第 27 卷 第 4 期 自 动 化 学 报 Vol.27,No.4
2001 年 7 月 ACTA AUTOMATICA SINICA July, 2001

Optimal Fuzzy Tracking Controller Design for Discrete-time Fuzzy Systems

Shinq-Jen WU Chin-Teng LIN

(*Dept. of Electrical and Control Engineering, Chiao-Tung University, Hsinchu*)

(E-mail: ctlin@fnn.cn.nctu.edu.tw, Tel: 886-3-5712121EXT54315, Fax: 886-3-5739497)

Abstract In this paper, we propose a systematic and theoretically sound way to design a global optimal fuzzy tracking controller for discrete-time fuzzy systems with the aim of solving the discrete-time quadratic tracking problems with moving or model-following targets under finite or infinite horizon (time). A *linear-like* dynamical system representation of discrete-time fuzzy system is proposed to mature the theoretical design scheme of the discrete-time optimal fuzzy tracking controller which can achieve *global minimum* effect. A multistage decomposition of optimization scheme is proposed to simplify the computation, and then a *segmental* recursive Riccati-like equation and a difference equation in tracing the variation of the target are derived. Moreover, in the case of time-invariant fuzzy tracking systems, we show that the optimal tracking controller can be obtained by just solving discrete-time algebraic Riccati-like equations and algebraic matrix equations. An example is given to illustrate the proposed optimal fuzzy tracker design scheme.

Key words Degree of stability, gain margin, global minimum, Riccati-like equation, moving target, model-following.

1 Introduction

Although the researches in fuzzy modeling and fuzzy control have been quite matured[1~8], the field of *optimal fuzzy control* is nearly open[9]. In particular, although fuzzy logic concept has been introduced into tracking control[10~15], the field of theoretical approach of *optimal fuzzy tracking control* is fully open. The goal of this work is to propose a systematic and theoretically sound scheme for designing a global optimal fuzzy tracking controller to control and stabilize a discrete-time fuzzy system in solving the discrete-time quadratic tracking problems with moving or model-following targets under finite or infinite horizon.

Up to date, the fuzzy tracking controller is used in conceptual design only, and is always grounded on a conventional tracker. For example, Ott and coworkers[13] included fuzzy logic into an $\alpha-\beta$ tracker algorithm; Lea and coworkers[11] used fuzzy concept to develop the software algorithm of a camera tracking system. No theoretical demonstration has been developed for fuzzy tracking controller design in the literatures.

Stability and optimality are the most important requirements for any control system. Most of the existed works on the stability analysis of fuzzy control are based on Takagi-Sugeno (T-S) type fuzzy model combined with parallel distribution compensation (PDC) concept[1] and apply Lyapunov's method to do stability analysis. Tanaka and coworkers reduced the stability analysis and control design problems to linear matrix inequality (LMI) problems[2,4]. They also dealt with uncertainty issue[3]. This approach had been applied to several control problems such as control of chaos[4] and of articulated vehicle[5]. A frequency shaping method for systematic design of fuzzy controllers was also done by them[16]. Sun and coworkers developed a separation

Received 31 August, 2000; in revised form 5 November, 2000
收稿日期 2000-08-31 收修改稿日期 2000-11-05

scheme to design fuzzy observer and fuzzy controller independently[6]. Methods based on grid-point approach[17] and circle criteria[18,19] were introduced to do stability analysis of fuzzy control, too. Wang adopted a supervisory controller and introduced stability and robustness measures[20]. Cao proposed a decomposition principle to design a discrete-time fuzzy control system and an equivalent principle to do stability analysis[8]. On the issue of optimal fuzzy control, Wang developed an optimal fuzzy controller to stabilize a linear continuous time-invariant system via Pontryagin minimum principle[9]. Although fuzzy control of linear systems could be a good starting point for a better understanding of some issues in fuzzy control synthesis, it does not have much practical implications since using the fuzzy controller designed for a linear system directly as the controller may not be a good choice[9]. Moreover, the cited stability criteria may be simple, but rough to do systematic analysis and also may result in a controller with less flexibility.

Even with the aforementioned research results on the theoretic aspect of fuzzy control, the field of optimal fuzzy control for continuous system is still nearly open[9] and that for discrete-time system is fully open. Tanaka and others' works mentioned in the above always treat the stability of general linear feedback fuzzy controllers. The continuous optimal controller constructed by Wang[9] is suitable only to be a rough or initial controller, since the system concerned is linear. All of them viewed the fuzzy system by individual rules, i.e., from the local concept. It is difficult for researchers to provide a theoretical demonstration on that the designed controller can reach global minimum effect, if the design scheme is based on local concept approach.

Technical contributions of this paper can be described as follows. A linear-like dynamical system representation of a discrete-time fuzzy system is proposed, which makes materialize the global optimal fuzzy tracking controller design scheme for a discrete-time fuzzy tracking system from the global concept approach. The design scheme meets the necessary and sufficient condition of global optimum. The derived discrete-time fuzzy tracking law is theoretically demonstrated to be the best for the entire system to reach the optimal performance index. The optimal closed-loop fuzzy tracking system is guaranteed to be exponentially stable. Furthermore, we elicit that this kind of fuzzy tracking controller can stabilize a discrete-time fuzzy tracking system to any prescribed degree of stability, and the corresponding closed-loop fuzzy tracking system possesses an infinite gain margin. The design methodology is illustrated by one example.

2 System Representation and Problem Statement

We adopt the following T-S type fuzzy model as the fuzzy tracking system describing the given nonlinear plant:

$$\begin{aligned} R^i \ : \text{If } x_1 \text{ is } T_{1i},\ \ldots,\ x_n \text{ is } T_{ni}, \text{ then } \boldsymbol{X}(k+1) &= A_i(k)\boldsymbol{X}(k)+B_i(k)\boldsymbol{u}(k), \\ \boldsymbol{Y}(k) &= C(k)\boldsymbol{X}(k),\ i=1,\ \ldots,r, \end{aligned} \tag{1}$$

where R^i denotes the ith rule of the fuzzy model; $x_1, \ldots, x_n$ are system states; $T_{1i}, \ldots, T_{ni}$ are the input fuzzy terms in the ith rule; $\boldsymbol{X}(k)=[x_1, \ldots, x_n]^{\mathrm{T}} \in \mathcal{R}^n$ is the state vector, $\boldsymbol{Y}(k) \in \mathcal{R}^{n'}$ is the system output vector, and $\boldsymbol{u}(k) \in \mathcal{R}^m$ is the system input (i.e., control output); and sequences $A_i(k), B_i(k)$ and $C(k)$ are, respectively, $n \times n, n \times m$ and $n' \times n$ matrices whose elements are real-valued functions defined on nonnegative real numbers, N. We, throughout this paper, assume $A_i(k)$ is nonsingular for all k to ensure no deadbeat response; in that case, $\boldsymbol{X}(k+1)$ and $\boldsymbol{u}(k)$ cannot define $\boldsymbol{X}(k)$ uniquely, and the poles of the resultant closed-loop system are all located at zero point.

The desired tracking controller is then assumed to be a rule-based nonlinear fuzzy form

$$R^i \ : \text{ If } y_1 \text{ is } S_{1i}, \ \ldots, \ y_{n'} \text{ is } S_{n'i}, \text{ then } \boldsymbol{u}(k) = \boldsymbol{r}_i(k), \qquad i = 1, \ldots, \delta, \tag{2}$$

where $y_1, \ldots, y_{n'}$ are the elements of output vector $\boldsymbol{Y}(k), S_{1i}, \ldots, S_{n'i}$ are the input fuzzy terms in the ith control rule, and $\boldsymbol{u}(k)$ or $\boldsymbol{r}_i(k) \in \mathcal{R}^m$ is the plant input (i.e., control output) vector. We can view each control rule, R^i, in the above as an individual controller. Fuzzy blending of these individual controllers, R^i, $i = 1, \ldots, \delta$, gives the whole fuzzy controller

$$\boldsymbol{u}(k) = \sum_{i=1}^{\delta} w_i(\boldsymbol{Y}(k))\boldsymbol{r}_i(k), \tag{3}$$

where $w_i(\boldsymbol{Y}(k))$ denotes the normalized firing-strength of the ith fuzzy control rule; i.e., $w_i(\boldsymbol{Y}(k)) = \beta_i(\boldsymbol{Y}(k))/\sum_{i=1}^{\delta}\beta_i(\boldsymbol{Y}(k))$ with $\beta_i(\boldsymbol{Y}(k)) = \Pi_{j=1}^{n'}\mu_{S_{ji}}(y_j(k))$, where $\mu_{S_{ji}}(y_j(k))$ is the membership function of fuzzy term S_{ji}.

In the T-S type fuzzy tracking model, the local dynamics in each individual subspace is described by a linear model corresponding to a rule in Eq. (1) (called fuzzy subsystem), and the overall behavior is captured by the fuzzy blending of these linear models. Hence, the overall modeling of the closed-loop fuzzy tracking system is

$$\begin{aligned}
\boldsymbol{X}(k+1) &= \sum_{i=1}^{r} h_i(\boldsymbol{X}(k))[A_i(k)\boldsymbol{X}(k) + B_i(k)\boldsymbol{u}(k)] = \\
&\quad H(\boldsymbol{X}(k))A(k)\boldsymbol{X}(k) + H(\boldsymbol{X}(k))B(k)W(\boldsymbol{Y}(k))R(k), \\
\boldsymbol{Y}(k) &= \sum_{i=1}^{r} h_i(\boldsymbol{X}(k))C(k)\boldsymbol{X}(k) = C(k)\boldsymbol{X}(k),
\end{aligned} \tag{4}$$

where

$$H(\boldsymbol{X}(k)) = [h_1(\boldsymbol{X}(k))I_n \ldots h_r(\boldsymbol{X}(k))I_n], \ \ W(\boldsymbol{Y}(k)) = [w_1(\boldsymbol{Y}(k))I_m \ldots w_\delta(\boldsymbol{Y}(k))I_m],$$

$$A(k) = \begin{bmatrix} A_1(k) \\ \vdots \\ A_r(k) \end{bmatrix}, \ B(k) = \begin{bmatrix} B_1(k) \\ \vdots \\ B_r(k) \end{bmatrix}, \ R(k) = \begin{bmatrix} r_1(k) \\ \vdots \\ r_\delta(k) \end{bmatrix}$$

with I_n and I_m denoting the identity matrices of dimension n and m, respectively, $h_i(\boldsymbol{X}(k))$ denoting the normalized firing-strength of the ith rule of the discrete-time fuzzy tracking model, i.e., $h_i(\boldsymbol{X}(k)) = \alpha_i(\boldsymbol{X}(k))/\sum_{i=1}^{r}\alpha_i(\boldsymbol{X}(k))$ with $\alpha_i(\boldsymbol{X}(k)) = \Pi_{j=1}^{n}\mu_{T_{ji}}(x_j(k))$, where $\mu_{T_{ji}}(\boldsymbol{X}(k))$ is the membership function of fuzzy term T_{ji}. Therefore, the entire discrete-time fuzzy tracking system represented by Eq. (4) is a nonlinear but *linear-like* system. This kind of global system representation will be the foundation and kernel of the following fuzzy tracker design scheme.

