

# Parallel Distributed Compensation for Takagi-Sugeno Fuzzy Models: New Stability Conditions and Dynamic Feedback Designs<sup>1)</sup>

WANG Hua O.<sup>1,2</sup> LI Jing<sup>2</sup>

<sup>1)</sup>(*Lab. Intelligent and Nonlinear Control (LINC), Dept. Electrical and Computer Engineering, Duke University, Durham, NC 27708, U.S.A. )*

(E-mail: hua@ee.duke.edu, Tel: (919) 660-5273, Fax: (919) 660-5293 )

<sup>2)</sup>(*Center for Nonlinear and Complex Systems Dept. of Control Science and Engineering Huazhong University of Science and Technology, Wuhan 430074*)

**Abstract** This paper discusses the synthesis of compensators for plants which can be represented using a Takagi-Sugeno (T-S) fuzzy model. The resulting compensator designs, called PDC controllers, are parameter dependent and mirror the structure of the T-S plant model. This paper extends the existing work on state feedback PDC controllers by introducing the notion of a dynamic PDC controller. The paper contains three new results. The first result provides a more relaxed version of previously stated conditions that are sufficient for the existence of a quadratically stabilizing state feedback PDC controller. The second result provides analogous conditions that are sufficient for the existence of a quadratically stabilizing dynamic PDC controller. The third result deals with the performance-oriented controller synthesis for T-S fuzzy model.

**Key words** Fuzzy control, Takagi-Sugeno fuzzy model, LMI, dynamic feedback.

## 1 Introduction

In recent years, engineers have successfully utilized fuzzy logic in a variety of industrial control applications. As the interest in fuzzy systems has increased, researchers have considered the stability analysis of these systems using a variety of modeling and control frameworks<sup>[1~5]</sup>. One of the modeling techniques which has attracted a great deal of attention is the approach of Takagi-Sugeno fuzzy model<sup>[6]</sup>. In this approach, local dynamics in different state space regions are represented by linear models and the overall system is represented as the fuzzy interpolation of these linear models<sup>[6,7]</sup>. The appeal of the T-S model is that the stability and performance characteristics of the system can be analyzed using a Lyapunov approach<sup>[8~10]</sup>. It has also been demonstrated that sufficient conditions for the stability and performance of a system are stated in terms of the feasibility of a set of linear matrix inequalities (LMIs)<sup>[11~13]</sup>. A further and significant step has also been taken to utilize Lyapunov-function based control design techniques to the control synthesis problem for T-S models. The so-called parallel distributed compensation (PDC)<sup>[14,15]</sup> is one such control design framework that has been proposed and developed over the last few years. The PDC control structure utilizes a nonlinear state feedback controller which mirrors the structure of the associated T-S model. It has also been shown that within the framework of T-S fuzzy model and PDC control design, design conditions for the stability and performance of a system can be formulated into an LMI problem<sup>[14,15]</sup>. The gains of the controller can be determined automatically using this LMI formulation. This is a significant finding in the sense that there exist very efficient numerical algorithms for determining

<sup>1)</sup> Supported by the US Army Research Office grant (DAA G55-98-D-0002) and by Otis Elevator Company. Received 28 August, 2000; in revised form 12 November, 2000

收稿日期 2000-08-28 收修改稿日期 2000-11-12

the feasibility of LMIs, so even large-scale analysis and design problems are computationally tractable<sup>[16,17]</sup>. Applications of T-S model together with PDC controller have achieved many successes in real systems<sup>[18~20]</sup>.

Although the past few years have witnessed a rapid growth of interest in the controller synthesis of T-S models, only a few of these results discussed the output feedback controller<sup>[21,22]</sup>. Moreover, most of these results have concentrated only on the stabilization problem and do not address the design objectives such as disturbance attenuation, passivity, input constraint and so on. In this paper, we try to solve these problems. The notion of DPDC (dynamic parallel distributed compensation)<sup>[23]</sup> will be introduced in this paper. This DPDC controller is different from the fuzzy controller proposed in [22] where a fuzzy regulator and a fuzzy observer are designed separately and then combined together. Three results will be provided in this paper, The first result is a relaxed version of the LMI conditions stated in [22,24] that are sufficient for the existence of a quadratically stabilizing state feedback PDC controller. The second result is an analogous set of LMI conditions that are sufficient for the existence of a quadratically stabilizing dynamic PDC controller. The third result is the solution to the problem of performance-oriented controller synthesis for T-S models<sup>[25]</sup>. In this result, the controller synthesis is again formulated as an LMI problem. Multiple design objectives can also be achieved by finding a feasible solution to an augmented LMI problem as in [26].

Throughout the paper, the notation  $M > 0$  will mean that  $M$  is positive definite symmetric matrix, and the notation  $\mathcal{L}(A, P)$  will denote the mapping from  $\mathcal{R}^{n \times n} \times \mathcal{R}^{n \times n}$  to  $\mathcal{R}^{n \times n}$  defined such that  $(A, P) \rightarrow PA + A^T P^T$ . The same holds for  $\mathcal{L}(A^T, Q^T) = AQ + Q^T A^T$ .  $P^{-T}$  is the same as  $(P^{-1})^T$ .  $\mathcal{R}^+ = [0, \infty)$ .  $L_2^p(\mathcal{R}^+)$  is defined as the set of all  $p$ -dimensional vector valued functions  $\mathbf{u}(t), t \in \mathcal{R}^+$  such that  $\|\mathbf{u}\|_2 = (\int_0^\infty \|\mathbf{u}(t)\|^2 dt)^{1/2} < \infty$  and  $L_2^e(\mathcal{R}^+)$  is its extended space which is defined as the set of the vector-valued functions  $\mathbf{u}(t), t \in \mathcal{R}^+$  such that  $\|\mathbf{u}\|_2^e = (\int_0^T \|\mathbf{u}(t)\|^2 dt)^{1/2} < \infty$ , for all  $T \in \mathcal{R}^+$ .

The remaining of the paper is organized as follows: Section 2 describes the T-S fuzzy model and PDC controller. Section 3 presents a set of relaxed LMI design conditions which can be used to select the compensator gains for the state feedback PDC controller so that the closed loop system is globally stable. Section 4 introduces the notion of DPDC controller and provides an analogous set of LMI conditions for the stabilizing DPDC controller design. Section 5 addressed the problem of performance-oriented DPDC controller design for T-S Model. Section 6 contains an illustrative application. Concluding remarks are collected in Section 7.

## 2 Takagi-Sugeno Model & Parallel Distributed Compensators

### Takagi-Sugeno Model

A Takagi-Sugeno fuzzy model for a dynamic system consists of a finite set of fuzzy IF ... THEN rules expressed in the form:

*Dynamic Part :*

Rule  $i$ : IF  $p_1(t)$  is  $M_{i1}$  ... and  $p_l(t)$  is  $M_{il}$ ,

THEN  $\dot{\mathbf{x}}(t) = A_i \mathbf{x}(t) + B_i \mathbf{u}(t)$ .

*Output Part :*

Rule  $i$ : IF  $p_1(t)$  is  $M_{i1}$  ... and  $p_l(t)$  is  $M_{il}$ ,

THEN  $\mathbf{y}(t) = C_i \mathbf{x}(t)$ .

where  $\mathbf{x}(t)$ ,  $\mathbf{u}(t)$ ,  $\mathbf{y}(t)$ , and  $\mathbf{p}(t)$  respectively denote the state, input, output, and parameter vectors. The  $j$ th component of  $\mathbf{p}(t)$  is denoted by  $p_j(t)$ , and the fuzzy membership function associated with the  $i$ th rule and  $j$ th parameter component is denoted by  $M_{ij}$ . Each  $p_i(t)$  is a measurable time-varying quantity. In general, these parameters may be functions of the state

variables, external disturbances, and/or time.

There are two functions of  $\mathbf{p}(t)$  associated with each rule. The first function is called the truth value. The truth value for the  $i$ th rule is defined by the equation

$$w_i(\mathbf{p}(t)) = \prod_{j=1}^l M_{ij}(p_j(t)).$$

Throughout this paper, we will assume that each  $w_i$  is a non-negative function and that the truth value of at least one rule is always nonzero. The second function is called the firing probability. The firing probability for the  $i$ th rule is defined by the equation

$$h_i(\mathbf{p}(t)) = \frac{w_i(\mathbf{p}(t))}{\sum_{i=1}^r w_i(\mathbf{p}(t))},$$

where  $r$  denotes the number of rules in the rule base. Under the previously stated assumptions, this is always a well defined function taking values between 0 and 1, and the sum of all the firing probabilities is identically equal to 1.

The dynamics described by the T-S model evolve according to the system of equations

$$\dot{\mathbf{x}} = \sum_{i=1}^r h_i(\mathbf{p}) (A_i \mathbf{x} + B_i \mathbf{u}), \quad (1a)$$

$$\mathbf{y} = \sum_{i=1}^r h_i(\mathbf{p}) C_i \mathbf{x}. \quad (1b)$$

Thus the T-S model description can also be viewed as a parameter-dependent interpolation between linear models; however, the exact classification of the resultant system depends on the nature of the parameters. For example, if each  $p_i$  is a known function of time, then the T-S model describes a linear time-varying system. If, on the other hand, each  $p_i$  is a function of the state variables, then the T-S model describes an autonomous nonlinear system.

