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Wiener State Estimators Based on Kalman Filtering[1)]

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Abstract Using classical steady-state Kalman filtering theory, a new approach of designing Wiener state estimators is presented, whose principle is that based on steady-state Kalman filter and predictor given in the Wiener filter form, and using the autoregressive moving average (ARMA) innovation model, the recursive version of non-recursive steady-state optimal state estimators yields the Wiener state estimators. The proposed Wiener state estimators can handle the state filtering, smoothing and prediction problems in a unified framework. They have the ARMA recursive form, and have asymptotic stability and optimality. A simulation example shows their effectiveness.

Key words Wiener state estimators, filtering, prediction, smoothing, Kalman filtering method

1 Introduction

It is well known that classical Kalamn filtering method can't handle the state filtering, prediction and smoothing problems in a unified framework, particularly, the computation of Kalman smoother is complex[1]. Though a class of unified steady-state Kalman estimators is presented based on the modern time series analysis method in [2], but its disadvantage is that the estimators are not asymptotically stable for unstable systems, and require the computation of the optimal initial values. The Kalman smoother in [3] also requires the computation of the optimal initial value. Only the Wiener state prediction problem is handled based on the polynomial method on frequency-domain in [4], and the solution of Diophantine equations is required. Unified Wiener state estimators are presented based on the modern time series analysis method in [5], but they require the computations of the pseudo-exchange and pseudo-inverse. In this paper, using the Kalman filtering theory, a class of Wiener state estimators with ARMA recursive form is presented for completely observable and completely controllable systems or stable systems, which can handle the prediction, filtering and smoothing problems in a unified framework, and has the asymptotic stability and optimality. It avoids the solution of Diophantine equations and the computations of the pseudo-exchange, pseudo-inverse and the optimal initial values, so the computational burden may be reduced, and the disadvantages of above mentioned methods are overcome.

Consider the discrete time-invariant linear stochastic system

$$\boldsymbol{x}(t+1) = \Phi \boldsymbol{x}(t) + \Gamma \boldsymbol{w}(t) \tag{1}$$

$$\boldsymbol{y}(t) = H\boldsymbol{x}(t) + \boldsymbol{v}(t) \tag{2}$$

where the state $\boldsymbol{x}(t) \in R^n$, measurement $\boldsymbol{y}(t) \in R^m$, measurement noise $\boldsymbol{v}(t) \in R^m$, model noise $\boldsymbol{w}(t) \in R^r$, Φ, Γ, H are the constant matrices with compatible dimensions.

Assumption 1. $\boldsymbol{w}(t)$ and $\boldsymbol{v}(t)$ are independent white noises with zero mean and variance matrices $Q>0$ and $R>0$, respectively.

Assumption 2. The initial state $\boldsymbol{x}(t_0)$ is independent of $\boldsymbol{w}(t)$ and $\boldsymbol{v}(t)$.

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Assumption 3. The system is completely observable and completely controllable, or is stable (i. e., Φ is a stable matrix).

The problem is to find the Wiener state estimators $\hat{x}(t|t+N)$ of state $x(t)$ based on measurements $(y(t+N), y(t+N-1), \cdots)$. They have a transfer function matrix form with the measurement signal $y(t+N)$ as input. For $N=0, N>0$ and $N<0$, $\hat{x}(t|t+N)$ is called Wiener state filter, smoother and predictor, respectively.

2 Lemmas

Lemma 1[1]. For the system of (1) and (2) with Assumptions 1~3, the steady-state optimal Kalman filter and predictor are given by

$$\hat{x}(t \mid t) = \Psi_f \hat{x}(t-1 \mid t-1) + Ky(t) \tag{3}$$

$$\hat{x}(t+1 \mid t) = \Psi_P \hat{x}(t \mid t-1) + K_P y(t) \tag{4}$$

where $\Psi_f = (I_n - KH)\Phi$ and $\Psi_P = (\Phi - K_P H)$ are stable matrices. I_n is an $n \times n$ unity matrix. K and K_P are steady-state filtering and prediction gains respectively, and $K = \Sigma H^T [H\Sigma H^T + R]^{-1}$, T denotes the transpose, $K_P = \Phi K$. Σ is the steady-state prediction error variance matrix, and it is the unique positive definite solution of the following Riccati equation

$$\Sigma = \Phi[\Sigma - \Sigma H^T(H\Sigma H^T + R)^{-1} H\Sigma]\Phi^T + \Gamma Q \Gamma^T \tag{5}$$