Then, the discrete-time optimal fuzzy tracker design scheme is to control the discrete-time fuzzy tracking system in such a way to push the output $\boldsymbol{Y}(t)$ close to any desired target $\boldsymbol{Y}^d(t)$ without excessive control-energy consumption. Hence, the performance index is defined, over all sequence $[u(k)]_{k_0}^{k_1-1}$, as[21]

$$\begin{aligned}
J(u(\cdot)) &= \sum_{k=k_0}^{k_1-1}[\boldsymbol{u}^{\mathrm{T}}(k)S(k)\boldsymbol{u}(k) + \boldsymbol{X}^{\mathrm{T}}(k)L_1(k)\boldsymbol{X}(k) + \\
&\quad (\boldsymbol{Y}(k) - \boldsymbol{Y}^d(k))^{\mathrm{T}}L_2(k)(\boldsymbol{Y}(k) - \boldsymbol{Y}^d(k))],
\end{aligned} \tag{5}$$

where

$$L_1(k) = [I_n - C^{\mathrm{T}}(k)(C(k)C^{\mathrm{T}}(k))^{-1}C(k)]^{\mathrm{T}}L_3(k)[I_n - C^{\mathrm{T}}(k)(C(k)C^{\mathrm{T}}(k))^{-1}C(k)]; \tag{6}$$

$S(k), L_2(k)$ and $L_3(k)$ are, respectively, $m\times m, n'\times n'$ and $n\times n$ nonnegative symmetric matrices; $\boldsymbol{X}^{\mathrm{T}}(k)L_1(k)\boldsymbol{X}(k)$ is the state-trajectory penalty to produce smooth response; $\boldsymbol{u}^{\mathrm{T}}(k)S(k)\boldsymbol{u}(k)$ is fuel consumption; and the last term in $J(u(\cdot))$ is related to error cost. Moreover, the performance index in Eq. (5) with $L_1(k)$ in Eq. (6) may be rewritten as[21]

$$J(u(\cdot)) = \sum_{k=k_0}^{k_1-1}[\boldsymbol{u}^{\mathrm{T}}(k)S(k)\boldsymbol{u}(k) + (\boldsymbol{X}(k)-\boldsymbol{X}^d(k))^{\mathrm{T}}L(k)(\boldsymbol{X}(k)-\boldsymbol{X}^d(k))], \tag{7}$$

where $L(k) = L_1(k) + C^{\mathrm{T}}(k)L_2(k)C(k)$ and the desired trajectory $\boldsymbol{X}^d(k) = C^{\mathrm{T}}(k)[C(k)\cdot C^{\mathrm{T}}(k)]^{-1}\boldsymbol{Y}^d(k)$. Therefore, aiming at the fuzzy tracking system in Eq. (4), the quadratic optimal tracking problem is to find the controllers, $\boldsymbol{u}^*(\cdot)$, which can minimize the quadratic cost functional, $J(u(\cdot))$, in Eq. (7) over all sequence $[\boldsymbol{u}(k)]_{k_0}^{k_1-1}$, or, more precisely, to find the individual rule-based fuzzy controllers, $R^*(\cdot)$, which can minimize the quadratic cost functional,

$$J(R(\cdot)) = \sum_{k=k_0}^{k_1-1}[(\boldsymbol{X}(k)-\boldsymbol{X}^d(k))^{\mathrm{T}}L(k)(\boldsymbol{X}(k)-\boldsymbol{X}^d(k)) + R^{\mathrm{T}}(k)W^{\mathrm{T}}(\boldsymbol{Y}(k))S(k)W(\boldsymbol{Y}(k))R(k)], \tag{8}$$

over all sequences $[R(k)]_{k_0}^{k_1-1}$.

3 Discrete-time Optimal Fuzzy Tracker Design

We are going to design the optimal fuzzy tracking controllers for discrete-time fuzzy tracking system with moving target in Subsection 3.1 and for that with model-following target in Subsection 3.2. For brevity, we shall not state discrete-time explicitly in the following presentation.

3.1 Moving-Target Tracking Problem

In this section, we shall discuss the finite-horizon tracking problem first, and then generalize the results into infinite-horizon tracking solutions.

3.1.1 Finite-horizon Tracking Problem

By describing the fuzzy system from the global concept in Section 2, we can simplify our quadratic optimal fuzzy tracking problem as the issue below.

Problem 1. Given a fuzzy tracking system in Eq. (4) with $\boldsymbol{X}(k_0) = \boldsymbol{X}_0 \in \mathcal{R}^n$, $\boldsymbol{Y}^d(t) \in \mathcal{R}^{n'}$ and $k \in [k_0, k_1-1]$, find $R^*(\cdot)$ to minimize $J(R(\cdot))$ in Eq. (8).

The calculus-of-variations method combined with Lagrange-multiplier method can be adopted directly to obtain the necessary and sufficient condition for global optimum of the above problem. However, solving the derived nonlinear two-point-boundary-problem is at length in computational aspect. Therefore, we pursue another circumvent approach, a multistage decomposition of optimization scheme, to overcome this difficulty.

Lemma 1. (Multistage Decomposition) A foregoing optimization scheme is a dynamic allocation process or a successive multistage decision process. In other words, if we let $k_0 = k_0^1, k_1 = k_1^N, k_0^i = k_1^{i-1}, i = 2,\ldots,N; \triangle k^i = k_1^i - k_0^i, i = 1,\ldots,N$, and define

$$\Phi(\boldsymbol{X}(\cdot),\boldsymbol{u}(\cdot)) = \min_{u_{[k_0,k_1-1]}}\sum_{k=k_0}^{k_1-1}[(\boldsymbol{X}(k)-\boldsymbol{X}^d(k))^{\mathrm{T}}L(k)(\boldsymbol{X}(k)-\boldsymbol{X}^d(k)) + \boldsymbol{u}^{\mathrm{T}}(k)S(k)\boldsymbol{u}(k)],$$

$$\Phi^i(\boldsymbol{X}(\cdot),\boldsymbol{u}(\cdot)) = \min_{u_{[k_0^i,k_1^i-1]}} \sum_{k=k_0^i}^{k_1^i-1}[(\boldsymbol{X}(k)-\boldsymbol{X}^d(k))^{\mathrm{T}}L(k)(\boldsymbol{X}(k)-\boldsymbol{X}^d(k))+\boldsymbol{u}^{\mathrm{T}}(k)S(k)\boldsymbol{u}(k)],$$

$$i=1,\dots,N,$$

with regard to the state resulting from the previous decision, i.e., $\boldsymbol{X}(k_0^1)=\boldsymbol{X}_0$; $\boldsymbol{X}(k_0^i)=\boldsymbol{X}^*(k_1^{i-1})$, $i=2,\dots,N$, then

$$\Phi(\boldsymbol{X}(\cdot),\boldsymbol{u}(\cdot)) = \Phi^1(\boldsymbol{X}(\cdot),\boldsymbol{u}(\cdot))+\cdots+\Phi^N(\boldsymbol{X}(\cdot),\boldsymbol{u}(\cdot)). \tag{9}$$

Since the membership functions in the fuzzy tracking controller and fuzzy tracking system are piecewise continuous, it is reasonable to make the following assumption.

Assumption 1. All the membership functions are invariant under small perturbation; that is, $H(\boldsymbol{X}(k)+\epsilon\boldsymbol{Z}(k))=H(\boldsymbol{X}(k))$ and $W(\boldsymbol{Y}(k)+\epsilon\boldsymbol{\nu}(k))=W(\boldsymbol{Y}(k))$, where $\boldsymbol{Z}(k)\in\mathcal{R}^n$ and $\boldsymbol{\nu}(k)\in\mathcal{R}^{n'}$ are perturbation vectors with respect to $\boldsymbol{X}(k)$ and $\boldsymbol{Y}(k)$, respectively.

Moreover, if we enlarge N to the extent that $H(\boldsymbol{X}(k))$ and $W(\boldsymbol{Y}(k))$ are almost invariant during the *whole single stage*, and use H_i and W_i to denote them at the ith stage, then Problem 1 can be translated into the following N-stage optimal fuzzy tracking issue.

Problem 2. Given a fuzzy tracking system,

$$\begin{aligned}\boldsymbol{X}(k+1) &= H_iA(k)\boldsymbol{X}(k)+H_iB(k)W_iR(k),\\ \boldsymbol{Y}(k) &= C(k)\boldsymbol{X}(k),\ k\in[k_0^i,k_1^i-1],\ i=1,\dots,N,\end{aligned} \tag{10}$$

with $k_0^1=k_0$, $k_1^N=k_1$, $k_0^i=k_1^{i-1}$, $i=2,\dots,N$; $\boldsymbol{X}(k_0^1)=\boldsymbol{X}_0$, $\boldsymbol{X}(k_0^i)=\boldsymbol{X}^*(k_1^{i-1})$, $i=2,\dots,N$; $H_i=H(\boldsymbol{X}(k_0^i))$, $W_i=W(\boldsymbol{Y}(k_0^i))$, find $R^*(\cdot)$ to minimize $J^i(R(\cdot))$,

$$J^i(R(\cdot))=\sum_{k=k_0^i}^{k_1^i-1}[(\boldsymbol{X}(k)-\boldsymbol{X}^d(k))^{\mathrm{T}}L(k)(\boldsymbol{X}(k)-\boldsymbol{X}^d(k))+R^t(k)W_i^{\mathrm{T}}S(k)W_iR(k)]. \tag{11}$$

Thereupon, by decomposing the optimization problem into N stages, we can successively focus on only one stage at a time. For convenience, we now define $\bar{N}$ to be the value of the number of stages at which membership functions can be assumed to be invariant during the whole single stage, and then, we shall use calculus-of-variations method combined with Lagrange-multiplier method to derive the optimal fuzzy tracking controller.

Theorem 1. (Time-varying case) For the fuzzy tracking system and fuzzy tracking controller represented, respectively, by Eqs. (1) and (2), let $(\boldsymbol{X}^*(k),R^*(k))$, $k\in[k_0,k_1-1]$, be the optimal solution with respect to $J(R(\cdot))$ in Eq. (8), and $(\boldsymbol{X}^{i^*}(k),R^{i^*}(k))$, $k\in[k_0^i,k_1^i-1]$, be the ith-stage optimal solution with respect to $J^i(R(\cdot))$ in Eq. (11). If $N>\bar{N}$ then

1) $(\boldsymbol{X}^*(k),R^*(k))=(\boldsymbol{X}^{i^*}(k),R^{i^*}(k))$, for all $k\in[k_0^i,k_1^i-1]$, $i=1,\dots,N$, where $k_0^1=k_0$, $k_1^N=k_1$, $k_0^i=k_1^{i-1}$, $i=2,\dots,N$;

2) for the ith stage, $k\in[k_0^i,k_1^i-1]$, the optimal tracking controller is

$$\boldsymbol{u}^{i^*}(k)=G_1^i(k)\boldsymbol{X}^{i^*}(k)+\boldsymbol{u}_{ext}^i(k), \tag{12}$$

and the corresponding optimal tracking law is

$$R^{i^*}(k)=W_i^t[W_iW_i^t]^{-1}[G_1^i(k)\boldsymbol{X}^{i^*}(k)+\boldsymbol{u}_{ext}^i(k)], \tag{13}$$

where $G_1^i(k)$, the feedback gain, and $\boldsymbol{u}_{ext}^i(k)$, the introduced external input, are given by

$$\begin{aligned}G_1^i(k) \;=\; & -S^{-1}(k)B^t(k)H_i^t\pi^i(k+1,k_1^i)[I_n+\\ & H_iB(k)S^{-1}(k)B^t(k)H_i^t\pi^i(k+1,k_1^i)]^{-1}H_iA(k),\end{aligned} \tag{14}$$

$$u_{ext}^i(k) = -S^{-1}(k)B^t(k)H_i^t[I_n + \pi^i(k+1, k_1^i)H_iB(k)S^{-1}(k)B^t(k)H_i^t]^{-1}b(k+1), \tag{15}$$

where $\pi^i(k+1, k_1^i)$ is the symmetric positive semidefinte solution of the following segmental recursive Riccati-like equation

$$K(k) = L(k) + A^{\mathrm{T}}(k)H_i^{\mathrm{T}}K(k+1)[I_n + H_iB(k)S^{-1}(k)B^{\mathrm{T}}(k)H_i^{\mathrm{T}}K(k+1)]^{-1}H_iA(k), \tag{16}$$

with $K(k_1^i) = 0_{n\times n}$, zero matrix of dimension $n \times n$, and $b(k+1)$ being the introduced target-dependent variable, satisfying

$$b(k) = A^{\mathrm{T}}(k)H_i^{\mathrm{T}}[I_n + \pi^i(k+1, k_1^i)H_iB(k)S^{-1}(k)B^{\mathrm{T}}(k)H_i^{\mathrm{T}}]^{-1}\boldsymbol{b}(k+1) - L(k)\boldsymbol{X}^d(k) \tag{17}$$

with $\boldsymbol{b}(k_1^i) = \mathbf{0}_{n\times 1}$; the ith-stage optimal tracking trajectory is

$$\begin{aligned}\boldsymbol{X}^{i^*}(k+1) \;=\; & [I_n + H_iB(k)S^{-1}(k)B^{\mathrm{T}}(k)H_i^{\mathrm{T}}\pi^i(k+1, k_1^i)]^{-1}H_iA(k)\boldsymbol{X}^{i^*}(k) + \\ & H_iB(k)\boldsymbol{u}_{ext}^i(k);\end{aligned} \tag{18}$$

3) the minimum performance index is

$$\min_{R_{[k_0,k_1-1]}} J(R(\cdot)) = \sum_{i=1}^{N} \boldsymbol{X}^{i^{*\mathrm{T}}}(k_0^i)\pi^i(k_0^i, k_1^i)\boldsymbol{X}^{i^*}(k_0^i) + b^{\mathrm{T}}(k_0^i)\boldsymbol{X}^{i^*}(k_0^i) + \varphi^i(k_0^i), \tag{19}$$

where the introduced variable $\varphi^i(k)$ satisfies

$$\varphi^i(k) = \varphi^i(k+1) - (\boldsymbol{X}^{i^{*\mathrm{T}}}(k) - \boldsymbol{X}^d(k))^{\mathrm{T}}L(k)\boldsymbol{X}^d(k) \tag{20}$$

with $\varphi^i(k_1^i) = 0$, $\forall i = 1, \ldots, N$.

