Robustness of the Design by Unlocking Unstable Zero-dynamics[1)]

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Abstract   The method of unlocking zero dynamics in output feedback control is analyzed. It is pointed out that this method is based on the transformation of system configurations. By using a series of transformations a stabilizing controller for an unstable plant can be rearranged to become a controller for a non-minimum-phase plant. Although the stability of various configurations remains unchanged, the robustness of the system is not the same. Detailed analysis of a typical non-minimum-phase example is given.

Key words   Unstable zero-dynamics, non-minimum-phase, robust stability, output feedback

1   Introduction

Zero dynamics is a concept in nonlinear control theory. If the zero dynamics of a system is asymptotically stable, the system is also called minimum phase. Zero dynamics generally appears in the feedback path of the system. As indicated in the nonlinear control theory, the control law can be designed based on the feedforward part of the system only, if the zero dynamics is asymptotically stable[1,2]. And most recent design methods for nonlinear systems rely on hypothesis that the zero dynamics of the controlled plant is asymptotically stable. Or, in other words these methods are suitable only for minimum phase systems. In order to deal with non-minimum phase systems a new method by unlocking the zero dynamics was proposed by Isidori[3]. It will be pointed out in this paper that although this approach can stabilize the non-minimum phase plant, the non-minimum-phase nature of the system remains unchanged and the final design may not be robust.

2   Control design by unlocking the zero dynamics

Consider a single-input/single-output linear system with no input-output feedthrough. Let n denote the dimension of its state space and let r denote its relative degree. The system equations are as follows.

$$\dot{z} = F_0 z + G_0 x_1$$
$$\dot{x}_1 = x_2$$
$$\vdots$$
$$\dot{x}_{r-1} = x_r$$
$$\dot{x}_r = H_0 z + a_1 x_1 + \cdots + a_{r-1} x_{r-1} + a_r x_r + bu$$
$$y = x_1 \tag{1}$$

where

$$\dot{z} = F_0 z, \quad z \in R^{n-r}$$

is the zero dynamics, which is assumed to be unstable in this paper. It can be seen from (1), that zero dynamics forms a feedback loop in the system: $x_r \rightarrow x_1 \rightarrow z \rightarrow \dot{x}_r$. The method for locking the zero dynamics proposed in [3] consists of the following four design steps.

Step 1.   Break down the zero-dynamics loop before x_r, and insert a controller into the

loop. This forms an auxiliary loop used only in this step.

The controlled plant of this closed-loop is called auxiliary plant $P_a(s)$, its input is x_r, but here we relabel x_r as u_a. The feedback signal appeared on the right-hand side of the $\dot{x}_r$-formula of (1) is the output of the auxiliary plant, i. e., the output of $P_a(s)$ is

$$y_a = H_0 z + (a_1 + bh)x_1 + \cdots + a_{r-1}x_{r-1} + a_r u_a \tag{2}$$

The object of this step is to design a controller of the form

$$\begin{aligned}
\dot{\eta} &= L\eta + My_a \\
u_a &= N\eta
\end{aligned} \tag{3}$$

Step 2. By using an additional variable ς to modify controller (3) as follows.

$$\begin{aligned}
\dot{\eta} &= L\eta + Mk(\varsigma - N\eta) \\
\dot{\varsigma} &= N\dot{\eta} - k(\varsigma - N\eta) + y_a \\
u_a &= \varsigma
\end{aligned} \tag{4}$$

As pointed out in [3], if k is large enough, the unstable plant $P_a(s)$ can still be stabilized by the modified controller (4).

Step 3. Reconstruct a new controller from (4), i. e.,

$$\begin{aligned}
\dot{\eta} &= L\eta + Mk(x_r - N\eta) \\
u &= \frac{1}{b}(N\dot{\eta} - k(x_r - N\eta))
\end{aligned} \tag{5}$$

Now the input of this controller is x_r and its output is just the input u of the original system. Notice that from now on we restore our discussion with the original system (1). Controller (5) takes the signal from the original system at point x_r and feeds back to system (1) at the control input u. So it is already a stabilizing controller for a plant with unstable zero dynamics. The only difference is on the input of the controller——the input is x_r in this case.

Step 4. Take x_1 as the controller input and use the $(r-1)$th derivative of x_1 to replace x_r in (5). In practice, a "bench of rough differentiators" is used to give an appropriate estimate of x_r.