Lemma 2. For the system of (1) and (2) with Assumptions 1~3, the ARMA innovation model is given by

$$A(q^{-1})y(t) = D(q^{-1})\varepsilon(t) \tag{6}$$

where q^{-1} is the backward shift operator, i. e., $q^{-1}x(t) = x(t-1)$, and

$$\begin{cases} A(q^{-1}) = D(q^{-1}) - H\Phi \mathrm{adj}(I_n - \Psi_f q^{-1}) K q^{-1} \\ D(q^{-1}) = \psi(q^{-1}) I_m \\ \psi(q^{-1}) = \det(I_n - \Psi_f q^{-1}) \end{cases} \tag{7}$$

or

$$\begin{cases} A(q^{-1}) = D(q^{-1}) - H\mathrm{adj}(I_n - \Psi_P q^{-1}) K_P q^{-1} \\ D(q^{-1}) = \psi(q^{-1}) I_m \\ \psi(q^{-1}) = \det(I_n - \Psi_P q^{-1}) \end{cases} \tag{8}$$

where the definitions of K, K_P, Ψ_f and Ψ_P are the same as Lemma 1, and (7) and (8) are equivalent, $\psi(q^{-1})$ is a stable polynomial. The white noise $\varepsilon(t)$ with zero mean is the innovation process of $y(t)$.

Proof. For the system of (1) and (2) with Assumptions 1~3, we have the steady-state optimal Kalman predictor[1]

$$\hat{x}(t+1 \mid t) = \Phi \hat{x}(t \mid t), \quad y(t) = H\hat{x}(t \mid t-1) + \varepsilon(t) \tag{9}$$

Using (3) yields, we have

$$\hat{x}(t \mid t) = (I_n - \Psi_f q^{-1})^{-1} K y(t) \tag{10}$$

Using (9) and (10) yields, we have

$$y(t) = H\Phi(I_n - \Psi_f q^{-1})^{-1} K q^{-1} y(t) + \varepsilon(t) \tag{11}$$

Let the determinant and adjoint matrix of matrix X be det X and adj$X$, respectively. Hence, using (11), we obtain

$$\det(I_n - \Psi_f q^{-1})\varepsilon(t) = [\det(I_n - \Psi_f q^{-1}) I_m - H\Phi \mathrm{adj}(I_n - \Psi_f q^{-1}) K q^{-1}] y(t) \tag{12}$$

Therefore (6) and (7) hold. Additionally, from (4) and (9) we have

$$y(t) = H(I_n - \Psi_P q^{-1})^{-1} K_P q^{-1} y(t) + \varepsilon(t) \tag{13}$$

which in twin leads to

$$\det(I_n - \Psi_P q^{-1})\varepsilon(t) = [\det(I_n - \Psi_P q^{-1}) I_m - H\mathrm{adj}(I_n - \Psi_P q^{-1}) K_P q^{-1}] y(t) \tag{14}$$

therefore, (6) and (8) hold. From [1] we have $\det(I_n - \Psi_f q^{-1}) = \det(I_n - \Psi_P q^{-1})$, and $\psi(q^{-1})$ is stable. And using (12) and (14), we have (7) is equivalent to (8). The proof is completed. □

Lemma 3. For $N>0$, the steady-state optimal non-recursive smoother is given by

$$\hat{x}(t \mid t+N) = \hat{x}(t \mid t) + \sum_{i=1}^{N} M(i)\varepsilon(t+i) \tag{15}$$

where the smoothing gain $M(i)$ is computed by

$$M(i) = \Sigma[(I_n - KH)^{\mathrm{T}}\Phi^{\mathrm{T}}]^i H^{\mathrm{T}}[H\Sigma H^{\mathrm{T}} + R]^{-1}, \quad (i \geqslant 1) \tag{16}$$

Proof. Taking the finite initial time t_0, in the linear space generated by the finite measurements $(y(t_0), y(t_0+1), \cdots, y(t+N))$, from [1] we have the optimal non-recursive smoother

$$\hat{x}(t \mid t+N) = \hat{x}(t \mid t) + \sum_{i=1}^{N} M(t \mid t+i)\varepsilon(t+i) \tag{17}$$

$$M(t \mid t+i) = \prod_{j=t}^{t+i-1} A(j)K(t+i), \quad A(i) = P(i \mid i)\Phi^{\mathrm{T}}P^{-1}(i+1 \mid i) \tag{18}$$

where $P(i|i)$ and $P(i+1|i)$ are filtering and prediction error variance matrices, respectively. $K(t+i)$ is the filtering gain.