Proof. 1) Since $N > \bar{N}$, we, via Lemma 1, know $(\boldsymbol{X}^*(k), R^*(k))$ is coincident with $(\boldsymbol{X}^{i^*}(k), R^{i^*}(k))$ for each stage segment. Then, for the i-th stage, $k \in [k_0^i, k_1^i - 1]$, we define

$$\Phi^i(\boldsymbol{X}(\cdot), R(\cdot)) \triangleq \sum_{l=k}^{k_1^i-1}[(\boldsymbol{X}(l) - \boldsymbol{X}^d(l))^{\mathrm{T}}L(l)(\boldsymbol{X}(l) - \boldsymbol{X}^d(l)) + R^{\mathrm{T}}(l)W_i^{\mathrm{T}}S(l)W_iR(l)], \tag{21}$$

where $\boldsymbol{X}(k) = \boldsymbol{X}^{i^*}(k)$ is the initial state at time k. To simplify notations, we shall omit the explicit time- and state-dependence; e.g., we write $\boldsymbol{X}$ for $\boldsymbol{X}(l)$ and $\boldsymbol{X}_{l+1}$ for $\boldsymbol{X}(l+1)$ in the following derivation. Then, by Lagrange-multiplier method, Problem 2 can be turned into the problem of minimizing

$$\bar{\Phi}^i(\boldsymbol{X}(\cdot), R(\cdot)) = \Phi^i(\boldsymbol{X}(\cdot), R(\cdot)) - 2\sum_{l=k}^{k_1^i-1} P^{\mathrm{T}}(l+1)[\boldsymbol{X}_{l+1} - H_iA\boldsymbol{X} - H_iBW_iR], \tag{22}$$

where $P_{l+1} \in \mathcal{R}^n$ is the Langrange-multiplier vector. Now, we assume the optimal solutions $(\boldsymbol{X}^{i^*}(\cdot), R^{i^*}(\cdot))$ exist, and, according to the calculus-of-variations method, let $\boldsymbol{X}(l) = \boldsymbol{X}^{i^*}(l) + \epsilon\boldsymbol{Z}(l)$, $R(l) = R^{i^*}(l) + \epsilon\boldsymbol{V}(l)$, $l \in [k, k_1^i - 1]$, where $\boldsymbol{V}(l) \in \mathcal{R}^{m\delta}$ is the perturbation vector with respect to $R(l)$, and $\boldsymbol{Z}(k) = 0$ since the initial state at time k is $\boldsymbol{X}(k) = \boldsymbol{X}^{i^*}(k)$. Substituting these variables into Eq. (22), we have

$$\begin{aligned}\bar{\Phi}^i(\boldsymbol{X}(\cdot), R(\cdot)) \;=\; & \bar{\Phi}^i(\boldsymbol{X}^{i^*}(\cdot), R^{i^*}(\cdot)) + \epsilon^2\sum_{l=k}^{k_1^i-1}[\boldsymbol{Z}^{\mathrm{T}}L\boldsymbol{Z} + \boldsymbol{V}^{\mathrm{T}}W_i^{\mathrm{T}}SW_i\boldsymbol{V}] + \\ & 2\epsilon\sum_{l=k}^{k_1^i-1}\{\boldsymbol{Z}^{\mathrm{T}}L(\boldsymbol{X}^{i^*} - \boldsymbol{X}^d) + \boldsymbol{V}^{\mathrm{T}}W_i^{\mathrm{T}}SW_iR^{i^*} - \\ & P_{l+1}^{\mathrm{T}}[\boldsymbol{Z}_{l+1} - H_iA\boldsymbol{Z} - H_iBW_i\boldsymbol{V}]\}.\end{aligned}$$

We know that a minimum of $\bar{\Phi}^i(\boldsymbol{X}(\cdot), R(\cdot))$ requires

$$\frac{\partial \bar{\Phi}^i(\boldsymbol{X}(\cdot), R(\cdot))}{\partial \epsilon}\Big|_{\epsilon=0} = 0, \quad \frac{\partial^2 \bar{\Phi}^i(\boldsymbol{X}(\cdot), R(\cdot))}{\partial \epsilon^2}\Big|_{\epsilon=0} > 0.$$

The second criteria holds positively since

$$\frac{\partial^2 \bar{\Phi}^i(\boldsymbol{X}(\cdot), R(\cdot))}{\partial \epsilon^2} = 2\sum_{l=k}^{k_1-1}[\boldsymbol{Z}^{\mathrm{T}} L \boldsymbol{Z} + \boldsymbol{V}^{\mathrm{T}} W_i^{\mathrm{T}} S W_i \boldsymbol{V}] > 0.$$

Hence, the necessary and sufficient condition for optimality is

$$\sum_{l=k}^{k_1^i-1}\{\boldsymbol{Z}^{\mathrm{T}} L(\boldsymbol{X}^{i^*} - \boldsymbol{X}^d) - P_{l+1}^{\mathrm{T}} \boldsymbol{Z}_{l+1} + P_{l+1}^{\mathrm{T}} H_i A \boldsymbol{Z} + \boldsymbol{V}^{\mathrm{T}} W_i^{\mathrm{T}} S W_i R^{i^*} + P_{l+1}^{\mathrm{T}} H_i B W_i \boldsymbol{V}\} = 0. \quad (23)$$

Due to the facts $\sum_{l=k}^{k_1^i-1} P_{l+1}^{\mathrm{T}} \boldsymbol{Z}_{l+1} = \sum_{l=k}^{k_1^i-1}[P^{\mathrm{T}} \boldsymbol{Z}] + P_{k_1^i}^{\mathrm{T}} \boldsymbol{Z}_{k_1^i} - P_k^{\mathrm{T}} \boldsymbol{Z}_k$ and $\boldsymbol{Z}_k = 0$, we have

$$\sum_{l=k}^{k_1^i-1} \boldsymbol{V}^{\mathrm{T}} W_i^{\mathrm{T}}[S W_i R^{i^*} + B^{\mathrm{T}} H_i^{\mathrm{T}} P_{l+1}] + \sum_{l=k}^{k_1-1} \boldsymbol{Z}^{\mathrm{T}}[L(\boldsymbol{X}^{i^*} - \boldsymbol{X}^d) + A^{\mathrm{T}} H_i^{\mathrm{T}} P_{l+1} - P] - \boldsymbol{Z}_{k_1^i}^{\mathrm{T}} P_{k_1^i} = 0. \quad (24)$$

Since $\boldsymbol{Z}(\cdot)$ and $\boldsymbol{V}(\cdot)$ are independent, we conclude that $P_{k_1^i}$ is a zero vector and

$$P = A^{\mathrm{T}} H_i^{\mathrm{T}} P_{l+1} + L(\boldsymbol{X}^{i^*} - \boldsymbol{X}^d), \quad (25)$$

$$\boldsymbol{u}^{i^*} = W_i R^{i^*} = -S^{-1} B^{\mathrm{T}} H_i^{\mathrm{T}} P_{l+1}. \quad (26)$$

2) Let $P(l) = K(l)\boldsymbol{X}^{i^*}(l) + \boldsymbol{b}(l)$. Substituting this into Eq. (26) and regarding the term unrelated to $\boldsymbol{X}^{i^*}$ as an external input, $\boldsymbol{u}_{ext}^i$, we can obtain $\boldsymbol{u}^{i^*}(l)$ in Eq. (12), $R^{i^*}(l)$ in Eq. (13), and then, $\boldsymbol{X}^{i^*}(l)$ in Eq. (18). Accordingly, aiming at Eq. (25), we have

$$\begin{aligned} K\boldsymbol{X}^{i^*} + \boldsymbol{b} = {} & L(\boldsymbol{X}^{i^*} - \boldsymbol{X}^d) + A^{\mathrm{T}} H_i^{\mathrm{T}} K_{l+1}[I_n + H_i B S^{-1} B^{\mathrm{T}} H_i^{\mathrm{T}} K_{l+1}]^{-1} H_i A \boldsymbol{X}^{i^*} + \\ & A^{\mathrm{T}} H_i^{\mathrm{T}}[I_n + K_{l+1} H_i B S^{-1} B^{\mathrm{T}} H_i^{\mathrm{T}}]^{-1} \boldsymbol{b}_{l+1}. \end{aligned} \quad (27)$$

Hence, K and b satisfy, respectively, Eqs. (16) and (17) with $K(k_1^i) = 0_{n\times n}$ and $\boldsymbol{b}(k_1^i) = \boldsymbol{0}_{n\times 1}$ positively to ensure that the equality in Eq. (27) holds no matter what $K(l)$ and $\boldsymbol{b}(l)$ are.

3) Moreover, We know $P_{l+1}^{\mathrm{T}} H_i A = P^{\mathrm{T}} - (\boldsymbol{X}^{i^*} - \boldsymbol{X}^d)^{\mathrm{T}} L$ and $P_{l+1}^{\mathrm{T}} H_i B = -\boldsymbol{u}^{i^{*\mathrm{T}}} S$ from Eqs. (25) and (26). Substituting these into Eq. (22), we obtain

$$\begin{aligned} \min_{R_{[k,k_1^i-1]}} \Phi^i(\boldsymbol{X}(\cdot), R(\cdot)) = {} & \sum_{l=k}^{k_1^i-1}\{P^{\mathrm{T}} \boldsymbol{X}^{i^*} - P_{l+1}^{\mathrm{T}} \boldsymbol{X}_{l+1}^{i^*} - (\boldsymbol{X}^{i^*} - \boldsymbol{X}^d)^{\mathrm{T}} L \boldsymbol{X}^d\} = \\ & P^{\mathrm{T}}(k)\boldsymbol{X}^{i^*}(k) - \sum_{l=k}^{k_1^i-1}(\boldsymbol{X}^{i^*} - \boldsymbol{X}^d)^{\mathrm{T}} L \boldsymbol{X}^d = \\ & \boldsymbol{X}^{i^{*\mathrm{T}}} K(k)\boldsymbol{X}^{i^*} + \boldsymbol{b}^{\mathrm{T}}(k)\boldsymbol{X}^{i^*} + \varphi^i(k), \end{aligned} \quad (28)$$

for all $k \in [k_0^i, k_1^i - 1]$, where $\varphi^i(k)$ is the introduced variable defined as

$$\varphi^i(k) \triangleq -\sum_{l=k}^{k_1^i-1}(\boldsymbol{X}^{i^*} - \boldsymbol{X}^d)^{\mathrm{T}} L \boldsymbol{X}^d \quad (29)$$

with $\varphi^i(k_1^i)=0$. Obviously, $\varphi(k)$ satisfies Eq. (20), and accordingly, the minimum performance index is obtained as in Eq. (19). □

So far, we have solved the optimal fuzzy tracking problem by finding the optimal solution to the general time-varying case. We are now eager to know if a time-invariant fuzzy tracking system will give rise to time-invariant *linear* optimal tracking law in each stage. Some useful lemmas are demonstrated below in order to develop the design scheme of optimal fuzzy tracking law regarding to the time-invariant fuzzy tracking systems.

