C^{\mathrm{T}}PD)(R + B^{\mathrm{T}}PB + D^{\mathrm{T}}PD)^{-1}(L^{\mathrm{T}} + B^{\mathrm{T}}PA + D^{\mathrm{T}}PC)$.

Proof. Let $\{Z_{\tilde{P}}(\cdot)\}$ be the solution sequence to the GDRE

$$\begin{cases} Z_{\tilde{P}}(t) = A^{\mathrm{T}}Z_{\tilde{P}}(t+1)A + C^{\mathrm{T}}Z_{\tilde{P}}(t+1)C + Q(\tilde{P}) - \\ \qquad H(t)G^{-1}(t)H^{\mathrm{T}}(t) \\ H(t) = L(\tilde{P}) + A^{\mathrm{T}}Z_{\tilde{P}}(t+1)B + C^{\mathrm{T}}Z_{\tilde{P}}(t+1)D \\ G(t) = R(\tilde{P}) + B^{\mathrm{T}}Z_{\tilde{P}}(t+1)B + D^{\mathrm{T}}Z_{\tilde{P}}(t+1)D > 0 \\ Z_{\tilde{P}}(T) = 0 \\ t = t_0, t_0+1, \cdots, T-1 \end{cases} \tag{8}$$

where

$$\begin{cases} L(\tilde{P}) = L + A^{\mathrm{T}}\tilde{P}B + C^{\mathrm{T}}\tilde{P}D \\ Q(\tilde{P}) = -\tilde{P} + A^{\mathrm{T}}\tilde{P}A + C^{\mathrm{T}}\tilde{P}C + Q \\ R(\tilde{P}) = R + B^{\mathrm{T}}\tilde{P}B + D^{\mathrm{T}}\tilde{P}D \end{cases}$$

According to Lemma 1, GDRE (8) corresponds to the LQ problem for system (1) with the cost functional

$$J(t_0, \boldsymbol{x}_0; \boldsymbol{u}(t_0), \boldsymbol{u}(t_0+1), \cdots, \boldsymbol{u}(T-1)) = \sum_{t=t_0}^{T-1} \mathrm{E}\left\{ \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{u}(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} Q(\tilde{P}) & L(\tilde{P}) \\ L^{\mathrm{T}}(\tilde{P}) & R(\tilde{P}) \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{u}(t) \end{bmatrix} \right\}$$

The value function $V(t_0, \boldsymbol{x}_0) = \boldsymbol{x}_0^{\mathrm{T}} Z_{\tilde{P}}(t_0, T)\boldsymbol{x}_0$. Since the weighting matrix in the cost functional is positive semi-definite, the time-invariant property of the solution to (8) yields

$$Z_{\tilde{P}}(t_2, T) \leq Z_{\tilde{P}}(t_1, T) \tag{9}$$

where $t_1 \leq t_2, t_1, t_2 \in Z_T$. By choosing a mean-square stabilizing control $\boldsymbol{u}(t) = K\boldsymbol{x}(t)$, we obtain for any $t \in Z_T$,

$$\boldsymbol{x}_0^{\mathrm{T}} Z_{\tilde{P}}(t, T)\boldsymbol{x}_0 = \boldsymbol{x}_0^{\mathrm{T}} Z_{\tilde{P}}(0, T-t)\boldsymbol{x}_0 \leq \lambda_{\max}\left\{ \begin{bmatrix} I \\ K \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} Q(\tilde{P}) & L(\tilde{P}) \\ L^{\mathrm{T}}(\tilde{P}) & R(\tilde{P}) \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix} \right\} \times \sum_{s=0}^{\infty} \mathrm{E}\{\boldsymbol{x}^{\mathrm{T}}(s)\boldsymbol{x}(s)\} < \infty$$

where $\lambda_{\max}\{\cdot\}$ represents the maximal eigenvalue of its argument. Hence, $\lim_{t\to-\infty} Z_{\tilde{P}}(t, T)$ exists, and we denote it by $\bar{Z}_{\tilde{P}}$. For any $t \in Z_T$, let $P(t, T) = \tilde{P} + Z_{\tilde{P}}(t, T)$. Then, $\{P(\cdot, T)\}$ must be the solution sequence to GDRE (4) and $\lim_{t\to-\infty} P(t, T) = \tilde{P} + \bar{Z}_{\tilde{P}}$ holds. Finally, direct computation shows that $\tilde{P} + \bar{Z}_{\tilde{P}}$ satisfies GARE (7). □