Using Assumption 3 yields that there exists a steady-state Kalman filter, so that letting $t_0 \rightarrow -\infty$, we have $P(i|i) \rightarrow P, P(i+1|i) \rightarrow \Sigma, K(i) \rightarrow K$, where P, Σ, K are the corresponding steady-state filtering, prediction error variance matrices and steady-state filtering gain, respectively. Hence we have the corresponding steady-state values A and $M(i)$ for $A(i)$ and $M(t|t+i)$, *i.e.*,

$$A = P\Phi^{\mathrm{T}}\Sigma^{-1}, \quad M(i) = (P\Phi^{\mathrm{T}}\Sigma^{-1})^i K, \quad (i \geqslant 1) \tag{19}$$

From [1] we have $P=(I_n - KH)\Sigma$. Hence from the symmetry of the variance matrices we have $P=\Sigma(I_n - KH)^{\mathrm{T}}$. And substituting it into (19) yields

$$M(i) = \Sigma[(I_n - KH)^{\mathrm{T}}\Phi^{\mathrm{T}}]^i \Sigma^{-1}K \tag{20}$$

From Lemma 1 we have $\Sigma^{-1}K = H^{\mathrm{T}}[H\Sigma H^{\mathrm{T}} + R]^{-1}$, and substituting it into (20) we have (16). The proof is completed. □

3 Wiener state estimators

Theorem 1. For the system of (1) and (2) with Assumptions 1~3, the asymptotically stable Wiener state estimators with ARMA recursive form are given by

$$\psi(q^{-1})\hat{x}(t \mid t+N) = K_N(q^{-1})y(t+N) \tag{21}$$

where

i) For $N \leqslant 0$, $$K_N(q^{-1}) = \Phi^{-N}\mathrm{adj}(I_n - q^{-1}\Psi_f)K \tag{22}$$

ii) For $N>0$, $$K_N(q^{-1}) = \mathrm{adj}(I_n - q^{-1}\Psi_f)Kq^{-N} + L_N(q^{-1})A(q^{-1}) \tag{23}$$

$$L_N(q^{-1}) = \sum_{i=1}^{N} \Sigma[(I_n - KH)^{\mathrm{T}}\Phi^{\mathrm{T}}]^i H^{\mathrm{T}}[H\Sigma H^{\mathrm{T}} + R]^{-1} q^{i-N} \tag{24}$$

Proof. Substituting $(I_n - \Psi_f q^{-1})^{-1} = \mathrm{adj}(I_n - \Psi_f q^{-1})/\det(I_n - \Psi_f q^{-1})$ into (10), we have

$$\psi(q^{-1})\hat{x}(t \mid t) = [\mathrm{adj}(I_n - q^{-1}\Psi_f)K]y(t) \tag{25}$$

i) For $N \leqslant 0$, from [1] we have

$$\hat{x}(t \mid t+N) = \Phi^{-N}\hat{x}(t+N \mid t+N) \tag{26}$$

Substituting (25) into (26) leads to

$$\psi(q^{-1})\hat{x}(t \mid t+N) = \Phi^{-N}\mathrm{adj}(I_n - q^{-1}\Psi_f)Ky(t+N) \tag{27}$$

which means that (21) and (22) hold.