Lemma 2. Consider $\boldsymbol{X}(k+1)=\boldsymbol{f}(\boldsymbol{X}(k),\boldsymbol{u}(k),t)$ with $\boldsymbol{X}(k_0^i)=\boldsymbol{X}_0^i$ as the fixed-target tracking system. Let the pair $(\boldsymbol{X}^*(\cdot),\boldsymbol{u}^*(\cdot))$ be the infinite-horizon optimal solution with the performance index $J(\boldsymbol{u}(\cdot))=\sum_{k=k_0^i}^{\infty} f_0(\boldsymbol{X}(k),\boldsymbol{u}(k),t)$, and the pair $(\bar{\boldsymbol{X}}^*(\cdot),\bar{\boldsymbol{u}}^*(\cdot))$ be the finite-horizon optimal solution with respect to $\bar{J}(u(\cdot))=\sum_{k=k_0^i}^{k_1^i-1}\boldsymbol{f}_0(\boldsymbol{X}(k),\boldsymbol{u}(k),t)$, where both $\boldsymbol{f}(\cdot,\cdot,\cdot)$ and $\boldsymbol{f}_0(\cdot,\cdot,\cdot)$ are mapping from $\mathcal{R}^n\times\mathcal{R}^m\times\mathcal{R}$ to piecewise-continuous real-valued functions. If $\boldsymbol{X}(k_1^i)$ is a free point, then $(\bar{\boldsymbol{X}}^*(k),\bar{\boldsymbol{u}}^*(k))=(\boldsymbol{X}^*(k),\boldsymbol{u}^*(k))$ for all $k\in[k_0^i,k_1^i-1]$.

Lemma 3. (A_i,B_i) is completely controllable (c.c.) for all $i=1,\ldots,r$, if and only if $\operatorname{rank}[\lambda I_n-H(\boldsymbol{X}(k))A \quad H(\boldsymbol{X}(k))B]=n$, for all $\lambda\in\sigma(H(\boldsymbol{X}(k))A)$, where $\sigma(H(\boldsymbol{X}(k))A)$ denotes the spectrum of $H(\boldsymbol{X}(k))A$.

Lemma 4. (A_i,C) is completely observable (c.o.) for all $i=1,\ldots,r$, if and only if

$$\operatorname{rank}\begin{bmatrix}\lambda I_n-H(\boldsymbol{X}(k))A\\ C\end{bmatrix}=n,\quad \forall\lambda\in\sigma(H(\boldsymbol{X}(k))A).$$

Then, a more implementable and important theorem for the time-invariant fuzzy tracking system can be extracted on the ground of the aforementioned lemmas and Theorem 1, which concerns the time-varying fuzzy tracking system.

Theorem 2. (Time-invariant case) Consider the time-invariant fuzzy tracking system and fuzzy tracking controller described, respectively, by Eqs. (1) and (2). For the case that the moving target remains invariant during the whole single stage, i.e., $\boldsymbol{X}^d(k)=\bar{\boldsymbol{X}}_i^d$, $\forall k\in[k_0^i,k_1^i-1]$, let $(\boldsymbol{X}^*(k),R^*(k))$, $k\in[k_0,k_1-1]$, denote the optimal solution with respect to $J(R(\cdot))$ in Eq. (8), $(\boldsymbol{X}^{i^*}(k),R^{i^*}(k))$, $k\in[k_0^i,k_1^i-1]$, denote the ith-stage optimal solution with respect to $J^i(R(\cdot))$ in Eq. (11), and $(\boldsymbol{X}_\infty^{i^*}(k),R_\infty^{i^*}(k))$, $k\in[k_0^i,k_1^i-1]$, be the ith-stage asymptotically optimal solution with respect to

$$J_\infty^i(R(\cdot))=\sum_{k=k_0^i}^{\infty}[(\boldsymbol{X}(k)-\bar{\boldsymbol{X}}_i^d)^{\mathrm{T}}L(\boldsymbol{X}(k)-\bar{\boldsymbol{X}}_i^d)+R^{\mathrm{T}}(k)W_iSW_iR(k)]. \tag{30}$$

If $N>\bar{N}$, (A_i,B_i) is c.c. and (A_i,C) is c.o., for all $i=1,\ldots,r$, then,

1) $(\boldsymbol{X}^*(k),R^*(k))=(\boldsymbol{X}_\infty^{i^*}(k),R_\infty^{i^*}(k))$, $\forall k\in[k_0^i,k_1^i-1]$, $i=1,\ldots,N$, where $k_0^i=k_1^{i-1}$, $i=2,\ldots,N$, and $k_0^1=k_0$;

2) for the ith stage, $k\in[k_0^i,k_1^i-1]$, the optimal tracking law is

$$R_\infty^{i^*}(k)=W_i^{\mathrm{T}}[W_iW_i^{\mathrm{T}}]^{-1}[G_1^i\boldsymbol{X}_\infty^{i^*}(k)+\boldsymbol{u}_{ext}^i] \tag{31}$$

with the constant feedback gain, G_1^i, and the external input, $\boldsymbol{u}_{ext}^i$, calculated by

$$G_1^i = -S^{-1}B^{\mathrm{T}}H_i^{\mathrm{T}}\bar{\pi}^i[I_n+H_iBS^{-1}B^{\mathrm{T}}H_i^{\mathrm{T}}\bar{\pi}^i]^{-1}H_iA, \tag{32}$$

$$\boldsymbol{u}_{ext}^i = -S^{-1}B^{\mathrm{T}}H_i^{\mathrm{T}}[I_n+\bar{\pi}^iH_iBS^{-1}B^{\mathrm{T}}H_i^{\mathrm{T}}]^{-1}\bar{\boldsymbol{b}}^i, \tag{33}$$

$$\bar{\boldsymbol{b}}^i = -[I_n-A^{\mathrm{T}}H_i^{\mathrm{T}}(I_n+\bar{\pi}^iH_iBS^{-1}B^{\mathrm{T}}H_i^{\mathrm{T}})^{-1}]^{-1}L\bar{\boldsymbol{X}}_i^d, \tag{34}$$

where $\bar{\pi}^i$ is the unique symmetric positive semidefinte solution of the following discrete-time algebraic Riccati-like equation

$$K = L + A^{\mathrm{T}}H_i^{\mathrm{T}}K[I_n + H_iBS^{-1}B^{\mathrm{T}}H_i^{\mathrm{T}}K]^{-1}H_iA; \tag{35}$$

the ith-stage optimal tracking trajectory is

$$\boldsymbol{X}_{\infty}^{i^*}(k+1) = [I_n + H_iBS^{-1}B^{\mathrm{T}}H_i^{\mathrm{T}}\bar{\pi}^i]^{-1}H_iA\boldsymbol{X}^*(k) + H_iB\boldsymbol{u}_{ext}^i; \tag{36}$$

3) the minimum performance index is

$$\min_{R_{[k_0,k_1-1]}} J(R(\cdot)) = \sum_{i=1}^{N} \boldsymbol{X}_{\infty}^{i^{*\mathrm{T}}}(k_0^i)\bar{\pi}^i\boldsymbol{X}_{\infty}^{i^*}(k_0^i) + \bar{\boldsymbol{b}}^{i^{\mathrm{T}}}\boldsymbol{X}_{\infty}^{i^*}(k_0^i) + \varphi^i(k_0^i), \tag{37}$$

where $\varphi^i(k_1^i) = 0$ for all $i = 1,\dots,N$ and

$$\varphi^i(k) = \varphi^i(k+1) - (\boldsymbol{X}^{i^*}(k) - \bar{\boldsymbol{X}}_i^d)^{\mathrm{T}}L\bar{\boldsymbol{X}}_i^d. \tag{38}$$

Proof. 1) Based on Theorem 1, we know $(\boldsymbol{X}^*(k), R^*(k)) = (\boldsymbol{X}^{i^*}(k), R^{i^*}(k)), k \in [k_0^i, k_1^i - 1]$. Furthermore, via Lemma 2, we obtain $(\boldsymbol{X}^{i^*}(k), R^{i^*}(k)) = (\boldsymbol{X}_{\infty}^{i^*}(k), R_{\infty}^{i^*}(k))$ for the stage with constant target. Hence, the equivalence $(\boldsymbol{X}^*(k), R^*(k)) = (\boldsymbol{X}_{\infty}^{i^*}(k), R_{\infty}^{i^*}(k))$ exists positively on the time in period $k \in [k_0^i, k_1^i - 1]$, $i = 1,\dots,N$, if the target remain invariant during the whole single stage.

2) The optimal solution for each stage indeed follows the optimal solution in Theorem 1 except that all parameters in Eqs. (10) and (11) are constant now. It is easy to show that the asymptotically solution of the recursive Riccati-like equation in Eq. (16) is also the steady-state solution, i.e., $\lim_{k_1^i\to\infty}\pi(k, k_1^i) = \bar{\pi}^i$, which results in $\lim_{k_1^i\to\infty}\boldsymbol{b}(k) = \bar{\boldsymbol{b}}^i$.

3) Moreover, we know, from Lemmas 3 and 4, (A_i, B_i) is c.c., $\forall i = 1,\dots,r$, if and only if $\mathrm{rank}[\lambda I_n - H(\boldsymbol{X}(k))A \quad H(\boldsymbol{X}(k))B] = n$, $\forall\lambda \in \sigma(H(\boldsymbol{X}(k))A)$, and accordingly, $\mathrm{rank}[\lambda I_n - H_iA \ \ H_iB] = n$, $\forall\lambda \in \sigma(H_iA)$, $i = 1,\dots,N$. Also, we know (A_i, C) is c.o., $\forall i = 1,\dots,r$, if and only if $\mathrm{rank}\begin{bmatrix}\lambda I_n - H(\boldsymbol{X}(k))A \\ C\end{bmatrix} = n$, $\forall\lambda \in \sigma(H(\boldsymbol{X}(k))A)$, which ensures $\mathrm{rank}\begin{bmatrix}\lambda I_n - H_iA \\ C\end{bmatrix} = n$, $\forall\lambda \in \sigma(H_iA)$. Therefore, (A_i, B_i) c.c. and (A_i, C) c.o., $\forall i = 1,\dots,r$, guarantee, respectively, (H_iA, H_iB) c.c. and (H_iA, C) c.o., $\forall i = 1,\dots,N$, where r and N are, respectively, the number of rules of the fuzzy system in Eq. (1), and the number of stages of the process described by the dynamical fuzzy system in Eq. (10). Accordingly, by Lemma 5 in Appendix, we know the algebraic Riccati-like equation in Eq. (35) has unique symmetric positive semidefinte solution. Hence, the proof is completed. □

Thereupon, a time-invariant fuzzy tracking system can still give rise to the time-invariant linear optimal fuzzy tracking law for the stage with constant target.

3.1.2 Infinite-horizon Tracking Problem

The purpose of this section is to design the optimal fuzzy tracking controller concerning with the infinite-horizon tracking problem, which is the case that the operating time goes to infinity or is much larger than the time-constant of the dynamic system. We notice, as in optimal control problem, the issue: Does the minimal tracking performance index finitely exist? We introduce the concept proposed by Jack Machi and Aaron Strauss[22]: If the linearized system of a nonlinear system with respect to ($w.r.t$) some state $\boldsymbol{X}_o \in \mathcal{R}^n$ is c.c., then X_o is an interior point of the controllable set (the set of all initial points which can be steered to the target). Now, the linearized system of the fuzzy system in Eq. (4) with respect to point $\boldsymbol{X}_o$ is

$$\boldsymbol{X}(k+1) = H(\boldsymbol{X}_o)A(k)\boldsymbol{X}(k) + H(\boldsymbol{X}_o)B(k)\boldsymbol{u}(k). \tag{39}$$

Therefore, to ensure finite tracking cost, it is necessary that the pair $(H(\boldsymbol{X}_o)A(\cdot), H(\boldsymbol{X}_o)B(\cdot))$ is controllable at all time and for all $\boldsymbol{X}_o \in \mathcal{R}^n$. Furthermore, there is another obvious constraint that the moving target will remain invariant as time increasing to some extent, i.e., $\boldsymbol{X}^d(k) = \boldsymbol{X}^d(k+1)$, $\forall k \geqslant K_o$, where K_o is a large positive number. We can now find out the design scheme of the infinite-horizon optimal fuzzy tracking controller.