The unlocking approach directly deals with the unstable poles of the zero-dynamics loop, so the stability of the system is guaranteed. This design idea is further applied to nonlinear systems as presented in [3], where it is assumed that there exist a controller similar to (3) and a Lyapunov function $V(x_a, \eta)$, where $x_a = [z \quad x_1 \quad \cdots \quad x_{r-1}]^T$ (Assumption 2 in [3]). Because the proposed Lyapunov function is independent of the unknown vector-valued parameter p, the resulting design is robust. Notice that the system configuration of the auxiliary loop in Step1 is different from that of the real system. Although the nominal systems are both stable, the robustness of each design is not equivalent. If the stability of the Step1-design is robust, surely it does not mean that the final design has the same robustness.

3 Design example

An inverted pendulum on a cart is a typical non-minimum phase system. Let M and m be the masses of the cart and of the pendulum, respectively, l the length of the pendulum. And let θ denote the angle of the pendulum with respect to the vertical axis, x the distance of the cart from a reference point, and u the horizontal force applied to the cart. Then the linearized equations of the inverted pendulum can be expressed as follows.

$$\begin{aligned}
(M + m)\ddot{x} + ml\ddot{\theta} &= u \\
\frac{4}{3}ml^2\ddot{\theta} + ml\ddot{x} &= mgl\theta
\end{aligned} \tag{6}$$

By choosing appropriate state variables, the following state space equations can be ob-

tained

$$\dot{z}_1 = z_2 - \frac{3}{4l}x_2$$

$$\dot{z}_2 = \frac{3g}{4l}z_1$$

$$\dot{x}_1 = x_2 \tag{7}$$

$$\dot{x}_2 = -\frac{3mg}{(4M+m)}z_1 + \frac{4}{(4M+m)}u$$

Let $M=0.145$kg, $m=0.03$kg, $l=0.125$m; then the transfer function from u to $y = x_1$ has the form

$$P_1(s) = \frac{5.715[(0.13s)^2 - 1]}{s^2[(0.1217s)^2 - 1]} \tag{8}$$

(8) shows that this is an unstable non-minimum phase system. Now we use the unlocking method to design the controller.

Step 1. Note that from (8) its relative degree r equals 2. Let $h = -0.01$. Then from (1)(2) an auxiliary plant can be formed as

$$P_a(s) = \frac{Y_a(s)}{U_a(s)} = \frac{8.6188(s^2 + 0.4478)}{s(s + 7.672)(s - 7.672)} \tag{9}$$

By using the classical method, the controller (3) of the auxiliary loop is obtained as follows.

$$K_1(s) = \frac{-300(s+6)}{(s+50)(s-3)} \tag{10}$$

The closed-loop poles of the auxiliary loop formed from $P_a(s)$ and $K_1(s)$ are

$$-20.55 \pm i41.32, \ -5.11, \ -0.39 \pm i0.70 \tag{11}$$

And its Nyquist locus keeps an enough distance from the critical point (-1) (The Nyquist plot is omitted), i. e., the robustness of the design is quite well.

Step 2. By using an additional variable ς, a modified controller $K_2(s)$ can be obtained according to (4).

$$K_2(s) = \frac{(s - 1.5 \times 10^6)(s+6)}{(s+5000)(s+50)(s-3)} \tag{12}$$

where the value of k in (4) is 5000. Now the closed-loop poles of the system made up of $K_2(s)$ and $P_a(s)$ are

$$-5000.52, \ -20.29 \pm i41.45, \ -5.12, \ -0.39 \pm i0.70 \tag{13}$$

And the auxiliary system is still stable.

Step 3. According to (5) an output feedback controller $K_3(s)$ is then constructed.

$$K_3(s) = \frac{-229511.6967(s + 6.216)(s - 0.08017)}{(s - 1.5 \times 10^6)(s+6)} \tag{14}$$

The input of the controller is x_2 from the original system. Now the plant is from u to x_2, which is closed by the $K_3(s)$. The closed-loop poles are the same as in Step 2, see (13).

Step 4. Shift the output of the plant to point x_1 and use an approximate derivative of x_1 as the controller's input. Suppose, first, that the ideal derivative of x_1 is available. Then we replace it by a rough derivative and examine the change in the system's behavior.