ii) For $N>0$, from Lemma 3 we have

$$\hat{x}(t \mid t+N) = \hat{x}(t \mid t) + L_N(q^{-1})\varepsilon(t+N) \tag{28}$$

where $L_N(q^{-1})$ is defined by (24). Noting (6), (7) and (10), (28) can be expressed as

$$\psi(q^{-1})\hat{x}(t \mid t+N) = [\mathrm{adj}(I_n - q^{-1}\Psi_f)Kq^{-N} + L_N(q^{-1})A(q^{-1})]y(t+N) \tag{29}$$

which leads to that (21) and (23) hold. From Lemma 2 we have $\psi(q^{-1})$ to be stable, hence the Wiener state estimators (21) are asymptotically stable. Applying the projection property yields (21) to be steady-state optimal. The proof is completed. □

4 Simulation example

Consider the system of (1) and (2) with Assumptions 1~3, where $\boldsymbol{x}(t)=[x_1(t), x_2(t)]^{\mathrm{T}}$, $w(t)$ and $v(t)$ are independent white noise with zero mean and variance $Q=1$, $R=10$, respectively. $\boldsymbol{x}(0)=[0,0]^{\mathrm{T}}$, and

$$\Phi=\begin{bmatrix}1 & 0\\ 0.2 & 0.8\end{bmatrix},\quad \Gamma=\begin{bmatrix}1\\ -0.5\end{bmatrix},\quad H=[1\quad 1] \tag{30}$$

The problem is to find the Wiener state smoother $\hat{\boldsymbol{x}}(t|t+1)$.

By iteration for Riccati equation (5), we obtain

$$\Sigma=\begin{bmatrix}3.9408 & -0.1526\\ -0.1526 & 0.7146\end{bmatrix},\quad K=\begin{bmatrix}0.2640\\ 0.0392\end{bmatrix},\quad P=\begin{bmatrix}2.9408 & -0.3010\\ -0.3010 & 0.6926\end{bmatrix} \tag{31}$$

From (21) we have the asymptotically stable Wiener state smoother as

$$\psi(q^{-1})\hat{\boldsymbol{x}}(t\mid t+1)=K_1(q^{-1})\boldsymbol{y}(t+1) \tag{32}$$

where $\psi(q^{-1})=1-1.4519q^{-1}+0.5575q^{-2}$ is a stable polynomial, and

$$K_1(q^{-1})=\begin{bmatrix}0.2291-0.1484q^{-1}-0.0279q^{-2}\\ 0.0134+0.0150q^{-1}+0.0244q^{-2}\end{bmatrix} \tag{33}$$

Respectively, we take two sets of different initial values:

The initial values (a):

$$\hat{\boldsymbol{x}}^{(1)}(0\mid 1)=[-1,-1]^{\mathrm{T}},\quad \hat{\boldsymbol{x}}^{(1)}(1\mid 2)=[1\quad 1]^{\mathrm{T}} \tag{34}$$

The initial values (b):

$$\hat{\boldsymbol{x}}^{(2)}(0\mid 1)=[10,10]^{\mathrm{T}},\quad \hat{\boldsymbol{x}}^{(2)}(1\mid 2)=[-10,-10]^{\mathrm{T}} \tag{35}$$

The simulation results are shown in Fig. 1 and Fig. 2, where the solid line denotes the true value, the dashed line and dash dot line are the Wiener state smoothers $\hat{\boldsymbol{x}}(t|t+1)$ corresponding to the initial values (a) and (b), respectively. We see that the effect of the initial values (a) or (b) is gradually eliminated after a transition process, hence $\hat{\boldsymbol{x}}(t|t+1)$ has the asymptotic stability. And $\hat{\boldsymbol{x}}(t|t+1)$ has a higher precision and the asymptotic optimality.