3 Solutions to GARE

Based on the conclusions about the solvability of GARE (7), in this part, we further distinguish two kinds of solutions to it, i.e., the maximal solution and the feedback stabilizing solution.

Definition 4. A solution to GARE (7) is called a maximal solution, denoted by $P_{\max}$, if for any $P \in \mathcal{P}_{L,Q,R}$, $P_{\max} \geq P$.

Definition 5. A solution P to GARE (7) is called a feedback stabilizing solution, if $\boldsymbol{u}(t) = -(R + B^{\rm T}PB + D^{\rm T}PD)^{-1}(L^{\rm T} + B^{\rm T}PA + D^{\rm T}PC)\boldsymbol{x}(t)$ is a mean-square stabilizing control for system (1).

Through the application of the completion of squares technique, we get to know the intrinsic relationship between the defined solutions.

Theorem 2. If GARE (7) has a feedback stabilizing solution, then it must be unique and coincides with the maximal solution.

Proof. Let P be a feedback stabilizing solution to GARE (7). Then $P \in \mathcal{P}_{L,Q,R}$, according to Definition 4, $P_{\max} \geq P$. To get the conclusion, it only suffices to prove $P \geq P_{\max}$. From Definition 5, $\boldsymbol{u}(t) = -(R + B^{\rm T}PB + D^{\rm T}PD)^{-1}(L^{\rm T} + B^{\rm T}PA + D^{\rm T}PC)\boldsymbol{x}(t)$ is a mean-square stabilizing control for system (1). Thus, for any $T > t_0$, a completion of squares shows

$$\sum_{t=t_0}^{T-1} {\rm E}\{\boldsymbol{x}^{\rm T}(t)Q\boldsymbol{x}(t) + 2\boldsymbol{x}^{\rm T}(t)L\boldsymbol{u}(t) + \boldsymbol{u}^{\rm T}(t)R\boldsymbol{u}(t)\} = \boldsymbol{x}_0^{\rm T}P\boldsymbol{x}_0 - {\rm E}\{\boldsymbol{x}^{\rm T}(T)P\boldsymbol{x}(T)\} + \sum_{t=t_0}^{T-1}{\rm E}\{[\boldsymbol{u}(t) - K\boldsymbol{x}(t)]^{\rm T} \times (R + B^{\rm T}PB + D^{\rm T}PD)[\boldsymbol{u}(t) - K\boldsymbol{x}(t)]\}$$

where $K = -(R + B^{\rm T}PB + D^{\rm T}PD)^{-1}(L^{\rm T} + B^{\rm T}PA + D^{\rm T}PC)$. When $T \to \infty$, ${\rm E}\{\boldsymbol{x}^{\rm T}(T)P\boldsymbol{x}(T)\}$ vanishes, and

$$\boldsymbol{x}_0^{\rm T}P\boldsymbol{x}_0 + \sum_{t=t_0}^{\infty}{\rm E}\{[\boldsymbol{u}(t) - K\boldsymbol{x}(t)]^{\rm T}(R + B^{\rm T}PB + D^{\rm T}PD) \times [\boldsymbol{u}(t) - K\boldsymbol{x}(t)]\} = \boldsymbol{x}_0^{\rm T}P\boldsymbol{x}_0$$

Similarly, one can deduce that for any $\boldsymbol{x}_0 \in \mathbf{R}^n$, $\boldsymbol{x}_0^{\rm T}P\boldsymbol{x}_0 \geq \boldsymbol{x}_0^{\rm T}P_{\max}\boldsymbol{x}_0$. Therefore, $P = P_{\max}$ is derived. □

Some sufficient conditions are supplied here to assure the existence of the feedback stabilizing solution and the maximal solution.