Theorem 3.(Time-varying case) For the fuzzy tracking system and fuzzy tracking controller described by Eqs. (1) and (2), respectively, let $(\boldsymbol{X}^*_\infty(k), R^*_\infty(k))$, $k \in [k_0, \infty)$, be the optimal solution with respect to

$$J_\infty(R(\cdot)) = \sum_{k=k_0}^{\infty}[(\boldsymbol{X}(k)-\boldsymbol{X}^d(k))^{\mathrm{T}}L(k)(\boldsymbol{X}(k)-\boldsymbol{X}^d(k)) + R^{\mathrm{T}}(k)W^{\mathrm{T}}(Y(k))S(k)W(Y(k))R(k)], \tag{40}$$

and $(\boldsymbol{X}^{i^*}(k), R^{i^*}(k))$, $k \in [k_0^i, k_1^i - 1]$, be the ith-stage optimal solution with respect to $J^i(R(\cdot))$ in Eq. (11) except that $k_1^N = \infty$ now. If $N > \bar{N}$, $\boldsymbol{X}^d(k)$ remains invariant for all $k \geqslant K_o$, and the linearized fuzzy system in Eq. (39) is controllable, then

1) $(\boldsymbol{X}^*_\infty(k), R^*_\infty(k)) = (\boldsymbol{X}^{i^*}(k), R^{i^*}(k))$, $k \in [k_0^i, k_1^i - 1]$, $i = 1, \ldots, N$, where $k_0^i = k_1^{i-1}$, $i = 2, \ldots, N$, $k_0^1 = k_0$, $k_1^N = \infty$;

2) for the ith stage, $k \in [k_0^i, k_1^i - 1]$, $i = 1, \ldots, N$, the optimal tracking law, the corresponding optimal tracking controller and the optimal tracking trajectory satisfy the same corresponding equations in Theorem 1, except that $k_1^N = \infty$; the minimum performance index, $\min_{R_{[k_0,\infty)}} J_\infty(R(\cdot))$, is finite and calculated by Eq. (19) except that the boundary condition of $\varphi^N(k)$ in Eq. (20) for the Nth stage is $\varphi^N(\infty) = 0$ now.

Proof. This theorem obviously holds with Theorem 1. For the Nth stage, the controllability of the fuzzy system can ensure the existence of the limit value of $\pi^N(k, k_1)$; i.e., $\bar{\pi}^N(k) = \lim_{k_1 \to \infty} \pi^N(k, k_1)$ exists for all $k \in [k_0^N, k_1]$, and $\bar{\pi}^N(k)$ is still the symmetric positive semidefinite solution of the segmental recursive Riccati-like equation in Eq. (16). □

For the time-invariant case, the pair $(H(\boldsymbol{X}_o)A, H(\boldsymbol{X}_o)B)$ being completely controllable is equivalent to $rank[\lambda I_n - H(\boldsymbol{X}_o)A, \; H(\boldsymbol{X}_o)B] = n$, $\forall \lambda \in \sigma(H(\boldsymbol{X}_o)A)$, and this condition, by Lemma 3, can be satisfied if (A_i, B_i) is c.c., for all $i = 1, \ldots, r$. So, we need the following assumption as the prerequisite to ensure *finite* tracking cost in the time-invariant infinite-horizon tracking controller design.

Assumption 2. (A_i, B_i) is c.c., for all $i = 1, \ldots, r$.

Theorem 4. (Time-invariant case) Consider the time-invariant fuzzy tracking system and fuzzy tracking controller described, respectively, by Eqs. (1) and (2). For the case that the moving target remains invariant during the whole single stage, if $N > \bar{N}$, (A_i, B_i) is c.c. and (A_i, C) is c.o., for all $i = 1, \ldots, r$, then,

1) $(\boldsymbol{X}^*_\infty(k), R^*_\infty(k)) = (\boldsymbol{X}^{i^*}_\infty(k), R^{i^*}_\infty(k))$, $k \in [k_0^i, k_1^i - 1]$, $i = 1, \ldots, N$, where $k_0^1 = k_0$, $k_1^N = \infty$, $k_0^i = k_1^{i-1}$, $i = 2, \ldots, N$; $R^{i^*}_\infty(k)$ is the ith-stage asymptotically optimal tracking law in Eq. (31), and $\boldsymbol{X}^{i^*}_\infty(k)$ is the corresponding asymptotically optimal tracking trajectory in Eq. (36), where $\bar{\pi}^i$ is the unique symmetric positive semidefinte solution of the discrete-time algebraic Riccati-like equation in Eq. (35);

2) the minimum performance index, $\min_{R_{[k_0,\infty)}} J_\infty(R(\cdot))$, is finite and calculated by Eq. (37) except that the boundary condition of $\varphi^N(k)$ in Eq. (38) for the Nth stage is $\varphi^N(\infty) = 0$ now.

Proof. This theorem obviously holds according to Theorem 2. □

3.2 Model-Following Tracking Problem

Now, we shall be devoted to the model-following tracking problem, where the tracked target is the response of some reference model. Similar to the previous section, the finite-horizon tracking problem is discussed first. The derived optimal solutions can then be generalized into those for infinite-horizon problem as we did in the last section. We adopt the same T-S type fuzzy tracking system as in Section 2, and thereupon, the standard model-following tracking problem can be described as the following issue.

Problem 3. Given a fuzzy tracking system in Eq. (4) with $\boldsymbol{X}(k_0) = \boldsymbol{X}_0 \in \mathcal{R}^n$ and $k \in [k_0, k_1 - 1]$, find $R^*(\cdot)$ to minimize $J(R(\cdot))$ in Eq. (8), where the desired output $\boldsymbol{Y}^d(k)$ is the response of a linear system or model,

$$\begin{aligned} \boldsymbol{z}_1(k+1) &= F_1(k)\boldsymbol{z}_1(k) + J_1(k)\boldsymbol{\nu}(k), \\ \boldsymbol{Y}^d(k) &= E_1(k)\boldsymbol{z}_1(k) \end{aligned} \tag{41}$$

with $\boldsymbol{z}_1(k_0) = \boldsymbol{z}_{10}$, to the command input $\boldsymbol{\nu}(k) \in \mathcal{R}^{m'}$, which is the zero response of the system: $\boldsymbol{z}_2(k+1) = F_2(k)\boldsymbol{z}_2(k)$ and $\boldsymbol{\nu}(k) = E_2(k)\boldsymbol{z}_2(k)$ with $\boldsymbol{z}_2(k_0) = \boldsymbol{z}_{20}$ [21], where $\boldsymbol{z}_1(k) \in \mathcal{R}^h$ and $\boldsymbol{z}_2(k) \in \mathcal{R}^{h'}$ are system states; $F_1(k)$, $J_1(k)$, $E_1(k)$, $F_2(k)$ and $E_2(k)$ are matrices of $h \times h$, $h \times m'$, $n' \times h$, $h' \times h'$ and $m' \times h'$, respectively.

Accordingly, the desired tracked system, via letting $\boldsymbol{Z}(k) = [z_1^{\mathrm{T}}(k) \quad z_2^{\mathrm{T}}(k)]^{\mathrm{T}}$, can be rewritten as the following augmented system[21]

$$\begin{aligned} \boldsymbol{Z}(k+1) &= \begin{bmatrix} F_1(k) & J_1(k)E_2(k) \\ 0_{h'\times h} & F_2(k) \end{bmatrix} \boldsymbol{Z}(k) = F(k)\boldsymbol{Z}(k), \\ \boldsymbol{Y}^d(k) &= [E_1(k) \quad 0_{n'\times h'}]\boldsymbol{Z}(k) = E(k)\boldsymbol{Z}(k). \end{aligned} \tag{42}$$

We further define a new variable $\tilde{\boldsymbol{X}}(k) = [\boldsymbol{X}^{\mathrm{T}}(k) \quad \boldsymbol{Z}^{\mathrm{T}}(k)]^{\mathrm{T}}$[21], and then Problem 3 can be simplified as the issue below.

Problem 4. Given a fuzzy tracking system

$$\tilde{\boldsymbol{X}}(k+1) = \tilde{A}(k)\tilde{\boldsymbol{X}}(k) + \tilde{B}(k)W(Y(k))R(k) \tag{43}$$

with $\tilde{\boldsymbol{X}}(k_0) = \tilde{\boldsymbol{X}}_0 \in \mathcal{R}^{n+h+h'}$ and $k \in [k_0, k_1 - 1]$, find $R^*(\cdot)$ to minimize

$$J(R(\cdot)) = \sum_{k=k_0}^{k_1-1} [\tilde{\boldsymbol{X}}^{\mathrm{T}}(k)\tilde{L}(k)\tilde{\boldsymbol{X}}(k) + R^{\mathrm{T}}(k)W^{\mathrm{T}}(\boldsymbol{Y}(k))S(k)W(\boldsymbol{Y}(k))R(k)], \tag{44}$$

where the parameters are $\tilde{B}(k) = \begin{bmatrix} H(k)B(k) \\ 0_{(h+h')\times m} \end{bmatrix}$, $\tilde{A}(k) = \begin{bmatrix} H(k)A(k) & 0_{n\times(h+h')} \\ 0_{(h+h')\times n} & F(k) \end{bmatrix}$ and

$$\tilde{L}(k) = \begin{bmatrix} L(k) & -L(k)C^{\mathrm{T}}(k)[C(k)C^{\mathrm{T}}(k)]^{-1}E(k) \\ -E^{\mathrm{T}}(k)[C(k)C^{\mathrm{T}}(k)]^{-1}C(k)L(k) & E^{\mathrm{T}}(k)[C(k)C^{\mathrm{T}}(k)]^{-1}C(k)L(k)C^{\mathrm{T}}(k)[C(k)C^{\mathrm{T}}(k)]^{-1}E(k) \end{bmatrix}.$$

Obviously, the optimal solutions for the augmented optimal quadratic tracking problem in Problem 4 follow from Theorem 1 except that $\boldsymbol{X}^d(\cdot)$ in Problem 2 are zero vectors now. Then, via complicated matrix manipulations, we can obtain the optimal solutions for the original *optimal quadratic* tracking problem in Problem 3 as follows. The identity input weighting factor is set to get more concise formula in the remainder of this section, i.e., $S(\cdot) = I_m$ for all time steps.

Theorem 5. (Time-varying case) For the fuzzy tracking system and fuzzy tracking controller represented, respectively, by Eqs. (1) and (2), let $(\boldsymbol{X}^*(k), R^*(k))$, $k \in [k_0, k_1 - 1]$, be the optimal solution with respect to $J(R(\cdot))$ in Eq. (8), and $(\boldsymbol{X}^{i^*}(k), R^{i^*}(k))$, $k \in [k_0^i, k_1^i - 1]$, be the ith-stage optimal solution with respect to $J^i(R(\cdot))$ in Eq. (11), where the desired trajectory, $\boldsymbol{X}^d(k)$, comes from $\boldsymbol{Y}^d(k) = C\boldsymbol{X}^d(k)$ and $\boldsymbol{Y}^d(k)$ is the output of the tracked model in Eq. (41). If $N > \bar{N}$ then

1) $(\boldsymbol{X}^*(k), R^*(k)) = (\boldsymbol{X}^{i^*}(k), R^{i^*}(k))$, for all $k \in [k_0^i, k_1^i - 1]$, $i = 1, \ldots, N$, where $k_0^1 = k_0$, $k_1^N = k_1$, $k_0^i = k_1^{i-1}$, $i = 2, \ldots, N$;

2) for the ith stage, $k \in [k_0^i, k_1^i - 1]$, the optimal tracking controller is

$$\boldsymbol{u}^{i^*}(k) = G_1^i(k)\boldsymbol{X}^{i^*}(k) + G_2^i(k)\boldsymbol{Z}(k), \tag{45}$$

and the corresponding optimal tracking law is

$$R^{i^*}(k) = W_i^{\mathrm{T}}[W_i W_i^{\mathrm{T}}]^{-1}[G_1^i(k)\boldsymbol{X}^{i^*}(k) + G_2^i(k)\boldsymbol{Z}(k)], \tag{46}$$

where the feedback gain, $G_1^i(k)$, and the introduced matrix, $G_2^i(k)$, are calculated by

$$G_1^i(k) = -B^{\mathrm{T}}(k)H_i^{\mathrm{T}}\pi^i(k+1, k_1^i)[I_n + H_iB(k)B^{\mathrm{T}}(k)H_i^{\mathrm{T}}\pi^i(k+1, k_1^i)]^{-1}H_iA(k), \tag{47}$$

$$G_2^i(k) = -B^{\mathrm{T}}(k)H_i^{\mathrm{T}}[I_n + \pi^i(k+1, k_1^i)H_iB(k)B^{\mathrm{T}}(k)H_i^{\mathrm{T}}]^{-1}\pi_{21}^{i\mathrm{T}}(k+1, k_1^i)F(k), \tag{48}$$

where $\pi^i(k+1, k_1^i)$ is the symmetric positive semidefinte solution of the following *segmental* recursive Riccati-like equation

$$K(k) = L(k) + A^{\mathrm{T}}(k)H_i^{\mathrm{T}}K(k+1)[I_n + H_iB(k)B^{\mathrm{T}}(k)H_i^{\mathrm{T}}K(k+1)]^{-1}H_iA(k), \tag{49}$$

with $K(k_1^i) = 0_n$, and $\pi_{21}^i(k+1, k_1^i)$ satisfies

$$\begin{aligned} K_{21}(k) = {} & F^{\mathrm{T}}(k)K_{21}(k+1)[I_n + H_iB(k)B^{\mathrm{T}}(k)H_i^{\mathrm{T}}\pi^i(k+1, k_1^i)]^{-1}H_iA(k) - \\ & E^{\mathrm{T}}(k)[C(k)C^{\mathrm{T}}(k)]^{-1}C(k)L(k) \end{aligned} \tag{50}$$

with $K_{21}(k_1^i) = 0_{(h+h')\times n}$; the ith-stage optimal tracking trajectory is

$$\begin{aligned} \boldsymbol{X}^{i^*}(k+1) = {} & [I_n + H_iB(k)B^{\mathrm{T}}(k)H_i^{\mathrm{T}}\pi^i(k+1, k_1^i)]^{-1}H_iA(k)\boldsymbol{X}^{i^*}(k) + \\ & H_iB(k)G_2^i(k)\boldsymbol{Z}(k); \end{aligned} \tag{51}$$