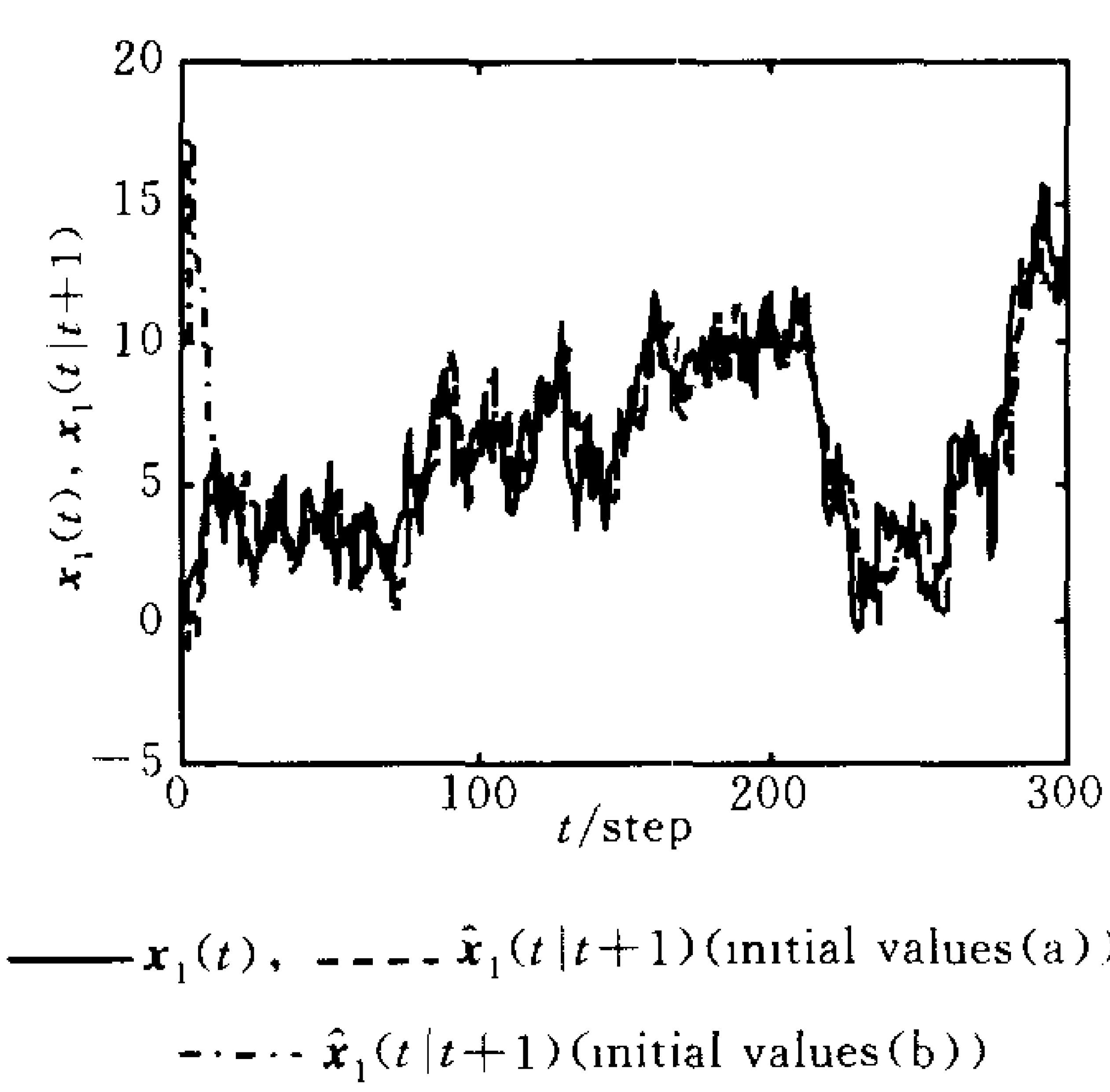

Fig. 1 $\boldsymbol{x}_1(t)$ and Wiener state smoother $\hat{\boldsymbol{x}}_1(t|t+1)$