Theorem 3. If $\mathcal{P}_{L,Q,R}$ has a nonempty interior, i.e., there is a matrix $\tilde{P} \in \mathcal{P}_{L,Q,R}$ satisfying $\mathcal{R}(\tilde{P}, L, Q, R) > 0$ and $R + B^{\rm T}\tilde{P}B + D^{\rm T}\tilde{P}D > 0$, then GARE (7) admits a feedback stabilizing solution.

Proof. If $\{Z_{\tilde{P}}(\cdot)\}$ satisfies GDRE (8), then Theorem 1 implies $Z_{\tilde{P}}(t)$ converges increasingly to $\bar{Z}_{\tilde{P}}$ as t decreases to $-\infty$. If $P = \tilde{P} + \bar{Z}_{\tilde{P}}$, then P is a solution to GARE (7) and the equality $(R + B^{\rm T}PB + D^{\rm T}PD)^{-1}(L^{\rm T} + B^{\rm T}PA + D^{\rm T}PC) = [R(\tilde{P}) + B^{\rm T}\bar{Z}_{\tilde{P}}B + D^{\rm T}\bar{Z}_{\tilde{P}}D]^{-1}[L^{\rm T}(\tilde{P}) + B^{\rm T}\bar{Z}_{\tilde{P}}A + D^{\rm T}\bar{Z}_{\tilde{P}}C]$ holds. We shall prove $\boldsymbol{u}(t) = -[R(\tilde{P}) + B^{\rm T}\bar{Z}_{\tilde{P}}B + D^{\rm T}\bar{Z}_{\tilde{P}}D]^{-1}[L^{\rm T}(\tilde{P}) + B^{\rm T}\bar{Z}_{\tilde{P}}A + D^{\rm T}\bar{Z}_{\tilde{P}}C]\boldsymbol{x}(t)$ is a mean-square stabilizing control for system (1). For any $(s, \boldsymbol{x}_0) \in N_{T-1} \times (\mathbf{R}^n \backslash \{0\})$, the inequality $\boldsymbol{x}_0^{\rm T}\bar{Z}_{\tilde{P}}\boldsymbol{x}_0 \geq \boldsymbol{x}_0^{\rm T}Z_{\tilde{P}}(s)\boldsymbol{x}_0 > 0$ indicates that $\bar{Z}_{\tilde{P}} > 0$. Since $K_{\tilde{P}} = -[R(\tilde{P}) + B^{\rm T}\bar{Z}_{\tilde{P}}B + D^{\rm T}\bar{Z}_{\tilde{P}}D]^{-1}[L^{\rm T}(\tilde{P}) + B^{\rm T}\bar{Z}_{\tilde{P}}A + D^{\rm T}\bar{Z}_{\tilde{P}}C]$ satisfies the GLI

$$-\bar{Z}_{\tilde{P}} + (A + BK_{\tilde{P}})^{\rm T}\bar{Z}_{\tilde{P}}(A + BK_{\tilde{P}}) + (C + DK_{\tilde{P}})^{\rm T}\bar{Z}_{\tilde{P}} \times (C + DK_{\tilde{P}}) = -Q(\tilde{P}) - L(\tilde{P})K_{\tilde{P}} - K_{\tilde{P}}^{\rm T}L^{\rm T}(\tilde{P}) - K_{\tilde{P}}^{\rm T}R(\tilde{P})K_{\tilde{P}} < 0$$

an application of Remark 2 shows the conclusion. □

Theorem 4. If L, $Q = Q^{\rm T}$, and $R = R^{\rm T}$ are the given matrices such that $\mathcal{P}_{L,Q,R} \neq \emptyset$, then GARE (7) admits a maximal solution.