3) the minimum performance index is

$$\begin{aligned} \min_{R_{[k_0, k_1-1]}} J(R(\cdot)) = {} & \sum_{i=1}^{N} \boldsymbol{X}^{i^*\mathrm{T}}(k_0^i)\pi^i(k_0^i, k_1^i)\boldsymbol{X}^{i^*}(k_0^i) + 2\boldsymbol{X}^{i^*\mathrm{T}}(k_0^i)\pi_{21}^{i\mathrm{T}}(k_0^i, k_1^i)\boldsymbol{Z}(k_0^i) + \\ & \boldsymbol{Z}^{\mathrm{T}}(k_0^i)\pi_{22}^i(k_0^i, k_1^i)\boldsymbol{Z}(k_0^i), \end{aligned} \tag{52}$$

where the introduced variable $\pi_{22}^i(k)$ satisfies

$$\begin{aligned} K_{22}(k) = {} & -F^{\mathrm{T}}(k)\pi_{21}^i(k+1, k_1^i)[I_n + H_iB(k)B^{\mathrm{T}}(k)H_i^{\mathrm{T}}\pi^i(k+1, k_1^i)]^{-1}H_iB(k)B^{\mathrm{T}}(k)\cdot \\ & H_i^{\mathrm{T}}\pi_{21}^{i\mathrm{T}}(k+1, k_1^i)\cdot F(k) + F^{\mathrm{T}}(k)K_{22}(k+1)F(k) + E^{\mathrm{T}}(k)[C(k)C^{\mathrm{T}}(k)]^{-1}\cdot \\ & C(k)L(k)C^{\mathrm{T}}(k)[C(k)C^{\mathrm{T}}(k)]^{-1}E(k), \end{aligned} \tag{53}$$

with $K_{22}(k_1^i) = 0_{(h+h')\times(h+h')}$, $\forall i = 1, \ldots, N$.

Proof. 1) For notation simplification, the identity and zero matrices of any dimension will be denoted by I and 0, respectively. Grounding on Theorem 1, we have

$$\tilde{\boldsymbol{X}}^{i^*}(k+1) = [I + \tilde{B}(k)\tilde{B}^{\mathrm{T}}(k)\tilde{\pi}^i(k+1, k_1^i)]^{-1}\tilde{A}(k)\tilde{\boldsymbol{X}}^{i^*}(k), \tag{54}$$

$$\boldsymbol{u}^{i^*}(k) = -\tilde{B}^{\mathrm{T}}(k)\tilde{\pi}^i(k+1, k_1^i)[I + \tilde{B}(k)\tilde{B}^{\mathrm{T}}(k)\tilde{\pi}^i(k+1, k_1^i)]^{-1}\tilde{A}(k)\tilde{\boldsymbol{X}}^{i^*}(k), \tag{55}$$

$$R^{i^*}(k) = W_i^{\mathrm{T}}[W_iW_i^{\mathrm{T}}]^{-1}\boldsymbol{u}^{i^*}(k), \tag{56}$$

$$\min_{R_{[k_0, k_1-1]}} J(R(\cdot)) = \sum_{i=1}^{N} \tilde{\boldsymbol{X}}^{i^*\mathrm{T}}(k_0^i)\tilde{\pi}^i(k_0^i, k_1^i)\tilde{\boldsymbol{X}}^{i^*}(k_0^i), \tag{57}$$

where $\tilde{\pi}^i(k,k_1^i)$ is the symmetric positive semidefinite solution of the the following segmental recursive Riccati-like equation

$$\tilde{K}(k)=\tilde{A}^{\mathrm{T}}(k)\tilde{K}(k+1)[I+\tilde{B}(k)\tilde{B}^{\mathrm{T}}(k)\tilde{K}(k+1)]^{-1}\tilde{A}(k)+\tilde{L}(k),\ \tilde{K}(k_1^i)=0. \tag{58}$$

Now, let $\tilde{K}(k)=\begin{bmatrix}K(k) & K_{21}^{\mathrm{T}}(k)\\ K_{21}(k) & K_{22}(k)\end{bmatrix}$. Then, we obtain $\boldsymbol{u}^{i^*}(k)$ in Eq. (45) from Eq. (55) via

$$\begin{bmatrix}I+H_iBB^{\mathrm{T}}H_i^{\mathrm{T}}K & H_iBB^{\mathrm{T}}H_i^{\mathrm{T}}K_{21}^{\mathrm{T}}\\ 0 & I\end{bmatrix}^{-1}=$$
$$\begin{bmatrix}[I+H_iBB^{\mathrm{T}}H_i^{\mathrm{T}}K]^{-1} & -[I+H_iBB^{\mathrm{T}}H_i^{\mathrm{T}}K]^{-1}H_iBB^{\mathrm{T}}H_i^{\mathrm{T}}K_{21}^{\mathrm{T}}\\ 0 & I\end{bmatrix},$$

and $B^{\mathrm{T}}H_i^{\mathrm{T}}K_{21}^{\mathrm{T}}-B^{\mathrm{T}}H_i^{\mathrm{T}}K[I+H_iBB^{\mathrm{T}}H_i^{\mathrm{T}}K]^{-1}H_iBB^{\mathrm{T}}H_i^{\mathrm{T}}K_{21}^{\mathrm{T}}=B^{\mathrm{T}}H_i^{\mathrm{T}}[I+KH_iBB^{\mathrm{T}}H_i^{\mathrm{T}}]^{-1}\cdot K_{21}^{\mathrm{T}}$, where the time-dependence is omitted for notation simplification; $K(k)$ in Eq. (49), K_{21} in Eq. (50) and K_{22} in Eq. (53) are derived from Eq. (58) with the aid of

$$I-K[I+H_iBB^{\mathrm{T}}H_i^{\mathrm{T}}K]^{-1}H_iBB^{\mathrm{T}}H_i^{\mathrm{T}}=[I+KH_iBB^{\mathrm{T}}H_i^{\mathrm{T}}]^{-1};$$

and then we have $\boldsymbol{X}^{i^*}(k)$ in Eq. (51) and $\min_{R_{[k_0,k_1-1]}}J(R(\cdot))$ in Eq. (52). □

As for the time-invariant system, since the target for Problem 4 can be viewed as staying at zero, we can use Lemma 2 to obtain the following equality

$$\boldsymbol{X}^*(k)\equiv\boldsymbol{X}^{i^*}(k)\equiv\boldsymbol{X}_{\infty}^{i^*}(k),\ \forall k\in[k_0^i,k_1^i-1], \tag{59}$$

where $i=1,\dots,N$ and $k_0^1=k_0,k_1^N=k_1,k_0^i=k_1^{i-1},i=2,\dots,N$. Then, via Lemmas 3 and 4 in Section 3.1 and Lemma 5 in the Appendix, we can derive the design scheme of time-invariant fuzzy tracker for model-following target.

Theorem 6. (Time-invariant case) Consider the time-invariant fuzzy tracking system and fuzzy tracking controller described, respectively, by Eqs. (1) and (2). Let $(\boldsymbol{X}^*(k),R^*(k))$, $k\in[k_0,k_1-1]$, denote the optimal solution with respect to $J(R(\cdot))$ in Eq. (8), $(\boldsymbol{X}^{i^*}(k),R^{i^*}(k))$, $k\in[k_0^i,k_1^i-1]$, denote the ith-stage optimal solution with respect to $J^i(R(\cdot))$ in Eq. (11), and $(\boldsymbol{X}_{\infty}^{i^*}(k),\boldsymbol{R}_{\infty}^{i^*}(k))$, $k\in[k_0^i,k_1^i-1]$, be the ith-stage asymptotically optimal solution with respect to $J_{\infty}^i(R(\cdot))$ in Eq. (30), where the desired trajectory, $\boldsymbol{X}^d(k)$, comes from $\boldsymbol{Y}^d(k)=C\boldsymbol{X}^d(k)$ and $\boldsymbol{Y}^d(k)$ is the output of the time-invariant tracked model in Eq. (41). If $N>\bar{N}$, (A_i,B_i) is c.c. and (A_i,C) is c.o., for all $i=1,\dots,r$, then,

1) $(\boldsymbol{X}^*(k),R^*(k))=(\boldsymbol{X}_{\infty}^{i^*}(k),R_{\infty}^{i^*}(k))$, $\forall k\in[k_0^i,k_1^i-1]$, $i=1,\dots,N-1$, where $k_0^i=k_1^{i-1}$, $i=2,\dots,N$, $k_0^1=k_0$;

2) for the ith stage, $k\in[k_0^i,k_1^i-1]$, the optimal tracking controller is

$$\boldsymbol{u}_{\infty}^{i^*}(k)=G_1^i\boldsymbol{X}_{\infty}^{i^*}(k)+G_2^i\boldsymbol{Z}(k), \tag{60}$$

and the corresponding optimal tracking law is

$$R_{\infty}^{i^*}(k)=W_i^{\mathrm{T}}[W_iW_i^{\mathrm{T}}]^{-1}[G_1^i\boldsymbol{X}_{\infty}^{i^*}(k)+G_2^i\boldsymbol{Z}(k)], \tag{61}$$

where the feedback gain, G_1^i, and the introduced matrix, G_2^i, are calculated by

$$G_1^i=-B^{\mathrm{T}}H_i^{\mathrm{T}}\bar{\pi}^i[I_n+H_iBB^{\mathrm{T}}H_i^{\mathrm{T}}\bar{\pi}^i]^{-1}H_iA, \tag{62}$$
$$G_2^i=-B^{\mathrm{T}}H_i^{\mathrm{T}}[I_n+\bar{\pi}^iH_iBB^{\mathrm{T}}H_i^{\mathrm{T}}]^{-1}\bar{\pi}_{21}^{i\mathrm{T}}F, \tag{63}$$

where $\bar{\pi}^i$ is the unique symmetric positive semidefinte solution of the following discrete-time algebraic Riccati-like equation

$$K=L+A^{\mathrm{T}}H_i^{\mathrm{T}}K[I_n+H_iBB^{\mathrm{T}}H_i^{\mathrm{T}}K]^{-1}H_iA, \tag{64}$$

and $\bar{\pi}_{21}^i$ satisfies

$$K_{21} = F^{\mathrm{T}}K_{21}[I_n + H_iBB^{\mathrm{T}}H_i^{\mathrm{T}}\bar{\pi}^i]^{-1}H_iA - E^{\mathrm{T}}[CC^{\mathrm{T}}]^{-1}CL, \tag{65}$$

the ith-stage optimal tracking trajectory is

$$\boldsymbol{X}_{\infty}^{i^*}(k+1) = [I_n + H_iBB^{\mathrm{T}}H_i^{\mathrm{T}}\bar{\pi}^i]^{-1}H_iA\boldsymbol{X}_{\infty}^{i^*}(k) + H_iBG_2^i\boldsymbol{Z}(k); \tag{66}$$

3) the minimum performance index is

$$\min_{R_{[k_0,k_1-1]}} J(R(\cdot)) = \sum_{i=1}^{N} \boldsymbol{X}_{\infty}^{i^*\mathrm{T}}(k_0^i)\bar{\pi}^i\boldsymbol{X}_{\infty}^{i^*}(k_0^i) + 2\boldsymbol{X}_{\infty}^{i^*\mathrm{T}}(k_0^i)\bar{\pi}_{21}^{i\mathrm{T}}\boldsymbol{Z}(k_0^i) + \boldsymbol{Z}^{\mathrm{T}}(k_0^i)\bar{\pi}_{22}^i\boldsymbol{Z}(k_0^i), \tag{67}$$

where the introduced variable $\bar{\pi}_{22}^i$ satisfies

$$\begin{aligned} K_{22} \ = \ & -F^{\mathrm{T}}\bar{\pi}_{21}^i[I_n + H_iBB^{\mathrm{T}}H_i^{\mathrm{T}}\bar{\pi}^i]^{-1}H_iBB^{\mathrm{T}}H_i^{\mathrm{T}}\bar{\pi}_{21}^{i\mathrm{T}}F + F^{\mathrm{T}}K_{22}F \\ & + E^i[CC^{\mathrm{T}}]^{-1}CLC^{\mathrm{T}}[CC^{\mathrm{T}}]^{-1}E. \end{aligned} \tag{68}$$

Proof. Grounded on Theorem 5, the proof follows the derivation in Theorems 2. □

For model-following target, the infinite tracking cost is unavoidable. Apart from this, the scheme of generalizing the optimal tracking solution from finite-horizon problem to infinite-horizon problem for model-following target is just the same as that for moving target in Section 3.1.2. Therefore, we shall not demonstrate the solutions of infinite-horizon problem with respect to model-following tracking here.