### Parallel Distributed Compensation

A T-S model rule base can also be used to describe a gain-scheduled static or dynamic compensator. This rule-based structure is particularly advantageous when the plant has also been described using a T-S model. The state feedback case was examined in [14,15] where they referred to this rule-based feedback structure as parallel distributed compensation (PDC). This controller structure incorporates a set of fuzzy rules expressed in the form:

Rule  $i$ : IF  $p_1(t)$  is  $M_{i1}$   $\cdots$  and  $p_l(t)$  is  $M_{il}$ ,  
THEN  $\mathbf{u}(t) = K_i \mathbf{x}(t)$ ,

where  $i = 1, 2, \dots, r$ .

The output of the PDC controller is determined by the summation

$$\mathbf{u} = \sum_{i=1}^r h_i K_i \mathbf{x}. \quad (2)$$

It is important to note that the T-S models of the plant and the PDC compensator contain the same number of rules, and that the membership functions for corresponding rules are the same. This mirrored structure is necessary for the LMI-based analysis and design procedures. Later in this paper, we will also extend these design procedures to include dynamic compensators.

### Stabilizing PDC Controller Design

For the state feedback case, the design variables are the gain matrices  $K_i$ ,  $1 \leq i \leq r$  in the PDC controller. The following result given in [14,15] states conditions which are sufficient for the existence of such a PDC controller. Taken together, these conditions form an LMI

feasibility problem. If a feasible solution to this problem can be found, then a set of stabilizing gain matrices can be computed directly from the solution.

**Lemma 1.** *The fuzzy control system of T-S model (1) is stabilizable in the large via PDC control (2) if there exist a  $Q > 0$  and  $M_i, i = 1, 2, \dots, r$  such that the following LMI conditions hold:*

$$\mathcal{L}(A_i^T, Q) + B_i M_i + M_i^T B_i^T < 0, \quad i = 1, \dots, r \tag{3}$$

$$\mathcal{L}(A_i^T + A_j^T, Q) + B_i M_j + M_j^T B_i^T + B_j M_i + M_i^T B_j^T < 0, \quad i < j \leq r. \tag{4}$$

And the PDC controller is given by :

$$K_i = M_i Q^{-1}. \tag{5}$$

The Lyapunov function is given by :

$$V = x^T Q^{-1} x. \tag{6}$$

In [20], a relaxed version of this result is given by adding a slack variable  $Y$ . Later this paper, we will give a more relaxed version of the LMI conditions for the stabilization of T-S model via PDC control.

The above concept of PDC control can also be applied to design fuzzy observer for T-S model<sup>[21,22]</sup>. The fuzzy observer is in the following form:

Rule  $i$ : IF  $p_1(t)$  is  $M_{i1} \dots$  and  $p_l(t)$  is  $M_{il}$ ,

$$\text{THEN } \begin{cases} \dot{\hat{x}}(t) = A_i \hat{x}(t) + B_i u(t) + L_i (y(t) - \hat{y}(t)), \\ \hat{y}(t) = C_i \hat{x}(t), \end{cases}$$

where  $i = 1, 2, \dots, r$ .

The output of the fuzzy observer is determined by the summation

$$\dot{\hat{x}}(t) = \sum_{i=1}^r h_i A_i \hat{x}(t) + \sum_{i=1}^r B_i h_i u(t) + \sum_{i=1}^r h_i L_i (y(t) - \hat{y}(t)), \tag{7a}$$

$$\hat{y} = \sum_{i=1}^r h_i C_i \hat{x}(t). \tag{7b}$$

It has been shown that the above fuzzy observer can also be designed via the approach of PDC. The following result<sup>[21,22]</sup> states the LMI conditions which are sufficient for the existence of fuzzy observer such that the steady error between  $x(t)$  and  $\hat{x}(t)$  converges to zero.

**Lemma 2.** *The fuzzy control system of T-S model (1) is observable in the large via fuzzy observer (7) if there exist a  $\hat{Q} > 0$  and  $N_i, i = 1, 2, \dots, r$  such that the following LMI conditions hold :*

$$\mathcal{L}(A_i, \hat{Q}) - N_i C_i - C_i^T N_i^T < 0, \quad i = 1, \dots, r \tag{8}$$

$$\mathcal{L}(A_i + A_j, \hat{Q}) - N_i C_j - C_j^T N_i^T - N_j C_i - C_i^T N_j^T < 0, \quad i < j \leq r. \tag{9}$$

And the fuzzy observer is given by :

$$L_i = \hat{Q}^{-1} N_i. \tag{10}$$

In [21,22], a controller structure combining both the state feedback PDC controller and the above fuzzy observer is adopted to handle the problem of output feedback problem. It has been proved that, when parameters  $p_i(t)$  are completely measurable, a separation principle also holds true. Therefore, as in the controller design for linear systems, we can design the fuzzy regulator and fuzzy observer separately and then combine them together as an overall controller for T-S model.

In this paper, we will take a different approach to tackle the problem of output feedback controller design for T-S problem. A special controller structure called dynamic parallel distributed compensator (DPDC) will be introduced. Compared to the previous results, there is no distinct regulator and observer part in the DPDC controller. Instead, the controller design problem is formulated into a single LMI problem. By doing that, performance-oriented controller design can also be incorporated into this framework.

### 3 Relaxed LMI Condition for State Feedback PDC Controller Design

In this section, we will present a relaxed version of the LMI conditions which are sufficient for the existence of a quadratically stabilizing state feedback PDC controller. First, we state a lemma which will be used repeatedly in the results which follow.

Suppose that we are given  $r$  functions of  $\mathbf{p}$ ,  $h_1(\mathbf{p}), \dots, h_r(\mathbf{p})$ , which satisfy the conditions that  $h_i(\mathbf{p}) \geq 0$  and that for every  $\mathbf{p}$ ,  $\sum_{i=1}^r h_i(\mathbf{p}) = 1$ . Furthermore, suppose that we are given  $r^2$  matrices  $A_{ij}$  indexed such that  $1 \leq i, j \leq r$ . The equation

$$\dot{\mathbf{x}} = \sum_{i=1}^r \sum_{j=1}^r h_i(\mathbf{p}) h_j(\mathbf{p}) A_{ij} \mathbf{x} \quad (11)$$

describes a parameter dependent differential equation with an equilibrium point at  $\mathbf{x} = 0$ . Recall that the origin is said to be quadratically stable if and only if there exists a symmetric matrix  $P > 0$  such that the time derivate of the associated function  $V(\mathbf{x}) = \mathbf{x}' P \mathbf{x}$  is negative for all  $\mathbf{x} \neq 0$ . The following lemma gives a sufficient condition for the quadratic stability of equation (11).

**Lemma 3.** *The nonlinear system described by differential equation (11) is quadratically stable if there exists a matrix  $P > 0$  and  $\frac{r(r+1)}{2}$  symmetric matrices  $T_{ij}$  with  $1 \leq i \leq j \leq r$  such that the following two conditions hold:*

1) *For every pair of indices  $1 \leq i \leq j \leq r$ , the equation*

$$\mathcal{L}(A_{ij} + A_{ji}, P) < T_{ij} \quad (12)$$

*is satisfied.*

2) *The matrix*

$$T = \begin{bmatrix} T_{11} & \dots & T_{1r} \\ \vdots & \ddots & \vdots \\ T_{1r} & \dots & T_{rr} \end{bmatrix} \quad (13)$$

*is negative definite.*

Proof. Omitted. □

Based on this lemma, we have the following result for the state feedback PDC controller design for T-S model:

**Theorem 1.** *A sufficient condition for the existence of a PDC controller (2) which quadratically stabilizes the T-S model (1) is that there exists a matrix  $Q > 0$ , matrices  $M_i$ ,  $1 \leq i \leq r$ , and symmetric matrices  $\hat{T}_{ij}$ ,  $1 \leq i \leq j \leq r$  such that the following two conditions hold:*

1) *For every pair of indices satisfying  $1 \leq i \leq j \leq r$ , the inequality*

$$\mathcal{L}(A_i^T + A_j^T, Q) + M_j^T B_i^T + B_i M_j + M_i^T B_j^T + B_j M_i < \hat{T}_{ij} \quad (14)$$

*is satisfied.*

2) *The matrix*

$$\hat{T} = \begin{bmatrix} \hat{T}_{11} & \dots & \hat{T}_{1r} \\ \vdots & \ddots & \vdots \\ \hat{T}_{1r} & \dots & \hat{T}_{rr} \end{bmatrix} \quad (15)$$

*is negative definite.*

Furthermore, if matrices exist which satisfy these inequalities, then the feedback gains  $K_i = M_i Q^{-1}$  will provide a quadratically stabilizing PDC controller.