Proof. Choose an element $\tilde{P} \in \mathcal{P}_{L,Q,R}$. Then, for any $\epsilon > 0, \mathcal{R}(\tilde{P}, L, Q + \epsilon I, R) > 0$. For an arbitrary positive decreasing sequence $\epsilon_i \to 0\,(i \to \infty)$, by applying Theorems 2 and 3, we obtain that for any $i \in \mathbf{N}$, there is a decreasing sequence $P_{\epsilon_0} \geq P_{\epsilon_1} \geq \cdots \geq P_{\epsilon_i} \geq \tilde{P}$, such that $\mathcal{R}(P_{\epsilon_i}, L, Q + \epsilon_i I, R) = 0$ and $R + B^{\rm T}P_{\epsilon_i}B + D^{\rm T}P_{\epsilon_i}D > 0$. Hence, $\lim_{i\to\infty} P_{\epsilon_i}$ exists, and we denote it by P^*. It seems certain that P^* satisfies $\mathcal{R}(P^*, L, Q, R) = 0$ and $R + B^{\rm T}P^*B + D^{\rm T}P^*D > 0$. Due to the arbitrariness of the choice of $\tilde{P}$ in $\mathcal{P}_{L,Q,R}$, we finish the proof by concluding that P^* is the maximal solution to GARE (7). □

4 Indefinite LQ optimal control

The conclusions deduced previously are used in this part to get the solution of the infinite horizon indefinite stochastic LQ problem.

Theorem 5. If $\mathcal{P}_{L,Q,R} \neq \emptyset$, then the infinite horizon indefinite LQ problem is well-posed and the value function $V(t_0, \boldsymbol{x}_0) = \boldsymbol{x}_0^{\rm T}P_{\max}\boldsymbol{x}_0$.

Proof. The justification of the well-posedness is similar to the proof of Theorem 2, which yields that $V(t_0, \boldsymbol{x}_0) \geq \boldsymbol{x}_0^{\rm T}P_{\max}\boldsymbol{x}_0$. Choose $P \in \mathcal{P}_{L,\,Q,\,R}$; then for any $\epsilon > 0$, $\mathcal{R}(P, L, Q + \epsilon I, R) > 0$. An application of Theorems 2 and 3 shows that there exists a maximal solution P_ϵ to the GARE

$$\mathcal{R}(P_\epsilon, L, Q + \epsilon I, R) = 0, \quad R + B^{\rm T}P_\epsilon B + D^{\rm T}P_\epsilon D > 0 \tag{10}$$

In addition, $\boldsymbol{u}(t) = -(R + B^{\rm T}P_\epsilon B + D^{\rm T}P_\epsilon D)^{-1}(L^{\rm T} + B^{\rm T}P_\epsilon A + D^{\rm T}P_\epsilon C)\boldsymbol{x}(t)$ is a mean-square stabilizing control. By performing the completion of squares technique, we obtain $V(t_0, \boldsymbol{x}_0) \leq \boldsymbol{x}_0^{\rm T}P_\epsilon\boldsymbol{x}_0$. From Theorem 4, $\lim_{\epsilon\to 0} P_\epsilon = P_{\max}$, which implies that $V(t_0, \boldsymbol{x}_0) \leq \boldsymbol{x}_0^{\rm T}P_{\max}\boldsymbol{x}_0$. Hence, $V(t_0, \boldsymbol{x}_0) = \boldsymbol{x}_0^{\rm T}P_{\max}\boldsymbol{x}_0$. □

Theorem 6. If $\mathcal{P}_{L,Q,R} \neq \emptyset$ and there exists an optimal state-control pair to the infinite horizon indefinite LQ problem, then the optimal control must be unique and can be expressed as

$$\boldsymbol{u}(t) = -(R + B^{\rm T}P_{\max}B + D^{\rm T}P_{\max}D)^{-1}(L^{\rm T} + B^{\rm T}P_{\max}A + D^{\rm T}P_{\max}C)\boldsymbol{x}(t) \tag{11}$$