4 Numerical Simulations

In this section, we consider a computer simulated trunk-trailer system to track a moving target or a model-following target. The computer simulated truck-trailer physical system was described by Tanaka and Sano[16] as

$$\begin{aligned} x_1(k+1) &= (1 - v\cdot t'/L')x_1(k) + v\cdot t'/l\cdot u(k), \\ x_2(k+1) &= x_2(k) + v\cdot t'/L'\cdot x_1(k), \\ x_3(k+1) &= x_3(k) + v\cdot t'\cdot \sin(x_2(k) + v\cdot t'/2L'\cdot x_1(k)), \end{aligned}$$

where l is the length of truck, L' is the length of trailer, t' is the sampling time, and v is the constant speed of the backward movement. Then, they used the following fuzzy model to represent the above mathematical model:

R^1 : If $z(k) \equiv x_2(k) + v\cdot t'/\{2L'\}\cdot x_1(k)$ is about 0, then $\boldsymbol{X}(k+1) = A_1\boldsymbol{X}(k) + \boldsymbol{B}_1u(k)$

R^2 : If $z(k) \equiv x_2(k) + v\cdot t'/\{2L'\}\cdot x_1(k)$ is about π or $-\pi$, then $\boldsymbol{X}(k+1) = A_2\boldsymbol{X}(k) + \boldsymbol{B}_2u(k)$, and the system output is $Y(k) = \boldsymbol{CX}(k)$ with $\boldsymbol{C} = [0\ 0\ 1]$, $l = 2.8$, $L' = 5.5$, $v = -1.0$, $t' = 2.0$ and $\boldsymbol{X}(k) = [x_1(k), x_2(k), x_3(k)]^{\mathrm{T}}$, where

$$A_1 = \begin{bmatrix} 1.3636 & 0 & 0 \\ -0.3636 & 1.0 & 0 \\ 0.0120 & -2.0 & 1.0 \end{bmatrix},\ A_2 = \begin{bmatrix} 1.3636 & 0 & 0 \\ -0.3636 & 1.0 & 0 \\ 0 & -0.0064 & 1.0 \end{bmatrix},\ \boldsymbol{B}_1 = \boldsymbol{B}_2 = \begin{bmatrix} -0.7143 \\ 0 \\ 0 \end{bmatrix}.$$

Grounding on this fuzzy system, we assume our fuzzy tracking controller as

R^1 : If $z(k) \equiv x_2(k) + v\cdot t'/\{2L'\}\cdot x_1(k)$ is about 0, then $u(k) = r_1(k)$

R^2 : If $z(k) \equiv x_2(k) + v\cdot t'/\{2L'\}\cdot x_1(k)$ is about π or $-\pi$, then $u(k) = r_2(k)$.

With the chosen membership functions[23], the firing-strengths are

$$\begin{aligned} h_1(\boldsymbol{X}(t)) &= \alpha_1(k) = (1 - 1/(1+\exp(-3(z(k)-\pi/2)))) \cdot (1/(1+\exp(-3(z(k)+\pi/2)))), \\ h_2(\boldsymbol{X}(t)) &= \alpha_2(k) = 1 - \alpha_1(k), \end{aligned}$$

which, in this case, are also the normalized firing-strengths of the rules for fuzzy system and controller. Therefore, the linear-like dynamical fuzzy system representation for the nonlinear truck-trailer system can be described by Eq. (4) with $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, $\boldsymbol{R} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$, $H(\boldsymbol{X}(k)) = [h_1(\boldsymbol{X}(k))\ h_2(\boldsymbol{X}(k))]$ and $W(\boldsymbol{Y}(k)) = [w_1(\boldsymbol{Y}(k))\ w_2(\boldsymbol{Y}(k))]$.

We sample the continuous signals in such a frequency that the membership functions are almost invariant during the single sampling period, and then, the decomposition stages will be coincident with the sampling sequences in finite-horizon case. Hence, by choosing the number of stages as $N = 100$, we have the following N linear dynamical fuzzy systems

$$\begin{aligned} \boldsymbol{X}(k+1) &= H_i A \boldsymbol{X}(k) + H_i B W_i R(k),\ i = 1, \dots, 100, \\ Y(k) &= C\boldsymbol{X}(k), \end{aligned} \tag{69}$$

where $H_1 = H(\boldsymbol{X}_0)$, $W_1 = W(Y(k_0))$, $k_0^1 = 0$; $H_i = H(\boldsymbol{X}^*(k_1^{i-1}))$, $W_i = W(\boldsymbol{X}^*(k_1^{i-1}))$, $i = 2, \dots, 100$. The performance index for finite-horizon tracking problem is

$$J(R(\cdot)) = \sum_{k=0}^{99} [e^{\mathrm{T}}(k) L e(k) + R^{\mathrm{T}}(k) W^{\mathrm{T}}(k) S W(k) R(k)], \tag{70}$$

and that for infinite-horizon tracking problem is

$$J_\infty(R(\cdot)) = \sum_{k=0}^{\infty} [e^{\mathrm{T}}(k) L e(k) + R^{\mathrm{T}}(k) W^{\mathrm{T}}(k) S W(k) R(k)]. \tag{71}$$

Since no final error penalty is emphasized, the optimal tracking trajectory with step'wise or model-following target for these two are coincident obviously. Now, we can design the optimal fuzzy tracking controllers for the trunk-trailer tracking system in both cases of moving target and modeling-following target by the proposed design scheme in Section 3.

Though the fuzzy subsystem is unstable (the spectrum of system matrix $\sigma(A_i) = \{1, 1, 1.36\}$, $i = 1, 2$), it is time-invariant and well-behaved; i.e., the fuzzy subsystem is c.c. and c.o. ($\mathrm{rank}[\lambda I_3 - A_i\ B_i] = \mathrm{rank}[\lambda I_3 - A_i\ C] = 3$, for all $\lambda \in \sigma(A_i)$). Hence, the dynamical system describing the dynamics of each stage is also well-behaved according to Lemmas 3 and 4.

For the moving-target tracking problem, we can obtain, by Theorem 2, the optimal trajectory of the closed-loop fuzzy tracking system with the designed optimal fuzzy tracking controller. The position responses of the resultant closed-loop fuzzy tracking system for step or step'wise targets with different input weighting factors ($S = 1$ or 0.01) are shown in Fig.1. As for the model-following-target case, since each fuzzy subsystem is well-behaved as mentioned above, the optimal fuzzy tracking controller and the corresponding tracking trajectory can be obtained according to Theorem 6. Fig.2 shows the position responses of the resultant closed-loop fuzzy tracking system for the targets from the tracked model with various parameters ($(F_1, F_2) = (1, 1)$, $(1, 0.9)$, $(0.85, 0.85)$ and $(0.45, 0.45)$). Our simulation results also show that the designed optimal fuzzy tracking controller can efficiently push the simulated trunk-trailer system to trace the target as soon as possible.

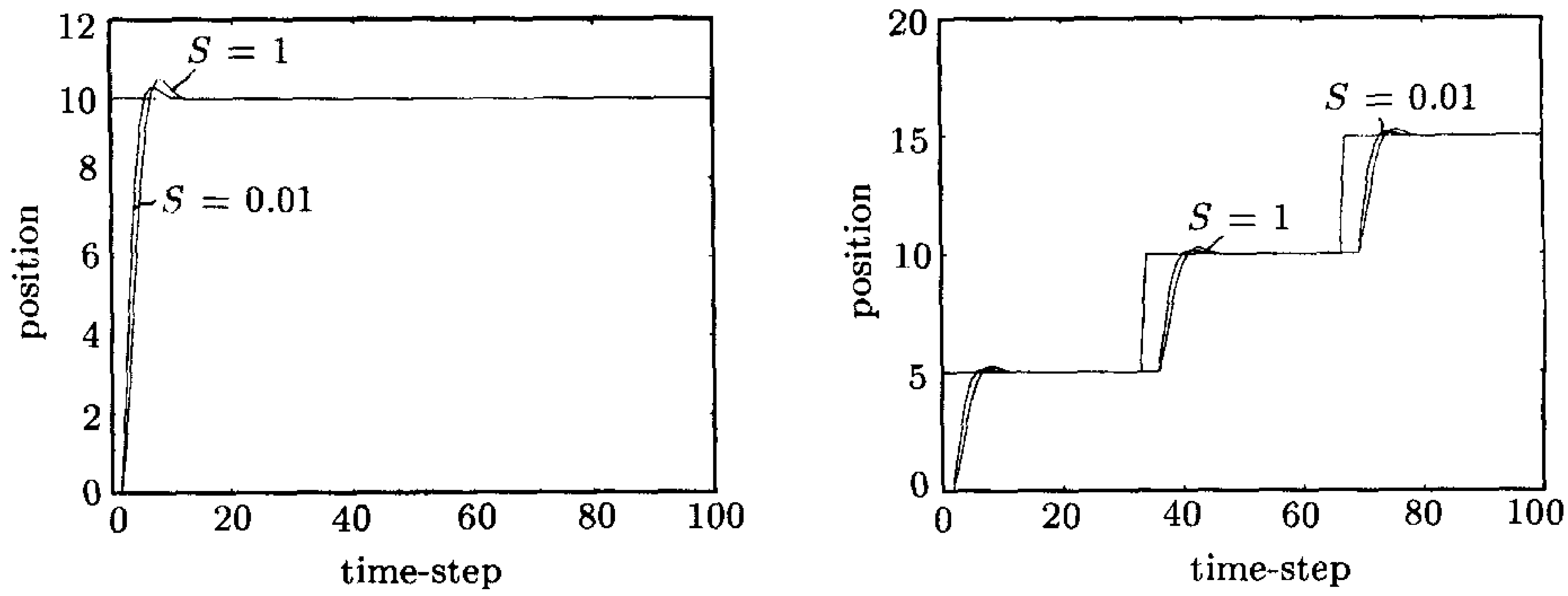

Fig.1 Position responses of the discrete-time fuzzy tracking system with the designed optimal fuzzy tracking controllers in Subsection 3.1 for step and step'wise targets at the two input weighting factors: $S = 1$ and $S = 0.01$