Proof. Suppose that the conditions in Theorem 1 are satisfied. If we pre- and post-multiply both sides of Eq. (14) by  $Q^{-1}$  the results can be expressed as

$$\mathcal{L}((A_i + B_i M_j Q^{-1}), Q^{-1}) + \mathcal{L}((A_j + B_j M_i Q^{-1}), Q^{-1}) < Q^{-1} \hat{T}_{ij} Q^{-1}. \quad (16)$$

Furthermore, if we pre- and post-multiply  $T$  by a block diagonal matrix which has a  $Q^{-1}$  at each of its diagonal elements, the second condition can be expressed as

$$\begin{bmatrix} Q^{-1} \hat{T}_{11} Q^{-1} & \dots & Q^{-1} \hat{T}_{1r} Q^{-1} \\ \vdots & \ddots & \vdots \\ Q^{-1} \hat{T}_{1r} Q^{-1} & \dots & Q^{-1} \hat{T}_{rr} Q^{-1} \end{bmatrix} < 0. \quad (17)$$

If we select the gains  $K_i = M_i Q^{-1}$  and apply the PDC controller (2) to the T-S model (1), then the resulting closed-loop dynamics are described by the equation

$$\dot{\mathbf{x}} = \sum_{i=1}^r \sum_{j=1}^r h_i h_j (A_i + B_i M_j Q^{-1}) \mathbf{x}. \quad (18)$$

By setting  $A_{ij} = (A_i + B_i M_j Q^{-1})$ ,  $P = Q^{-1}$ , and  $T_{ij} = Q^{-1} \hat{T}_{ij} Q^{-1}$ , it is apparent from Lemma 3 that Eq. (16) and (17) imply that this closed-loop system is quadratically stable. This proves the existence of a solution.  $\square$

**Remark.** The above theorem is a generalization of the results given in [15,22]. It also generalizes the LMI condition given by [24] in which case  $\hat{T}_{ij}$  is restricted to be of the form  $\hat{t}_{ij} I$ . The above theorem can also be further relaxed if the structure of the fuzzy membership function is known. For example,

1) Sometimes there is no overlap between two rules, i.e., the product of the  $h_i$  and the  $h_j$  may be identically zero. In this case, the above theorem can be relaxed by dropping the condition (14) corresponding to the  $i$  and  $j$  in (14).

2) If only  $s < r$  rules can fire at the same time, then the conditions of this theorem can be further relaxed to only require that all the diagonal  $s \times s$  principle submatrices of  $\hat{T}$  are negative definite.

#### 4 Stabilizing Controller Design of DPDC

For many systems, it is not possible to obtain a measurement of the entire state vector. If only the output measurements are available, dynamic elements must often be incorporated within the controller. Therefore, to tackle the problem of output feedback controller design for T-S models, we need to introduce a rule-based dynamic compensator structure which we will refer to as dynamic parallel distributed compensation (DPDC). The DPDC structure consists of a double index set of fuzzy rules:

*Dynamic Part :*

Rule  $ij$ : IF  $p_1(t)$  is  $M_{i1}$  and  $p_1(t)$  is  $M_{j1}$  and  
 $\dots$  and  $p_l(t)$  is  $M_{il}$  and  $p_l(t)$  is  $M_{jl}$ ,  
 THEN  $\dot{\mathbf{x}}(t) = A_c^{ij} \mathbf{x}(t) + B_c^j \mathbf{y}(t)$ .

Output Part :

Rule  $ij$ : IF  $p_1(t)$  is  $M_{i1}$  and  $p_1(t)$  is  $M_{j1}$  and  
 $\dots$  and  $p_l(t)$  is  $M_{il}$  and  $p_l(t)$  is  $M_{jl}$ ,  
 THEN  $\mathbf{u}(t) = C_c^j \mathbf{x}(t) + D_c \mathbf{y}(t)$ .

Note that if the T-S model of the plant contains  $r$  rules then the DPDC rule base contains  $r^2$  rules and that each rule uses the same feedthrough matrix  $D_c$ . It is possible to relax this restriction, but doing so necessitates the use of a triple indexed rule base, and the complexity increases greatly. The truth value for the  $ij$ th rule is the product

$$w_{ij}(\mathbf{p}(t)) = \prod_{k=1}^l \prod_{q=1}^l M_{ik}(\mathbf{p}_k(t)) M_{jq}(\mathbf{p}_q(t)),$$

and the firing probability for the  $ij$ th rule is defined by the equation

$$h_{ij}(\mathbf{p}(t)) = h_i(\mathbf{p}(t)) h_j(\mathbf{p}(t)) = \frac{w_i(\mathbf{p}(t)) w_j(\mathbf{p}(t))}{(\sum_{i=1}^r w_i(\mathbf{p}(t)))^2}.$$

For every value of  $\mathbf{p}$ , these firing probabilities satisfy the identity  $\sum_i \sum_j h_{ij}(\mathbf{p}) = 1$ .

The dynamics described by the DPDC compensator evolve according to the system of equations

$$\dot{\mathbf{x}} = \sum_{i=1}^r \sum_{j=1}^r h_i(\mathbf{p}) h_j(\mathbf{p}) (A_c^{ij} \mathbf{x} + B_c^j \mathbf{y}), \quad (19a)$$

$$\mathbf{u} = \sum_{i=1}^r \sum_{j=1}^r h_j(\mathbf{p}) (C_c^j \mathbf{x} + D_c \mathbf{y}). \quad (19b)$$

Next, we are going to derive a set of sufficient LMI conditions for the existence of a quadratically stabilizing DPDC. In order to derive the LMI design conditions, it is useful to begin with the closed-loop system for the T-S model with DPDC controller. It can be written as:

$$\dot{\mathbf{x}}_{cl} = \sum_{i=1}^r \sum_{j=1}^r h_i(\mathbf{p}) h_j(\mathbf{p}) A_{cl}^{ij} \mathbf{x}_{cl}, \quad (20)$$

where

$$A_{cl}^{ij} = \begin{bmatrix} A_i + B_i D_c C_j & B_i C_c^j \\ B_c^i C_j & A_c^{ij} \end{bmatrix}.$$

Based on Lemma 3, we know that the closed-loop system will be stable if the following two conditions hold:

$$1) \quad \mathcal{L}(A_{cl}^{ij} + A_{cl}^{ji}, P) < T_{ij}, \quad (21)$$

$$2) \quad \mathbf{T} = \begin{bmatrix} T_{11} & \dots & T_{1r} \\ \vdots & \ddots & \vdots \\ T_{1r} & \dots & T_{rr} \end{bmatrix} < 0. \quad (22)$$

However, Eq. (21) is not an LMI condition by itself since the DPDC controller parameters and the Lyapunov function parameters interweave with each other. So we have to find a linear matrix transformation to convert Eq. (21) into LMIs. To do that, we will first partition the constant matrices  $P$  and  $P^{-1}$  into components:

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}.$$

We will also define the matrices  $\Pi_1 = \begin{bmatrix} Q_{11} & I \\ Q_{12}^T & 0 \end{bmatrix}$  and  $\Pi_2 = P\Pi_1 = \begin{bmatrix} I & P_{11} \\ 0 & P_{12}^T \end{bmatrix}$ .

Eq. (21) will hold if and only if  $\Pi_1^T(\mathcal{L}(A_{cl}^{ij} + A_{cl}^{ji}, P))\Pi_1 < \Pi_1^T T_{ij} \Pi_1$ .

This equation can also be rewritten as

$$\Pi_2^T(A_{cl}^{ij} + A_{cl}^{ji})\Pi_1 + \Pi_1^T(A_{cl}^{ij} + A_{cl}^{ji})^T\Pi_2 < \Pi_1^T T_{ij} \Pi_1. \quad (23)$$

We can define the term  $\Pi_2^T A_{cl}^{ij} \Pi_1$  on the left hand side of the above inequality as  $E_{ij}$ . Writing

out this term, we have  $E_{ij} = \begin{bmatrix} I & 0 \\ P_{11} & P_{12} \end{bmatrix} \begin{bmatrix} A_i + B_i D_c C_j & B_i C_c^j \\ B_c^i C_j & A_c^{ij} \end{bmatrix} \begin{bmatrix} Q_{11} & I \\ Q_{12}^T & 0 \end{bmatrix}$ .