Proof. Let $(\boldsymbol{x}_*(\cdot), \boldsymbol{u}_*(\cdot))$ be an optimal state-control pair. As $\boldsymbol{u}_*(\cdot)$ is a mean-square stabilizing control, $\lim_{T\to\infty}{\rm E}\{\boldsymbol{x}_*^{\rm T}(T)P_{\max}\boldsymbol{x}_*(T)\} = 0$. A completion of squares implies

$$V(t_0, \boldsymbol{x}_0) = \boldsymbol{x}_0^{\rm T}P_{\max}\boldsymbol{x}_0 + \sum_{t=t_0}^{\infty}{\rm E}\{[\boldsymbol{u}_*(t) - K_{\max}\boldsymbol{x}_*(t)]^{\rm T}(R + B^{\rm T}P_{\max}B + D^{\rm T}P_{\max}D)[\boldsymbol{u}_*(t) - K_{\max}\boldsymbol{x}_*(t)]\}$$

where $K_{\max} = -(R + B^{\rm T}P_{\max}B + D^{\rm T}P_{\max}D)^{-1}(L^{\rm T} + B^{\rm T}P_{\max}A + D^{\rm T}P_{\max}C)$. According to Theorem 5, the conclusion is obtained. □

Theorems 2 and 6 imply the following result.

Corollary 1. If GARE (7) admits a feedback stabilizing solution P, then $P = P_{\max}$ holds and the unique optimal control can also be given by (11).

The fact that the value function and the optimal control are expressed in terms of the maximal solution to GARE (7) makes it meaningful to further study the asymptotic convergence from the GDRE′s solution to the GARE′s maximal solution. Here, we present some related results.

Lemma 3. Let $\{P(t), t \in Z_T\}$ be the solution sequence to GDRE (4). If there is a $t^* \in (N_T \backslash \{0\})$, such that $P(t^*) \geq P(t^*-1)$, then $P(t)$ is monotonically decreasing as t decreases to $-\infty$ in Z_{t^*}.

Proof. When $P(t^*) \geq P(t^*-1)$, let $\{P_{(1)}(\cdot)\}$ and $\{P_{(2)}(\cdot)\}$ satisfy the following GDREs

$$\begin{cases} P_{(1)}(t) = A^{\mathrm T}P_{(1)}(t+1)A + C^{\mathrm T}P_{(1)}(t+1)C + Q - \\ \qquad\qquad H_{(1)}(t)G_{(1)}^{-1}(t)H_{(1)}^{\mathrm T}(t) \\ H_{(1)}(t) = L + A^{\mathrm T}P_{(1)}(t+1)B + C^{\mathrm T}P_{(1)}(t+1)D \\ G_{(1)}(t) = R + B^{\mathrm T}P_{(1)}(t+1)B + D^{\mathrm T}P_{(1)}(t+1)D > 0 \\ P_{(1)}(t^*) = P(t^*) \\ t = 0, 1, \cdots, t^*-1 \end{cases}$$

$$\begin{cases} P_{(2)}(t) = A^{\mathrm T}P_{(2)}(t+1)A + C^{\mathrm T}P_{(2)}(t+1)C + Q - \\ \qquad\qquad H_{(2)}(t)G_{(2)}^{-1}(t)H_{(2)}^{\mathrm T}(t) \\ H_{(2)}(t) = L + A^{\mathrm T}P_{(2)}(t+1)B + C^{\mathrm T}P_{(2)}(t+1)D \\ G_{(2)}(t) = R + B^{\mathrm T}P_{(2)}(t+1)B + D^{\mathrm T}P_{(2)}(t+1)D > 0 \\ P_{(2)}(t^*) = P(t^*-1) \\ t = 0, 1, \cdots, t^*-1 \end{cases}$$

respectively. From Lemma 2, we get $P_{(1)}(t) \geq P_{(2)}(t)$, $t \in N_{t^*}$, which is equivalent to $P(t) \geq P(t-1)$, $t \in N_{t^*}$. The conclusion is obtained by the time-invariant property of $\{P(t), t \in Z_T\}$. □

Lemma 4. If M is an arbitrary symmetric matrix satisfying $R + B^{\mathrm T}MB + D^{\mathrm T}MD > 0$, then there is a scalar $\alpha \geq 0$, such that the GARE

$$\mathcal{R}(P, L, Q + \alpha I, R) = 0,\ R + B^{\mathrm T}PB + D^{\mathrm T}PD > 0 \quad (12)$$

admits a solution P_α with $P_\alpha \geq M$.