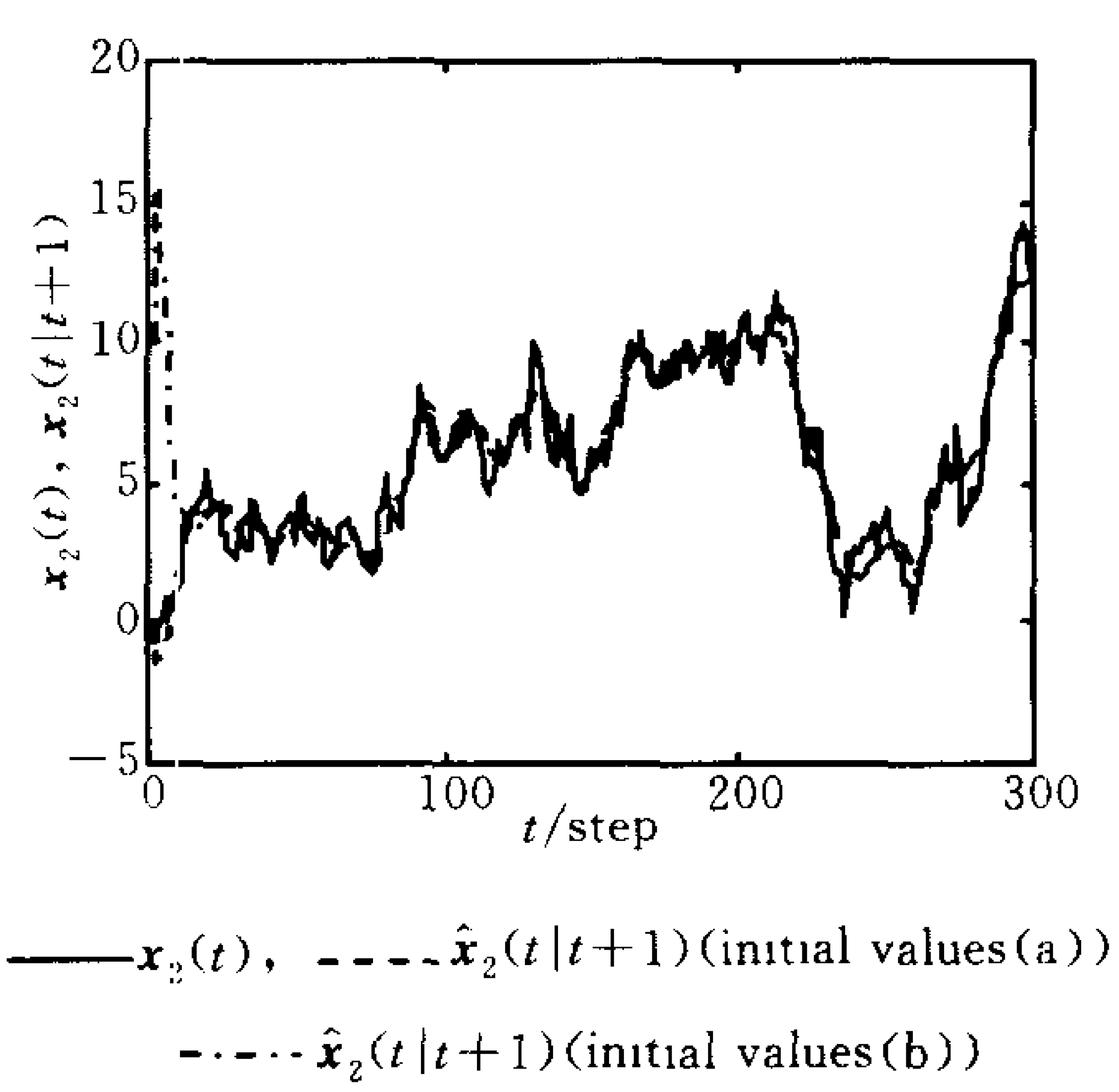

Fig. 2 $\boldsymbol{x}_2(t)$ and Wiener state smoother $\hat{\boldsymbol{x}}_2(t|t+1)$

5 Conclusion

The relation between the Kalman filter and the Wiener filter is discovered in this paper. The unified Wiener state estimators with asymptotic stability and optimality are presented. Comparing with other methods, they avoid the computations of the optimal initial values, the pseudo-exchange and pseudo-inverse, and avoid the solution of Diophantine equations, so that the computational burden can be reduced.

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基于 Kalman 滤波的 Wiener 状态估值器

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摘 要 应用经典稳态 Kalman 滤波理论提出了设计 Wiener 状态估值器的新方法，其原理是：基于在 Wiener 滤波器形式下的稳态 Kalman 滤波器和预报器及 ARMA 新息模型，由稳态最优非递推状态估值器的递推变形引出 Wiener 状态估值器. 所提出的 Wiener 状态估值器可统一处理状态滤波、预报和平滑问题. 它们具有 ARMA 递推形式，且具有渐近稳定性和最优性，仿真例子说明了它们的有效性.

关键词 Wiener 状态估值器，滤波，预报，平滑，Kalman 滤波方法

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