Matrix  $E_{ij}$  can also be rewritten as

$$E_{ij} = \begin{bmatrix} E_{ij}(1,1) & E_{ij}(1,2) \\ E_{ij}(2,1) & E_{ij}(2,2) \end{bmatrix}, \quad (24)$$

where

$$E_{ij}(1,1) \triangleq (A_i + B_i D_c C_j)Q_{11} + B_i C_c^j Q_{12}^T,$$

$$E_{ij}(1,2) \triangleq A_i + B_i D_c C_j,$$

$$E_{ij}(2,1) \triangleq P_{11}(A_i + B_i D_c C_j)Q_{11} + P_{12} B_c^i C_j Q_{11} + P_{11} B_i C_c^j Q_{12}^T + P_{12} A_c^{ij} Q_{12}^T,$$

$$E_{ij}(2,2) \triangleq P_{11}(A_i + B_i D_c C_j) + P_{12} B_c^i C_j.$$

And if we define

$$\mathcal{A}_{ij} \triangleq P_{11}(A_i + B_i D_c C_j)Q_{11} + P_{12} B_c^i C_j Q_{11} + P_{11} B_i C_c^j Q_{12}^T + P_{12} A_c^{ij} Q_{12}^T, \quad (25)$$

$$\mathcal{B}_i \triangleq P_{11} B_i D_c + P_{12} B_c^i, \quad (26)$$

$$\mathcal{C}_i \triangleq D_c C_i Q_{11} + C_c^i Q_{12}^T, \quad (27)$$

$$\mathcal{D} \triangleq D_c. \quad (28)$$

The matrix  $E_{ij}$  then becomes

$$E_{ij} = \begin{bmatrix} \mathcal{A}_{ij} Q_{11} + \mathcal{B}_i C_j & \mathcal{A}_{ij} + \mathcal{B}_i \mathcal{D} C_j \\ \mathcal{C}_i & P_{11} \mathcal{A}_{ij} + \mathcal{B}_i C_j \end{bmatrix}, \quad (29)$$

and the closed-loop stability condition Eq. (23) can be expressed as:

$$\mathcal{L}(E_{ij} + E_{ji}, I) < \Pi_1^T T_{ij} \Pi_1.$$

If we further define  $\hat{T}_{ij} = \Pi_1 T_{ij} \Pi_1$  and notice that the negative definiteness of matrix  $\mathbf{T} = \begin{bmatrix} T_{11} & \dots & T_{1r} \\ \vdots & \ddots & \vdots \\ T_{1r} & \dots & T_{rr} \end{bmatrix}$  is equivalent to the negative definiteness of matrix  $\hat{\mathbf{T}} = \begin{bmatrix} \hat{T}_{11} & \dots & \hat{T}_{1r} \\ \vdots & \ddots & \vdots \\ \hat{T}_{1r} & \dots & \hat{T}_{rr} \end{bmatrix}$ , we arrive at the next theorem:

**Theorem 2.** *A sufficient condition for the existence of an  $s$ -dimensional DPDC controller (19) which quadratically stabilizes the T-S model (1) is that there exist two matrices  $Q_{11} > 0$  and  $P_{11} > 0$ , matrices  $\mathcal{A}_{ij}$ ,  $\mathcal{B}_i$ ,  $\mathcal{C}_j$ ,  $\mathcal{D}$ ,  $1 \leq i \leq r$ , and symmetric matrices  $\hat{T}_{ij}$ ,  $1 \leq i \leq j \leq r$  such that the following three conditions hold:*

1) *The matrix*

$$\begin{bmatrix} Q_{11} & I \\ I & P_{11} \end{bmatrix} \quad (30)$$



is positive definite.

2) For every pair of indices satisfying  $1 \leq i \leq j \leq r$ , the equation

$$\mathcal{L}(E_{ij} + E_{ji}, I) < \hat{T}_{ij} \quad (31)$$

is satisfied where

$$E_{ij} = \begin{bmatrix} A_i Q_{11} + B_i C_j & A_i + B_i D C_j \\ \mathcal{A}_{ij} & P_{11} A_i + B_i C_j \end{bmatrix}.$$

3) The matrix

$$\hat{T} = \begin{bmatrix} \hat{T}_{11} & \dots & \hat{T}_{1r} \\ \vdots & \ddots & \vdots \\ \hat{T}_{1r} & \dots & \hat{T}_{rr} \end{bmatrix} \quad (32)$$

is negative definite.

Furthermore, if matrices exist which satisfy these inequalities, then the gains

$$\begin{aligned} A_c^{ij} &= P_{12}^{-1} (\mathcal{A}_{ij} - P_{12} B_c^i C_j Q_{11} - P_{11} B_i C_c^j Q_{12}^T - P_{11} (A_i + B_i D_c C_j) Q_{11}) Q_{12}^{-1}, \\ B_c^i &= P_{12}^{-1} (B_i - P_{11} B_i D_c), \\ C_c^i &= (C_i - D_c C_i Q_{11}) Q_{12}^{-T}, \\ D_c &= \mathcal{D}. \end{aligned}$$

will generate a measurement feedback DPDC controller which quadratically stabilizes the plant.  $P_{12}$  and  $Q_{12}$  are chosen so that the constraint  $P_{11} Q_{11} + P_{12} Q_{12}^T = I$  is satisfied.

In the above theorem, Eq. (30) is equivalent to the requirement that  $P > 0$ . This equivalence has been proven in [27]. The constraint  $P_{11} Q_{11} + P_{12} Q_{12}^T = I$  comes from the fact that  $PP^{-1} = I$ .

A further step can be taken to reduce the number of LMI variable in the above result. It is noted that the controller can be simplified as:

$$\begin{aligned} \dot{x} &= \sum_{i=1}^r \sum_{j=1}^r h_i(p) h_j(p) (\bar{A}_c^{ij} x + \bar{B}_c^j y), \\ u &= \sum_{i=1}^r \sum_{j=1}^r h_j(p) (\bar{C}_c^j x + \bar{D}_c y). \end{aligned} \quad (33)$$

where

$$\bar{A}_c^{ij} = \frac{1}{2} (A_c^{ij} + A_c^{ji}), \quad \bar{B}_c^i = B_c^i, \quad \bar{C}_c^i = C_c^i, \quad \bar{D}_c = D_c. \quad (34)$$

Since  $\bar{A}_c^{ij} = \bar{A}_c^{ji}$ , the number of controller parameters can be reduced nearly by half by only defining the case for  $i \leq j$ . So if we define variable

$$\bar{\mathcal{A}}_{ij} \triangleq \frac{1}{2} (\mathcal{A}_{ij} + \mathcal{A}_{ji}), \quad (35)$$

$$\bar{B}_i \triangleq B_i, \quad (36)$$

$$\bar{C}_i \triangleq C_i, \quad (37)$$

$$\bar{D} \triangleq D. \quad (38)$$

It can be shown that  $\bar{\mathcal{A}}_{ij}, \bar{B}_i, \bar{C}_i, \bar{D}$  will depend only on  $\bar{A}_c^{ij}, \bar{B}_c^i, \bar{C}_c^i, \bar{D}_c$ . And the matrix  $E_{ij} + E_{ji}$  can be written as:

$$E_{ij} + E_{ji} \triangleq \bar{E}_{ij} \quad (39)$$

$$= \begin{bmatrix} \bar{E}_{ij}(1,1) & \bar{E}_{ij}(1,2) \\ \bar{E}_{ij}(2,1) & \bar{E}_{ij}(2,2) \end{bmatrix} \quad (40)$$

where

$$\begin{aligned} \bar{E}_{ij}(1,1) &= A_i Q_{11} + A_j Q_{11} + B_i \bar{C}_j + B_j \bar{C}_i, \\ \bar{E}_{ij}(1,2) &= A_i + A_j + B_i \bar{D} C_j + B_j \bar{D} C_i, \\ \bar{E}_{ij}(2,1) &= 2\bar{A}_{ij}, \\ \bar{E}_{ij}(2,2) &= P_{11} A_i + P_{11} A_j + \bar{B}_i C_j + \bar{B}_j C_i. \end{aligned}$$

Therefore, we get the following corollary:

**Corollary.** *A sufficient condition for the existence of an s-dimensional DPDC controller (34) which quadratically stabilizes the T-S model (1) is that there exist two matrices  $Q_{11} > 0$  and  $P_{11} > 0$ , matrices  $\bar{A}_{ij}, \bar{B}_i, \bar{C}_j, \bar{D}$ ,  $1 \leq i \leq r$ , and symmetric matrices  $\hat{T}_{ij}$ ,  $1 \leq i \leq j \leq r$  such that the following three conditions hold :*

1) *The matrix*

$$\begin{bmatrix} Q_{11} & I \\ I & P_{11} \end{bmatrix} \tag{41}$$

*is positive definite.*

2) *For every pair of indices satisfying  $1 \leq i \leq j \leq r$ , the equation*

$$\mathcal{L}(\bar{E}_{ij}, I) < \hat{T}_{ij} \tag{42}$$

*is satisfied where*

$$\bar{E}_{ij} = \begin{bmatrix} A_i Q_{11} + A_j Q_{11} + B_i \bar{C}_j + B_j \bar{C}_i & A_i + A_j + B_i \bar{D} C_j + B_j \bar{D} C_i \\ 2\bar{A}_{ij} & P_{11} A_i + P_{11} A_j + \bar{B}_i C_j + \bar{B}_j C_i \end{bmatrix} \tag{43}$$

3) *The matrix*

$$\hat{T} = \begin{bmatrix} \hat{T}_{11} & \dots & \hat{T}_{1r} \\ \vdots & \ddots & \vdots \\ \hat{T}_{1r} & \dots & \hat{T}_{rr} \end{bmatrix} \tag{44}$$