Proof. By choosing a sufficiently large number $\alpha \geq 0$, such that $\mathcal{R}(M, L, Q + \alpha I, R) \geq 0$, we derive from Theorem 4 that GARE (12) admits a maximal solution P_α. Since $M \in \mathcal{P}_{L,\,Q+\alpha I,\,R}$, then $P_\alpha \geq M$. □

Theorem 7. Suppose $\mathcal{P}_{L,\,Q,\,R} \neq \emptyset$, if $M \in \mathcal{M}_{L,\,Q,\,R}$, where $\mathcal{M}_{L,\,Q,\,R} = \{M = M^{\mathrm T} \mid$ there exists $P \in \mathcal{P}_{L,\,Q,\,R}$ satisfying $M \geq P\}$, then there is a matrix $\tilde{M} \geq M$, such that GDRE (4) with terminal value $P(T) = \tilde{M}$ admits a solution sequence $\{P(t), t \in Z_T\}$. Moreover, $P(t)$ converges decreasingly to the maximal solution of GARE (7) as t decreases to $-\infty$.

Proof. Lemma 4 guarantees the existence of $\alpha \geq 0$ and P_α, which satisfy $\mathcal{R}(P_\alpha, L, Q + \alpha I, R) = 0, R + B^{\mathrm T}P_\alpha B + D^{\mathrm T}P_\alpha D > 0$, and $P_\alpha \geq M$. Set $\tilde{M} = P_\alpha$ and let $\{\tilde{P}(\cdot)\}$ be the solution sequence to GDRE (4) with $t_0 = 0$, $\tilde{P}(T) = \tilde{M}$. When $t = T-1$, direct computation shows $\tilde{P}(T) \geq \tilde{P}(T-1)$. By applying Lemma 3, we find $\tilde{P}(t)$ decreases as t decreases to $-\infty$ in Z_T. According to Theorem 4, $\tilde{P}(T) = \tilde{M} \geq P_{\max}$ holds. Then, we know from Lemma 2 and the time-invariant property of $\{\tilde{P}(t), t \in Z_T\}$ that for any $t \in Z_T, \tilde{P}(t) \geq P_{\max}$. Hence, $\lim_{t\to-\infty} \tilde{P}(t)$ exists. If $\bar{P} = \lim_{t\to-\infty} \tilde{P}(t)$, then $\bar{P} \geq P_{\max}$. Moreover, $\bar{P}$ is a solution to GARE (7). Since $P_{\max} \geq \bar{P}$, then $\bar{P} = P_{\max}$ is finally obtained. □

Next result devotes to discussing a special convergence property of the solution to GDRE (4).

Theorem 8. If the solution $P(t)$ to GDRE (4) with terminal value $P(T) = M \in \mathcal{P}_{L,\,Q,\,R}$ converges to $P_{\max}$ as t decreases to $-\infty$, then the corresponding solution of GDRE (4) with an arbitrary terminal value $P(T) \geq M$ also converges to $P_{\max}$ as t decreases to $-\infty$.

Proof. If $\{\bar{P}(t), t \in Z_T\}$ is the solution sequence to GDRE (4) with terminal value $\bar{P}(T) = \bar{M} \geq M$. Theorem 7 implies the existence of a matrix $\tilde{M} \geq \bar{M}$, such that the corresponding solution $\tilde{P}(t)$ of GDRE (4) with terminal value $\tilde{P}(T) = \tilde{M}$ converges to $P_{\max}$ as t decreases to $-\infty$. By applying Lemma 2 and the time-invariant property of the solution to GDRE (4), we obtain that for any $t \in Z_T$, $\tilde{P}(t) \geq \bar{P}(t) \geq P(t)$. Therefore, $\lim_{t\to-\infty} \bar{P}(t) = P_{\max}$ holds. □

In particular, since $P_{\max} \in \mathcal{P}_{L,\,Q,\,R}$, we obtain Corollary 2.