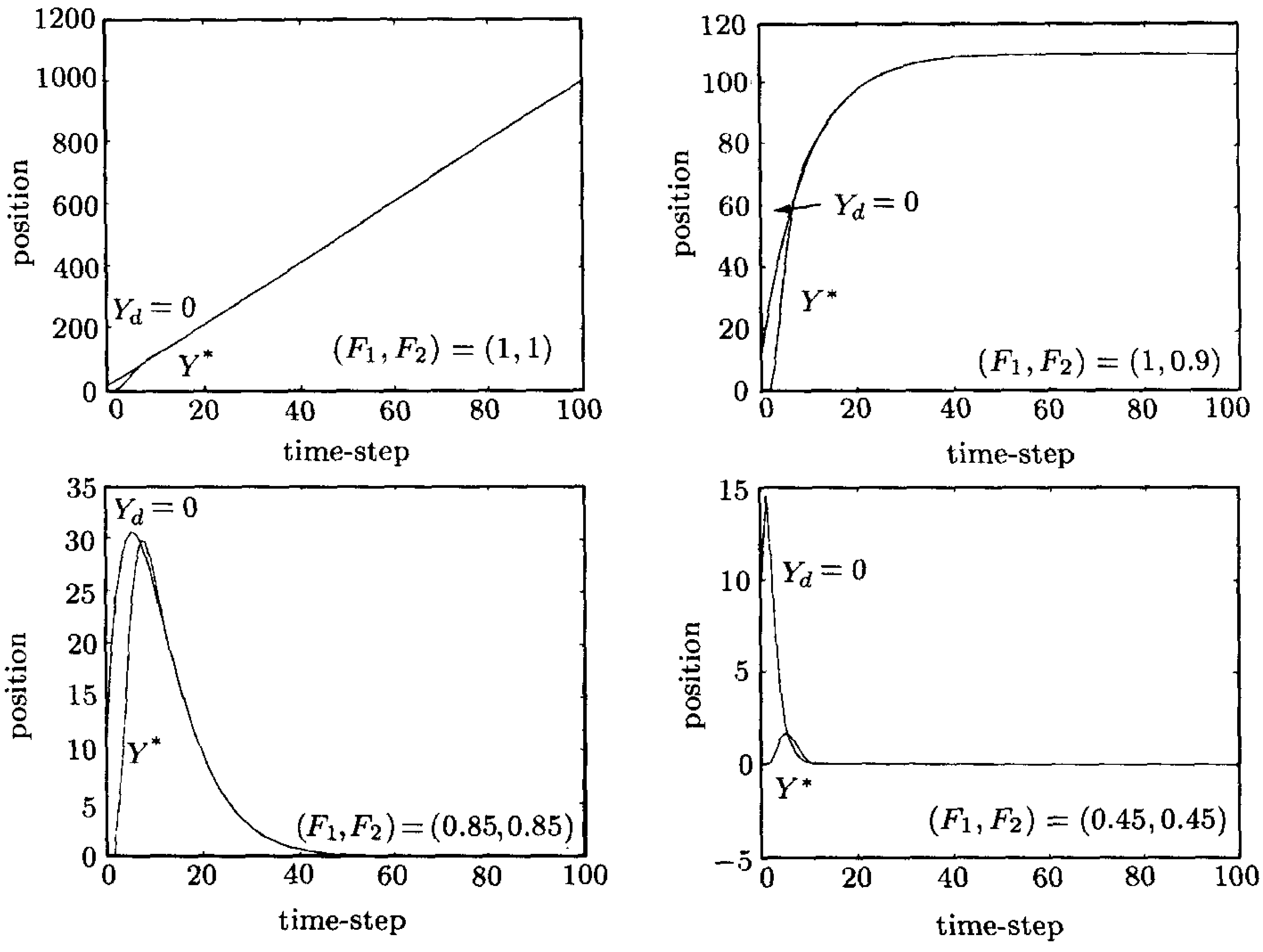

Fig.2 Position responses of the discrete-time fuzzy tracking system with the designed optimal fuzzy tracking controllers in Subsection 3.2 for the targets from the tracked model with the four different parameters: $(F_1, F_2) = (1, 0.9)$, $(1, 0.4)$, $(0.9, 0.4)$ and $(0.4, 0.4)$

5 Conclusions

A *linear-like* dynamical system representation of discrete-time fuzzy systems was proposed in this paper. Based on this representation, the design scheme of global optimal fuzzy tracking controllers for discrete-time fuzzy systems was derived theoretically. A multistage decomposition of optimization scheme was proposed to design the global optimal fuzzy tracking controller more efficiently. Simulation results have manifested that the designed optimal fuzzy tracking controllers can effectively drive a fuzzy system to trace the target profile in short time.

References

1 Wang H, Tanaka K, Griffin M. Parallel distributed compensation of nonlinear systems by Takagi and Sugeno's fuzzy model. In: Proc. FUZZ-IEEE'95, 1995. 531~538

2 Wang H O, Tanaka K, Griffin M F. An approach to fuzzy control of nonlinear system: stability and design issues. *IEEE Trans Fuzzy Systems*, 1996, **4**(1): 14~23

3 Tanaka K, Ikeda T, Wang H O. Robust stabilization of a class of uncertain nonlinear systems via fuzzy control: quadratic stabilizability, H^{∞} control theory and linear matrix inequalities. *IEEE Trans. Fuzzy Systems*, 1996, **4**(1): 1~13

4 Wang H O, Tanaka K. An LMI-based stable fuzzy control of nonlinear systems and its application to control of chaos. In: Proc. FUZZ-IEEE'96, 1996. 1433~1438

5 Tanaka K, Kosaki T. Design of a stable fuzzy controller for an articulated vehicle. *IEEE Trans. Syst. Man Cybern. B*, 1997, **27** (3): 552~558

6 Ma X J, Sun Z Q, He Y Y. Analysis and design of fuzzy controller and fuzzy observer. *IEEE Trans. Fuzzy Syst.*, 1998, **6**(1): 41~51

7 Cao S G, Rees N W, Feng G. Analysis and design for a class of complex control systems, Part I: Fuzzy modelling and identification. *Automatica*, 1997, **33**(6): 1017~1028

8 Cao S G, Rees N W, Feng G. Analysis and design for a class of complex control systems, Part II: Fuzzy controller design. *Automatica*, 1997, **33**(6): 1029~1039

9 Wang L X, Stable and optimal fuzzy control of linear systems. *IEEE Trans. Fuzzy Syst.*, 1998, **6**(1): 137~143

10 Huang L J, Tomizuka M. A self-paced fuzzy tracking controller for two-dimensional motion control. *IEEE Trans. Syst. Man & Cybern.*, **20**(5): 1115~1124

11 Lea R N, Chowdhury I, Jani Y, Shehadeh H. Design and performance of the fuzzy tracking controller in software simulation. In: Proc FUZZ-IEEE'92, 1992. 1123~1130

12 Wang W H, Lie W N, Chen Y C. A fuzzy-computing method for rotation-invariant image tracking. In: Proc. ICIP-IEEE'94, 1994. 540~544

13 Ott P, Chan K C C, Leung H. A study on fuzzy tracking of maneuvering aircraft using real data. In: Proc. FUZZ-IEEE'94, 1994. 1671~1675

14 Panda S K, Low C W, Dash P K. A hybrid feedback linearization -fuzzy controller for variable reluctance motors. In: Proc. PEDS-IEEE'95, 1995. 186~191

15 Tong W, Li T H S. Realization of two-dimensional target tracking problem via autonomous mobile robots using fuzzy sliding mode control. In: Proc. IECON-IEEE'98, 1998. 1158~1163

16 Tanaka K, Sano M. Frequency shaping for fuzzy control systems with unkown non-linear plants by a learning method of neural network. *Fuzzy Sets. and Systems*, 1995, **71**: 71~84

17 Lam H K, Leung F H F, Tam P K S. Stable and robust fuzzy control for uncertain nonlinear systems based on a grid-point approach. In: Proc FUZZ-IEEE'97, 1997. 281~285

18 Kang H J, Kwon C. L^{∞} robust stability analysis for the fuzzy feedback linearization regulator. In: Proc. FUZZ-IEEE'97, 1997. 277~280

19 Ray K S, Maunder D D. Application of the circle criteria for stability analysis of linear SISO and MIMO systems associated with fuzzy logic controllers. *IEEE Trans. Syst. Man Cybern.*, 1984, **2**: 345~349

20 Wang L X. A supervisory controller for fuzzy control systems that guarantees stability. In: Proc. FUZZ-IEEE'94, Orlando, 1994. 1035~1039

21 Anderson B D O, Moore J B. Optimal Control: Linear Quadratic Methods, New Jersey: Prentice-Hall, 1990

22 Macki J, Strauss A. Introduction to Optimal Control Theory, New York: Springer-Verlag, 1982

23 Cao S G, Rees N W, Feng G. Lyapunov-like stability theorems for discrete-time fuzzy control systems. *Int. J. Syst. Science*, 1997, **28**(3): 297~308

Appendix

Lemma 5. For a linear time-invariant dynamical fuzzy system

$$\begin{aligned} \boldsymbol{X}(k+1) &= H_iA\boldsymbol{X}(k) + H_iBW_iR(k), \\ \boldsymbol{Y}(k) &= C\boldsymbol{X}(k), \end{aligned} \tag{72}$$

with $\boldsymbol{X}(k_0^i)$ known. If (H_iA, H_iB) is stabilizable and (H_iA, C) is detectable, then,

1) there exists an unique $n \times n$ symmetric positive semidefinite solution, $\bar{\pi}^i$, of the discrete-time algebraic Riccati-like equation

$$K = L + A^{\mathrm{T}}H_i^{\mathrm{T}}K[I_n + H_iBS^{-1}B^{\mathrm{T}}H_i^{\mathrm{T}}K]^{-1}H_iA; \tag{73}$$

2) the asymptotically optimal control law is

$$R_\infty^{i*}(k) = -W_i^{\mathrm{T}}[W_iW_i^{\mathrm{T}}]^{-1}S^{-1}B^{\mathrm{T}}H_i^{\mathrm{T}}\bar{\pi}^i[I_n + H_iBS^{-1}B^{\mathrm{T}}H_i^{\mathrm{T}}\bar{\pi}^i]^{-1}H_iA\boldsymbol{X}_\infty^{i*}(k),\ k \in [k_0^i, \infty), \tag{74}$$

which minimizes $J_\infty^i(R(\cdot)) = \sum_{k=k_0^i}^{\infty}[\boldsymbol{X}^{\mathrm{T}}(k)L\boldsymbol{X}(k) + R^{\mathrm{T}}(k)W_i^{\mathrm{T}}SW_iR(k)]$;

3) and the optimal closed-loop fuzzy system

$$\boldsymbol{X}_\infty^{i*}(k+1) = [I_n + H_iBS^{-1}B^{\mathrm{T}}H_i^{\mathrm{T}}\bar{\pi}^i]^{-1}H_iA\boldsymbol{X}_\infty^{i*}(k),\ k \in [k_0^i, \infty) \tag{75}$$

is asymptotically and exponentially stable.

应用于离散模糊系统的最优模糊追踪器设计

吴幸珍 林进灯

(交通大学电机与控制工程学系 新竹)

(E-mail: ctlin@fnn.cn.nctu.edu.tw Tel: 886-3-5712121EXT54315, Fax: 886-3-5739497)

摘 要 针对二阶离散式模糊追踪系统，提出用于构造一种最优模糊追踪器的设计方式，并用于追踪一类时变目标或任意模型目标. 为了能让所提出的设计方法完整并达到总体最小化，提出了一种近似线性化的离散模糊动态系统的表示法. 另外，为了简化计算量，一个多层分解方式被用于本文所提出的最优设计程序中. 模拟结果显示，本文提出的最优模糊追踪设计方式确能达到预期效果.

关键词 稳定度，增益余量，全局最小，类 Riccati 方程，移动目标，模型跟随.

Shinq-Jen Wu Received the B.S. degree in chemical engineering from Taiwan University in 1986,the M.S. degree in chemical engineering from Tsing-Hua University(Hsinchu) in 1989,the M.S. degree in electrical engineering from the University of California, Los Angeles, CA, in 1994, and the Ph.D. degree in electrical and control engineering from Chiao-Tung University(Hsinchu) in 2000, respectively. Her research interests include thermodynamics, transport phenomena, process control and design, especially in VLSI and the petroleum industry, system and control theory, especially in optimal control, filtering theory, fuzzy system theory, and optimal fuzzy controller and tracker design.

Chin-Teng Lin Received the B.S. degree in control engineering from Chiao-Tung University(Hsinchu) in 1986 and the M.S.E.E. and Ph.D. degrees in electrical engineering from Purdue University, West Lafayette, IN, in 1989 and 1992, respectively. He is IEEE Senior Member (Member No: 00226571). Dr. Lin has published about 60 journal papers in the areas of soft computing, neural networks, and fuzzy systems, including 35 IEEE Transactions papers. His current research interests are fuzzy systems, neural networks, intelligent control, human-machine interface, image processing, pattern recognition, video and audio (speech) processing, and intelligent transportation system (ITS).