*is negative definite.*

*Furthermore, if matrices exist which satisfy these inequalities, then the gains*

$$\begin{aligned} \bar{A}_c^{ij} &= \frac{1}{2} P_{12}^{-1} (2\bar{A}_{ij} - P_{12} \bar{B}_c^i C_j Q_{11} - P_{12} \bar{B}_c^j C_i Q_{11} - P_{11} B_i \bar{C}_j^c Q_{12}^T - \\ & P_{11} B_j \bar{C}_i^c Q_{12}^T - P_{11} (A_i + B_i \bar{D}_c C_j) Q_{11} - P_{11} (A_j + B_j \bar{D}_c C_i) Q_{11}) Q_{12}^{-1}, \end{aligned} \tag{45}$$

$$\bar{B}_c^i = P_{12}^{-1} (\bar{B}_i - P_{11} B_i \bar{D}_c), \tag{46}$$

$$\bar{C}_c^i = (\bar{C}_i - \bar{D}_c C_i Q_{11}) Q_{12}^{-T}, \tag{47}$$

$$\bar{D}_c = \bar{D} \tag{48}$$

*will generate a measurement feedback DPDC controller which quadratically stabilizes the plant.  $P_{12}$  and  $Q_{12}$  are chosen so that the constraint  $P_{11} Q_{11} + P_{12} Q_{12}^T = I$  is satisfied.*

### 5 Performance-Oriented Controller Design of DPDC

The section presents the solution to the problem of DPDC controller design which meets a variety of useful performance criteria. These performance criteria is often needed in many systems for requirements such as disturbance attenuation, input constraint and so on. As in the previous sections, LMI conditions will be derived that are sufficient for the existence of a satisfactory DPDC controller. Due to limit of space, the performance specifications presented will only include generalized  $H_2$  performance and input constraint. Results for other performance

criteria can be derived following similar procedure. For each of the performance specification included here, we will first present a set of parameter dependent inequalities conditions that can guarantee the satisfaction of the performance specification. Then, we will restrict our consideration to the DPDC controller structure. This restriction allows us to convert the parameter dependent inequalities, which used to be difficult to solve, into LMIs.

In the subsection, we will consider the T-S models which can be represented by a set of fuzzy rules in the following form:

*Dynamic Part :*

Rule  $i$ : IF  $p_1(t)$  is  $M_{i1}$   $\cdots$  and  $p_l(t)$  is  $M_{il}$ ,  
THEN  $\dot{\mathbf{x}}(t) = A_i \mathbf{x}(t) + B_i \mathbf{u}(t) + B_w^i \mathbf{w}(t)$ .

*Output Part :*

Rule  $i$ : IF  $p_1(t)$  is  $M_{i1}$   $\cdots$  and  $p_l(t)$  is  $M_{il}$ ,  
THEN  $\mathbf{y}(t) = C_i \mathbf{x}(t) + D_w^i \mathbf{w}(t)$ .  
and  $\mathbf{z}(t) = C_z^i \mathbf{x} + D_z^i \mathbf{u} + D_{zw}^i \mathbf{w}$ .

where  $p_i(t)$  are some fuzzy variables,  $\mathbf{x}(t)$  are the system states,  $\mathbf{u}(t)$  are the control inputs,  $\mathbf{w}(t)$  are exogenous inputs such as disturbance signals, noises or reference signals,  $\mathbf{y}(t)$  represent the measurements and  $\mathbf{z}(t)$  stand for performance variables of the control systems. We can simplify the expressions of the T-S model as:

$$\dot{\mathbf{x}} = \sum_{i=1}^r h_i(\mathbf{p})(A_i \mathbf{x} + B_i \mathbf{u} + B_w^i \mathbf{w}), \quad (49a)$$

$$\mathbf{z} = \sum_{i=1}^r h_i(\mathbf{p})(C_z^i \mathbf{x} + D_z^i \mathbf{u} + D_{zw}^i \mathbf{w}), \quad (49b)$$

$$\mathbf{y} = \sum_{i=1}^r h_i(\mathbf{p})(C_i \mathbf{x} + D_w^i \mathbf{w}). \quad (49c)$$

The closed-loop system equations for a T-S model (49) with DPDC controller (19) have the form:

$$\dot{\mathbf{x}}_{cl} = \sum_{i=1}^r \sum_{j=1}^r h_i(\mathbf{p}) h_j(\mathbf{p})(A_{cl}^{ij} \mathbf{x}_{cl} + B_{cl}^{ij} \mathbf{w}), \quad (50)$$

$$\mathbf{z}_{cl} = \sum_{i=1}^r \sum_{j=1}^r h_i(\mathbf{p}) h_j(\mathbf{p})(C_{cl}^{ij} \mathbf{x}_{cl} + D_{cl}^{ij} \mathbf{w}), \quad (51)$$

where

$$\begin{aligned} A_{cl}^{ij} &= \begin{pmatrix} A_i + B_i D_c C_j & B_i C_c^j \\ B_c^i C_j & A_c^{ij} \end{pmatrix}, & B_{cl}^{ij} &= \begin{bmatrix} B_w^i + B_i D_c D_w^j \\ B_c^i D_w^j \end{bmatrix}, \\ C_{cl}^{ij} &= [C_z^i + D_z^i D_c C_j \quad D_z^i C_c^j], & D_{cl}^{ij} &= [D_{zw}^i + D_z^i D_c D_w^j]. \end{aligned}$$

Now we are ready to derive LMI conditions that can be used to design DPDC controllers which satisfy a variety of useful performance criteria.

### Generalized $H_2$ Performance

**Definition 1**<sup>[28]</sup>. A causal NLTIG:

$$\dot{\mathbf{x}}_{cl} = A_{cl}(\mathbf{p}) \mathbf{x}_{cl} + B_{cl}(\mathbf{p}) \mathbf{w}, \quad (52a)$$

$$\mathbf{z} = C_{cl}(\mathbf{p}) \mathbf{x}_{cl}, \quad (52b)$$

is said to have generalized  $H_2$  performance less or equal to  $\zeta$  if and only if

$$\|z(T)\| \leq \zeta, \quad \forall T \geq 0, \tag{53}$$

where  $x_{cl}(0) = 0$  and  $\int_0^T \|w(t)\|^2 dt \leq 1$ .

Define the function  $V(x_{cl}(t)) = x_{cl}' P x_{cl}$ , where  $P > 0$ . Suppose

$$\begin{pmatrix} \mathcal{L}(A_{cl}(p), P) & PB_{cl}(p) \\ B_{cl}'(p)P & -\zeta I \end{pmatrix} < 0. \tag{54}$$

Then  $\frac{d}{dt}V(x_{cl}(t)) < \zeta w'(t)w(t)$ . We will suppose  $D_{cl}(p) = 0$ . In this case, if the equation

$$\begin{pmatrix} P & C_{cl}'(p) \\ C_{cl}(p) & \zeta I \end{pmatrix} > 0 \tag{55}$$

is satisfied, then  $z'(t) z(t) < \zeta V(x_{cl}(t))$ . This leads to the following lemma:

**Lemma 4.** For system  $G$  that can be described by Eq. (52) the generalized  $H_2$  performance will be less than  $\zeta$  if there exists a matrix  $P = P^T > 0$  such that (54) and (55) are feasible.

To convert the inequality (54) and (55) into LMIs, We begin by applying a matrix congruence transform on them using the matrix  $\begin{pmatrix} \Pi_1 & 0 \\ 0 & I \end{pmatrix}$ , where the system  $G$  is defined as in (50) and (51). By utilizing the notation in (35)~(38) and following similar procedure as in the previous section, we get the following result:

**Theorem 3.** For a  $T$ - $S$  model (49) with DPDC controller (34), the generalized  $H_2$  performance will be less than  $\zeta$  if the following LMIs are feasible with LMI variables  $Q_{11}, P_{11}, T_{ij}, S_{ij}, \bar{A}_{ij}, \bar{B}_i, \bar{C}_i$  and  $\mathcal{D}$  for all  $i \leq j$ ,

$$1) \quad \begin{pmatrix} U_{ij}(1,1) & U_{ij}(1,2) & U_{ij}(1,3) \\ (U_{ij}(1,2))^T & U_{ij}(2,2) & U_{ij}(2,3) \\ (U_{ij}(1,3))^T & (U_{ij}(2,3))^T & U_{ij}(3,3) \end{pmatrix} < T_{ij}, \tag{56}$$

where

$$\begin{aligned} U_{ij}(1,1) &= \mathcal{L}(A_i, Q_{11}) + \mathcal{L}(A_j, Q_{11}) + B_i \bar{C}_j + (B_i \bar{C}_j)^T + B_j \bar{C}_i + (B_j \bar{C}_i)^T, \\ U_{ij}(1,2) &= A_i + A_j + B_i \bar{D} C_j + B_j \bar{D} C_i + 2 \bar{A}_{ij}^T, \\ U_{ij}(1,3) &= B_w^i + B_w^j + B_i \bar{D} D_w^j + B_j \bar{D} D_w^i, \\ U_{ij}(2,2) &= \mathcal{L}(A_i^T, P_{11}) + \mathcal{L}(A_j^T, P_{11}) + \bar{B}_i C_j + \bar{B}_j C_i + (\bar{B}_i C_j)^T + (\bar{B}_j C_i)^T, \\ U_{ij}(2,3) &= P_{11} B_w^i + P_{11} B_w^j + \bar{B}_i D_w^j + \bar{B}_j D_w^i, \\ U_{ij}(3,3) &= -2\zeta I, \end{aligned}$$