Corollary 2. The solution $P(t)$ to GDRE (4) with terminal value satisfying $P(T) \geq P_{\max}$ exists, and converges to $P_{\max}$ as t decreases to $-\infty$.

5 Illustrative examples

In this section, two examples are provided to demonstrate the effectiveness of our theoretical results.

Example 1. A numerical example is given here to compute a mean-square stabilizing control for a discrete-time stochastic system (1), which is specified as

$$A = \begin{bmatrix} 0.36 & -0.12 \\ -0.12 & 0.6 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0.25 \\ 0.21 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.7 & -0.12 \\ -0.12 & 0.72 \end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix} 0.4 \\ -0.25 \end{bmatrix}$$

Based on Remark 3 and LMI optimization technique, this problem is feasible, and a mean-square stabilizing control can be determined by

$$\boldsymbol{u}(t) = [-0.5860, 0.6670]\boldsymbol{x}(t) \quad (13)$$

Let us focus on a 1-dimensional infinite horizon indefinite stochastic LQ problem. The finite horizon counterpart is formulated as

$$\min\ J(u(\cdot)) = \mathrm{E}\left\{\sum_{t=0}^{T-1}[qx^2(t) + ru^2(t)] + mx^2(T)\right\}$$

$$\text{s.t.}\quad \begin{cases} x(t+1) = [ax(t) + bu(t)] + [cx(t) + du(t)]w(t) \\ x(0) = x_0, \quad x_0 \in \mathbf{R} \end{cases}$$

Example 2. For a set of specific parameters of the coefficients:

$$a = \frac{3}{4}, \quad b = 3, \quad c = -\frac{3}{4}, \quad d = 1, \quad q = 1, \quad r = -8 < 0$$

the corresponding GDRE is

$$\begin{cases} p(t) = \dfrac{[p(t+1)+1][9p(t+1)-8]}{10p(t+1)-8} \\ 10p(t+1) - 8 > 0 \\ p(T) = m \\ t = 0, 1, \cdots, T-1 \end{cases} \quad (14)$$

and the GARE is

$$\begin{cases} p^2 - 9p + 8 = 0 \\ 10p - 8 > 0 \end{cases} \tag{15}$$

Here, $\mathcal{P}_{L,Q,R}$ is the closed interval $[1, 8]$. According to Theorems 1 and 4, for any $m \in (1, 8)$, the solution of (14) converges increasingly to the maximal solution of (15), i.e., $p_{\max} = 8$, as t decreases to $-\infty$. Taking $T = 10$ and $m = 5$ as an example, we plot the trajectory of the solution to (14) in Fig. 1.

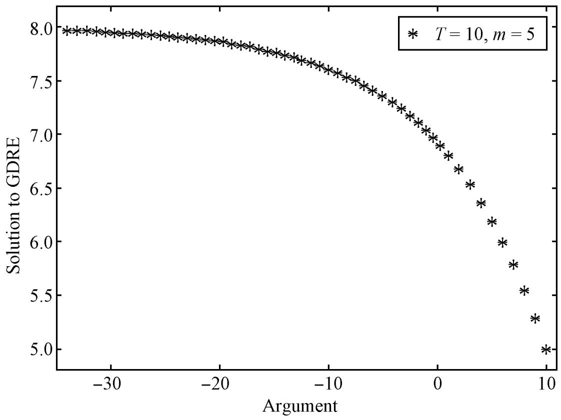

Fig. 1 Trajectory of the solution to (14)

It seems fairly certain to check that in this example, $p_{\max}$ is the feedback stabilizing solution to (15). Therefore, Theorem 5 and Corollary 1 imply that the value function is $8x_0^2$, and the unique optimal control is expressed as $u(t) = -\dfrac{1}{6}x(t)$.

6 Conclusion

The infinite horizon discrete-time indefinite stochastic LQ problem was discussed via the application of the asymptotic analysis method. The obtained results reveal that both the value function and the optimal control can be explicitly expressed in terms of the maximal solution to the GARE. Throughout this paper, the completion of squares technique played a central role. Actually, this technique can also be used to deal with many other problems in stochastic control theory.

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