If the derivative is ideal, the transfer function of the controller can be directly obtained from (14), namely $sK_3(s)$. But for this example an additional term of bhx_1 during the design with (2) must be added to bu to form the required controller. Since $bu + bhx_1 = b(u + hx_1)$, when we use x_1 as the feedback signal, the controller becomes

$$K_4(s) = sK_3(s) + h = \frac{-229511.6967(s + 6.217)(s - 0.2946)(s + 0.2141)}{(s - 1.5 \times 10^6)(s+6)} \tag{15}$$

This controller is just the output-feedback controller for the non-minimum phase plant (8)

(if an ideal differentiator is available). The closed-loop poles of the output feedback loop formed from (8) and (15) are still the same as in Steps 2 and 3. This is really the primary idea of [3]: A stabilization controller for an unstable plant is transformed into an output-feedback controller for a non-minimum plant *via* loop transformation. The closed-loop poles of the system remain unchanged during the transformation.

An ideal differentiator is unrealistic and, besides, the order of the numerator of $K_4(s)$ is higher then that of the denominator, therefore a rough differentiator is needed to replace the ideal one. According to the algorithms given by [3], with rough differentiators the controller of (15) is then modified as

$$K_5(s) = \frac{g^2 s}{(s+g)^2} K_3(s) + h \tag{16}$$

It was pointed out in [3] that if g is large enough, then P_1 of the equation(8) can be stabilized by this K_5. Let $g = 1000$ in this example. But it shows the system is unstable. Increase g to 10^4, the closed-loop poles in this case are 1.499×10^6, -19950.797, 47.949, $-15.624 \pm i22.308$, -5.109, and $-0.389 \pm i0.697$. The controller K_5 still can not stabilize plant P_1. Although as said in [3] that theoretically there exists a solution if $g \to \infty$, in this case $g^2 s$ in (16) has already reached $10^8 s$, a value that numerical calculation error might be considered, however the system is still unstable. So it is not a simple stability problem, it is a typical robust problem.

4 Robust analysis

It must be mentioned that the method for unlocking the zero dynamics does not ignore the robust consideration. When it is applied to nonlinear systems, it requires that the Lyapunov function is independent of the unknown parameter p in order to make a robust design. Notice that the unlocking method is based on the loop transformation. Therefore even though the closed-loop poles remain unchanged, the robustness is not equivalent after the transformation. It is because the system configuration is changed.

Because the main concepts of [3] are established on the basis of linear systems, and the robust theory for linear systems is well developed, we apply the results from [4] to robustness analysis of the example.

According to [4], for robust analysis the system is first separated into a plant $P(s)$ and a controller $K(s)$, and then if the uncertainty is multiplicative, we have the following robust stability condition

$$l_m(\omega) < \underline{\sigma}[I - PK(j\omega)^{-1}], \quad \forall \omega \geqslant 0 \tag{17}$$

where $l_m(\omega)$ is the upper limit of the multiplicative uncertainty, and $\underline{\sigma}$ denotes minimum singular value. For SISO system, the frequency response is also the singular value plot.

For the auxiliary loop, the plant is $P_a(s)$ and the controller is $K_2(s)$. But after the loop transformation, the plant is the original non-minimum plant $P_1(s)$ and the controller now is $K_4(s)$.