$$2) \quad T = \begin{bmatrix} T_{11} & \dots & T_{1r} \\ \vdots & \ddots & \vdots \\ T_{1r} & \dots & T_{rr} \end{bmatrix} < 0, \tag{57}$$

$$3) \quad \begin{pmatrix} V_{ij}(1,1) & V_{ij}(1,2) & V_{ij}(1,3) \\ (V_{ij}(1,2))^T & V_{ij}(2,2) & V_{ij}(2,3) \\ (V_{ij}(1,3))^T & (V_{ij}(2,3))^T & V_{ij}(3,3) \end{pmatrix} > S_{ij}, \tag{58}$$

where

$$\begin{aligned} V_{ij}(1,1) &= 2Q_{11}, \quad V_{ij}(1,2) = 2I, \\ V_{ij}(1,3) &= Q_{11}(C_z^i + C_z^j)^T + (D_z^i \bar{C}_j + D_z^j \bar{C}_i)^T, \quad V_{ij}(2,2) = 2P_{11}, \\ V_{ij}(2,3) &= (C_z^i + C_z^j + D_z^i \bar{D} C_j + D_z^j \bar{D} C_i)^T, \quad V_{ij}(3,3) = 2\zeta I, \end{aligned}$$

$$4) \quad \mathbf{S} = \begin{bmatrix} S_{11} & \dots & S_{1r} \\ \vdots & \ddots & \vdots \\ S_{1r} & \dots & S_{rr} \end{bmatrix} > 0,$$

together with the constraint

$$D_{zw}^i + D_{zw}^j + D_z^i \bar{D} D_w^j + D_z^j \bar{D} D_w^i = 0, \quad \forall i \leq j. \quad (59)$$

The controller is given by (45)~(48), where  $P_{12}$  and  $Q_{12}$  are chosen so that the constraint  $P_{11}Q_{11} + P_{12}Q_{12}^T = I$  is satisfied.

### Constraints on Control Input

**Definition 2**<sup>[17]</sup>. A causal NLTI  $G: \dot{\mathbf{x}}_{cl} = A_{cl}(\mathbf{p})\mathbf{x}_{cl}$  and  $\mathbf{u} = K(\mathbf{p})\mathbf{x}_{cl}$  with a specified initial condition  $\mathbf{x}_{cl}(0)$  satisfies an exponential constraint on the input if

$$\|\mathbf{u}(T)\| \leq \zeta e^{-\alpha T}, \quad \forall T \geq 0. \quad (60)$$

Define the function  $V(\mathbf{x}_{cl}) = \mathbf{x}'_{cl} P \mathbf{x}_{cl}$ , where  $P = P^T > 0$ . Suppose that the equation

$$\mathcal{L}(A_{cl}, P) + 2\alpha P < 0 \quad (61)$$

holds. In this case, the inequality  $V(\mathbf{x}_{cl}(t)) < e^{-2\alpha t} V(\mathbf{x}_{cl}(0))$  will be satisfied. Furthermore, if the equations

$$\begin{pmatrix} P & P\mathbf{x}_{cl}(0) \\ \mathbf{x}'_{cl}(0)P & \zeta I \end{pmatrix} > 0 \quad (62)$$

and

$$\begin{pmatrix} P & K(\mathbf{p})^T \\ K(\mathbf{p}) & \zeta I \end{pmatrix} < 0 \quad (63)$$

hold, then the inequality

$$\mathbf{u}'(t)\mathbf{u}(t) < \zeta(\mathbf{x}'_{cl}(t)P\mathbf{x}_{cl}(t)) < \zeta e^{-2\alpha t}(\mathbf{x}'_{cl}(0)P\mathbf{x}_{cl}(0)) < \zeta^2 e^{-2\alpha t}$$

will also be satisfied. Combining these results, we have the following lemma:

**Lemma 5.** For system  $G: \dot{\mathbf{x}}_{cl} = A_{cl}(\mathbf{p})\mathbf{x}_{cl}$  and  $\mathbf{u} = K(\mathbf{p})\mathbf{x}_{cl}$ , the exponential constraint  $\|\mathbf{u}(T)\| \leq \zeta e^{-\alpha T}$ ,  $\forall T \geq 0$  will be satisfied if there exists a matrix  $P = P^T > 0$  such that inequalities (61), (62) and (63) are satisfied.

Similar to the derivation for the previous theorem, we will apply a linear matrix transformation to convert the above inequalities into LMIs. For(61), we will apply  $\Pi_1$  defined previously for the congruence transformation. For (62) and (63), the matrix  $\begin{pmatrix} \Pi_1 & 0 \\ 0 & I \end{pmatrix}$  will be used as the congruence transformation matrix. Again, after we utilize the notation in (35)~(38) and follow similar procedure as in the previous section, we will get the following result:

**Theorem 4.** Consider a T-S model (1) with PDC controller (34). Suppose the initial state is given by  $[\mathbf{x}(0) \quad \mathbf{x}_c(0)]$ , then  $\|\mathbf{u}(t)\| < \zeta e^{-\alpha t}$  for all  $t \geq 0$  if the LMI conditions the following LMI conditions are feasible with LMI variables  $Q_{11}, P_{11}, P_{12}, T_{ij}, \bar{A}_{ij}, \bar{B}_i, \bar{C}_i$  and  $D$ .

$$1) \quad \begin{pmatrix} U_{ij}(1,1) & U_{ij}(1,2) \\ (U_{ij}(1,2))^T & U_{ij}(2,2) \end{pmatrix} < T_{ij}, \quad \forall i \leq j, \quad (64)$$

where

$$U_{ij}(1,1) = \mathcal{L}(A_i, Q_{11}) + \mathcal{L}(A_j, Q_{11}) + B_i \bar{C}_j + (B_i \bar{C}_j)^T + B_j \bar{C}_i + (B_j \bar{C}_i)^T + 2\alpha Q_{11},$$

$$U_{ij}(1,2) = A_i + A_j + B_i \bar{D} C_j + B_j \bar{D} C_i + 2\bar{A}_{ij}^T + 2\alpha I,$$

$$U_{ij}(2,2) = \mathcal{L}(A_i^T, P_{11}) + \mathcal{L}(A_j^T, P_{11}) + \bar{B}_i C_j + \bar{B}_j C_i + (\bar{B}_i C_j)^T + (\bar{B}_j C_i)^T + 2\alpha P_{11},$$

$$2) \quad T = \begin{bmatrix} T_{11} & \dots & T_{1r} \\ \vdots & \ddots & \vdots \\ T_{1r} & \dots & T_r \end{bmatrix} < 0, \tag{65}$$

$$3) \quad \begin{pmatrix} Q_{11} & I & \mathbf{x}(0) \\ I & P_{11} & P_{11}\mathbf{x}(0) + P_{12}\mathbf{x}_c(0) \\ \mathbf{x}'(0) & \mathbf{x}'(0)P_{11} + \mathbf{x}'_c(0)P_{12}^T & \zeta I \end{pmatrix} > 0, \tag{66}$$

$$4) \quad \begin{pmatrix} Q_{11} & I & \bar{C}_i^T \\ I & P_{11} & (\bar{D}C_i)^T \\ \bar{C}_i & \bar{D}C_i & \zeta I \end{pmatrix} > 0. \tag{67}$$

The controller is given by (45)~(48), where  $Q_{12}$  is chosen such that the constraint  $P_{11}Q_{11} + P_{12}Q_{12}^T = I$  is satisfied.

### 6 Example

In this section, a ball and beam system is considered which is commonly used as an illustrative application of various control schemes<sup>[29]</sup>. The system is shown in Fig.1. The beam is made to rotate in a vertical plane by applying a torque at the center of rotation and the ball is free to roll along the beam. We assume no slipping between the ball and the beam. Let  $\mathbf{x} = (r, \dot{r}, \theta, \dot{\theta})$  be the state of the system and  $y = r$  be the system output. The system can be expressed by the state-space model:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u \tag{68}$$

$$\text{where } \mathbf{f}(\mathbf{x}) = \begin{bmatrix} x_2 \\ B(x_1x_4^2 - G \sin x_3) \\ x_4 \\ 0 \end{bmatrix}, \quad \mathbf{g}(\mathbf{x}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

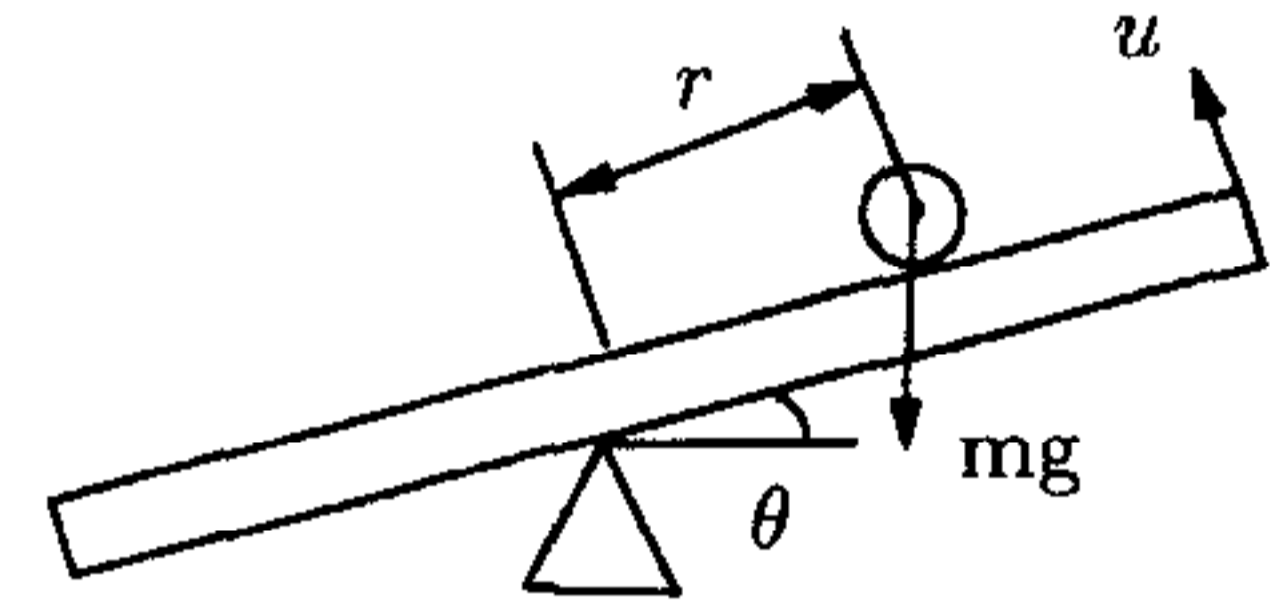


Fig.1 The ball and beam system

We begin by representing this system using a T-S model via sectorization. There are two nonlinearities in (68), the  $x_1x_4^2$  term and the  $\sin x_3$  term. We will sector bound these nonlinearities on the operating region. In this example, assume  $x_3 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$  and  $x_1, x_4 \in [-d, d]$ . This is the region that the system will operate within. It follows that  $\mathbf{f}(\mathbf{x})$  can be written as :

$$\mathbf{f}(\mathbf{x}) = M_{11}M_{21}v_1 + M_{11}M_{22}(v_1 + v_3) + M_{11}M_{23}(v_1 - v_3) + M_{12}M_{21}v_2 + M_{11}M_{22}(v_2 + v_3) + M_{12}M_{23}(v_2 - v_3),$$

where

$$M_{12}(x_3) = \frac{1 - \frac{\sin(x_3)}{\frac{x_3}{2}}}{1 - \frac{\pi}{x_3}}, \quad M_{11}(x_3) = 1 - M_{12}(x_3),$$

$$M_{22}(x_1x_4) = \begin{cases} 1 & x_1x_4 \geq d \\ \frac{x_1x_4}{d} & 0 < x_1x_4 < d \\ 0 & x_1x_4 \leq 0 \end{cases}, \quad M_{23}(x_1x_4) = \begin{cases} 0 & x_1x_4 \geq 0 \\ \frac{x_1x_4}{-d} & -d < x_1x_4 < 0 \\ 1 & x_1x_4 \leq -d \end{cases},$$

$$M_{21}(x_1x_4) = 1 - M_{22}(x_1x_4) - M_{23}(x_1x_4),$$

and  $v_1, v_2$ , and  $v_3$  are defined by

$$v_1 = (x_2, -BGx_3, x_4, 0)^T, \quad v_2 = (x_2, -\frac{2BG}{\pi}x_3, x_4, 0)^T, \quad v_3 = (0, Bdx_4, 0, 0)^T.$$

The T-S model follows directly as follows:

Rule  $ij$ : IF  $|x_3|$  is  $M_{1i}$  and  $x_1x_4$  is  $M_{2j}$ , where  $i = 1, 2, j = 1, 2, 3$ .  
THEN  $\dot{x}(t) = A_{ij}x(t) + B_{ij}u(t)$ ,

For example,

$$A_{11} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -BG & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_{11} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

An output feedback DPDC controller was designed for this system. LMI design conditions are solved via Matlab LMI Toolbox<sup>[24]</sup>. The simulation results are shown in Fig.2. The system parameters for simulation were chosen as  $B = 0.7143$ ,  $G = 9.81$ ,  $d = 5$  and the initial condition was  $[1, 0, 0.0564, 0]$ .

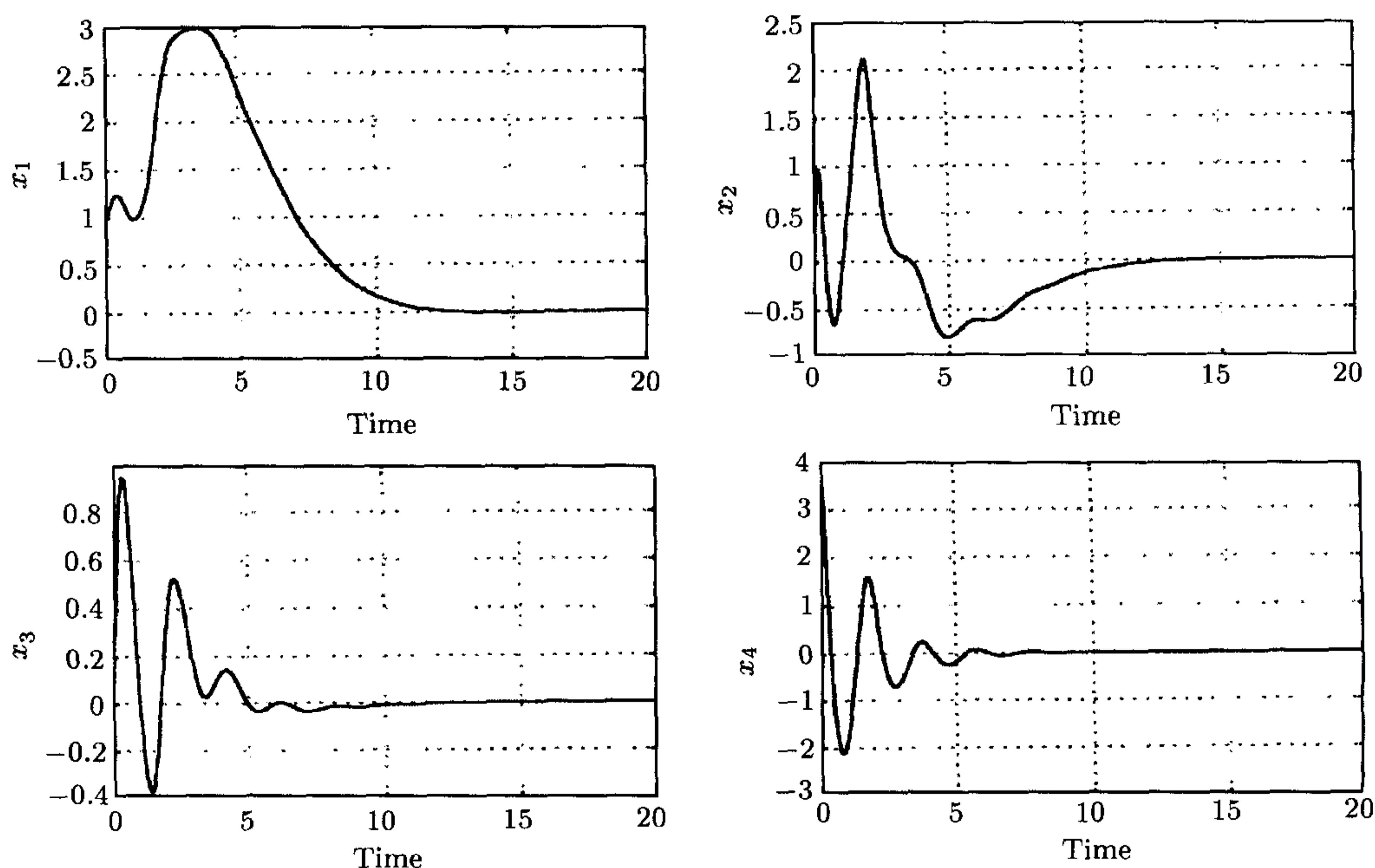


Fig.2 Response of Ball and Beam using DPDC with linear parameterization

## 7 Conclusion

This paper presents the LMI-based controller design for the T-S fuzzy models. The control laws are in the form of the so-called parallel distributed compensation (PDC) controller which is essentially a nonlinear controller. Both the state feedback and the dynamic feedback controller are considered. The controller parameters are obtained from the feasible solution of a set of sufficient LMI conditions.

If variables  $p$  comes from the output of the system, the dynamic feedback controller will become output feedback controller which is necessary if only part of the system states are available. However, if full system states are available, it is noted from the LMI conditions that the output feedback controller offers no advantage over the state feedback controller.

The framework used in this paper can be applied to the nonlinear uncertainty control. The basic tool for robustness analysis of such uncertainty system is the small gain theorem which can be related to the  $L_2$  gain. Thus by making the gain of nominal plant small enough, we

can guarantee the robust stability. The results in this paper are also applicable to hybrid and switching systems. Details will be presented in the sequels of this paper.

**Acknowledgments** The authors are grateful to Prof. Kazuo Tanaka and Dr. David Niemann for their great help in formatting this paper.

### References

- 1 Buckley J J. Universal fuzzy controllers. *Automatica*, 1992, **28**: 1245~1248
- 2 Cao S G, Rees N W. Identification of dynamic fuzzy models. *Fuzzy Sets & Systems*, 1995, **74**(2): 307~320
- 3 Chen G, Ying H. Stability analysis of nonlinear fuzzy PI control systems. In: *Proc. of the 3rd Int. Conf. on Industrial Fuzzy Control and Intelligent Systems*, Kobe, Japan: 1993. 128~133
- 4 Langari R, Tomizuka M. Stability of fuzzy linguistic control systems. In: *Proc. of the 29th IEEE Conf. Decision and Control*, Honolulu: HI, 1990. 35~42
- 5 Wang L X. Analysis and design of hierarchical fuzzy system. *IEEE Trans. Fuzzy Systems*, 1999, **7**(5): 617~624
- 6 Takagi T, Sugeno M. Fuzzy identification of systems and its applications to modeling and control. *IEEE Trans. Syst., Man & Cybern.*, 1985, **15**: 116~132
- 7 Ying H. Sufficient Conditions on uniform approximation of multivariate functions by general Takagi-Sugeno fuzzy systems with linear rule consequence. *IEEE Trans. Syst., Man & Cybern.*, 1998, **28**(4): 515~521
- 8 Cao S G, Rees N W, Feng G. Fuzzy control of nonlinear continuous-time systems. In: *Proc. 35th IEEE Conf. Decision and Control*, New York: NY, 1996. 592~597
- 9 Kang G, Lee W, Sugeno M. Design of TSK fuzzy controller based on TSK fuzzy model using pole placement. In: *Proc. FUZZ-IEEE'98*, 1998. 246~251
- 10 Zhao J, Wertz V, Gorez R. Fuzzy Gain Scheduling Controllers Based on Fuzzy Models. In: *Proc. Fuzzy-IEEE'96*, 1996. 1670~1676
- 11 Cao Y Y, Frank P M. Analysis and synthesis of nonlinear time-delay systems via fuzzy control. *IEEE Trans. Fuzzy Systems*, 2000, **8**(2): 200~211
- 12 Johansson M, Rantzer A, Arzen K E. Piecewise quadratic stability of fuzzy systems. *IEEE Trans. Fuzzy Systems*, 1999, **7**(6): 713~722
- 13 Tanaka K, Sugeno M. Stability analysis and design of fuzzy control systems. *Fuzzy Sets & Systems*, 1992, **45**(2): 135~156
- 14 Wang H O, Tanaka K, Griffin M F. Parallel distributed compensation of nonlinear systems by Takagi-Sugeno fuzzy model. In: *Proc. FUZZ-IEEE/IFES'95*, 1995. 531~538
- 15 Wang H O, Tanaka K, Griffin M F. An approach to fuzzy control of nonlinear systems: stability and design issues. *IEEE Trans. Fuzzy Systems*, 1996, **4**(1): 14~23
- 16 Gahinet P, Apkarian P. A linear matrix inequality approach to H1 Control. *Int. Journal of Robust & Nonlinear Control*, 1994, **4**(4): 421~428
- 17 Boyd S, Ghaoui L E, Feron E, Balakrishnan V. *Linear Matrix Inequalities in Systems and Control Theory*. SIAM, Philadelphia: PA, 1994
- 18 Li J, Niemann D, Wang H O. Robust tracking for high-rise/high-speed elevators. In: *Proc. 1998 American Control Conference*, 1998. 3445~3449
- 19 Ma X J, Sun Z Q. Output tracking and regulation of nonlinear system based on Takagi-Sugeno Fuzzy Model. *IEEE Trans. Syst., Man & Cybern.*, 1998, **30**(1): 47~59
- 20 Tanaka T, Sano M. A robust stabilization problem of fuzzy control systems and its applications to backing up control of a truck-trailer. *IEEE Trans. Fuzzy Systems*, 1994, **2**(3): 119~134.
- 21 Ma X J, Sun Z Q, He Y Y. Analysis and design of fuzzy controller and fuzzy observer. *IEEE Trans. Fuzzy Systems*, 1998, **6**(1): 41~51
- 22 Tanaka K, Ikeda T, Wang H O. Fuzzy Regulators and Fuzzy Observers: Relaxed Stability Conditions and LMI-Based Designs. *IEEE Trans. Fuzzy Systems*, 1998, **6**(2): 250~265
- 23 Niemann D, Li J, Wang H O. Parallel distributed compensation for Takagi-Sugeno fuzzy models: new stability conditions and dynamic feedback designs. In: *Proc. IFAC 1999*. Beijing: 1999, 207~212



- 24 Gahinet P, Nemirovski A, Laub A J, Chilali M. LMI Control Toolbox. The Math Works, 1995
- 25 Li J, Niemann D, Wang H O, Tanaka K. Multiobjective dynamic feedback control of Takagi-Sugeno model via LMIs. In: Proc. 6th Int. Conf. on Fuzzy Theory and Technology, 1998. 159~165
- 26 Scherer C, Gahinet P, Chilali M. Multiobjective Output-Feedback Control via LMI Optimization. *IEEE Trans. Automatic Control*, 1997, **42**(7): 896~911
- 27 Packard A. Gain scheduling via linear fractional transformation. *Systems Control Letters*, 1994, **22**: 79~92
- 28 Boyd S, Barratt C H. Linear Controller Design: Limits of Performance, Prentice Hall, 1991
- 29 Wang L X. Adaptive Fuzzy Systems and Control: Design and Stability Analysis. Prentice Hall, 1993

## Takagai-Sugeno模糊模型的并行分布补偿: 新的稳定条件和动态反馈设计

王 华<sup>1,2</sup> 李 竞<sup>2</sup>

<sup>1</sup>(美国 Duke 大学电子与计算机工程系智能和非线性控制实验室 美国达勒姆)

(E-mail: hua@ee.duke.edu, Tel: (919) 660-5273, Fax: (919) 660-5293)

<sup>2</sup>(华中科技大学控制科学与工程系非线性与复杂系统研究中心 武汉 430074)

**摘 要** 讨论可以用 Takagai-Sugeno 模糊模型描述的系统的补偿器设计. 这种被称之为并行分布补偿 (PDC) 的控制器是基于参数并且影像了 Takagai-Sugeno 模型的结构. 通过引进动态并行分布控制 (DPDC) 的概念, 扩展了关于此类控制器状态反馈的已有结果. 提出三个新的结果. 第一个结果给出了以往结果的宽松条件. 这种条件是存在二次型稳定状态反馈这类控制器的充分条件. 第二个结果给出了类似的二次型稳定动态反馈控制器存在的充分条件. 第三个结果适用于 Takagai-Sugeno 模糊模型的基于性能的控制设计.

**关键词** 模糊控制, Takagai-Sugeno 模糊模型, 线性不等式, 动态反馈.

**WANG Hua O.** Received the Ph.D.degree from the University of Maryland at College Park in 1993. He was with the United Technologies Research Center from 1993 to 1996. Since September 1996, he has been with the Duke University, Durham, NC, where he is an Assistant Professor in the Department of Electrical and Computer Engineering. He also holds a joint appointment as Cheung Kong Chair Professor and Director with the Center for Nonlinear and Complex Systems at Huazhong University of Science and Technology, Wuhan, China. Dr. Wang is a recipient of the 1994 O. Hugo Schuck Best Paper Award of the American Automatic Control Council, the 14th IFAC World Congress Poster Paper Prize, the 2000 IEEE Transactions on Fuzzy Systems Outstanding Paper Award. He co-authored (with Kazuo Tanaka) the book *Fuzzy Control Systems Design an Analysis: A Linear Matrix Inequality Approach* (Wiley-Interscience, 2001). Dr. Wang has served on the IEEE Control Systems Society Conference Editorial Board and is currently an Associate Editor for the *IEEE Transactions on Automatic Control*. His research interests include control of nonlinear dynamics, nonlinear control systems, intelligent systems and control, networked control systems, sensor fusion, and applications.

**LI Jing** Received the B.S.and M.S.degree in Automatic Control and System Engineering from Xi'an Jiao Tong University, China, in 1993 and 1996, respectively. He is currently a Ph.D.candidate in Dept. of Electrical and Computer Engineering at Duke University, Durham, NC, U.S.A.He is particularly interested in advanced control design for industrial systems with high performance and reliability. Mr. Li is a recipient of the Sunshine Fellowship in 1996. He is currently an IEEE student member and SIAM member. His research interests include fuzzy control, LPV control, signal processing, sensor fusion and pattern recognition.