Although the characteristic equations of these loops are the same, their P and K both are different. So according to (17), their robustness is different from each other. For the loop with K_4 and P_1, its singular value plot of $1 + [P_1(j\omega)K_4(j\omega)]^{-1}$ is shown in Fig. 1. The smallest value of σ is approximately -50dB. This means that multiplicative uncertainties as small as -50dB(approximately 0.003) could produce instability. Such a lack of stability robustness can not be seen from the closed-loop pole test given by (11) and (13). But it is clear if we use the Nyquist plot. The Nyquist plot of $P_1(j\omega)K_4(j\omega)$ is very close to the point (-1), this means that the design is really on the verge of instability. Because of serious lack of robustness, the system becomes unstable when the differentiator is replaced by a rough one as in (16). So it is not the problem of how large g must be, it really
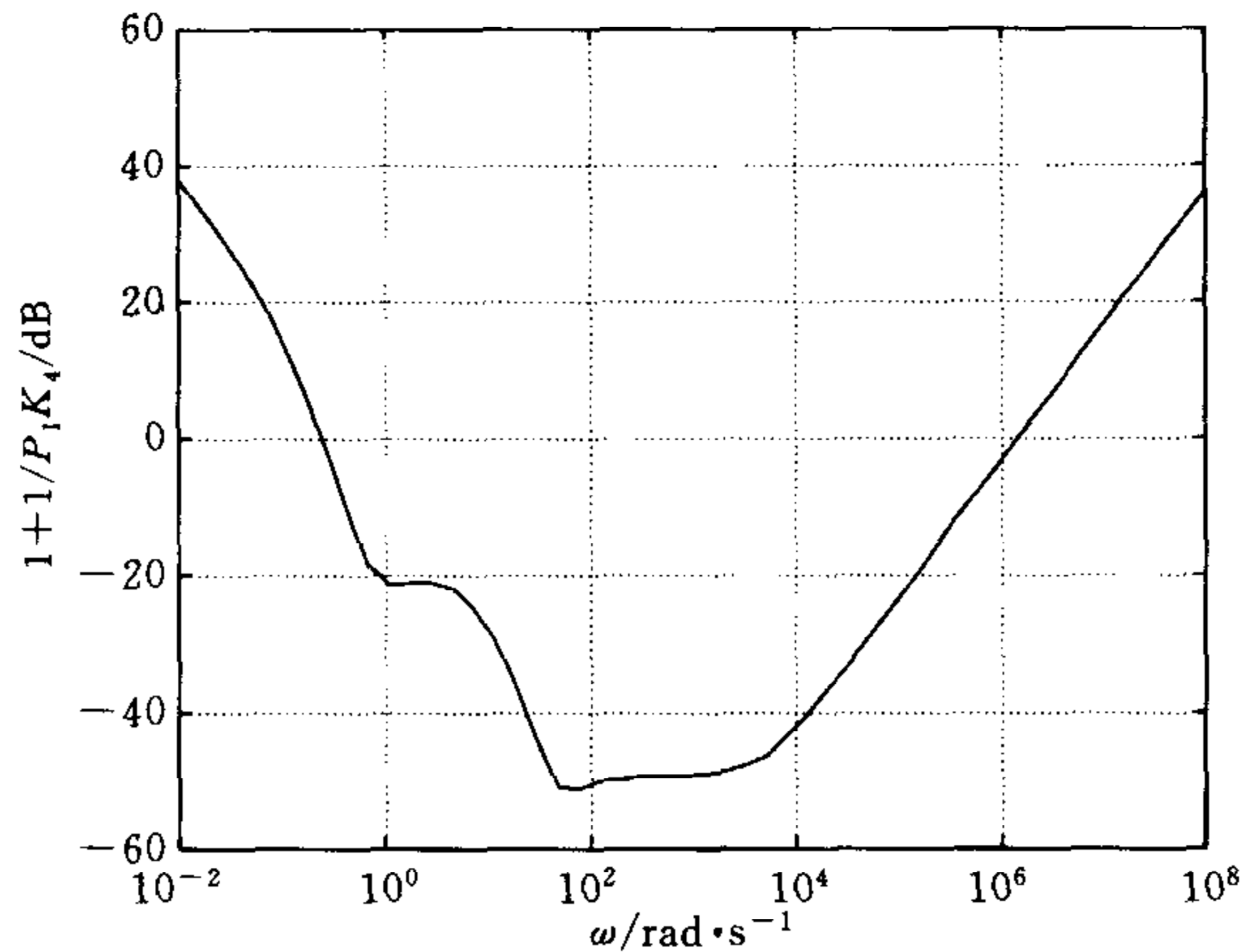
is a robust problem.

Fig. 1　Singular value Bode plot of $\underline{\sigma}[I+(P_1K_4)^{-1}]$

We will further examine the robust problem. Let the controller be K_4, which is made up of an ideal differentiator. Suppose now that the plant has a perturbation, i. e., a small additional term of damping $D_x\dot{x}$ is added to the system, $D_x=0.05$kg/s. Closed-loop pole analysis and impulse response simulation both show that the perturbed system is unstable. Therefore, even if we can get a large enough g in the above mentioned design to stabilize the system, the final design will be unstable if the real system is slightly different from the design model.

5　Conclusions

The method for unlocking the zero dynamics is mainly based on loop transformations, however the robustness of the system is not the same for different loop configurations. Therefore, as in the inverted pendulum example, though the stability of the nominal system can be guaranteed in design, it has no robustness.

The unstable zero of the inverted pendulum-cart system is close to its unstable pole, so this example reflects the typical design difficulties for non-minimum phase systems. It is clear that such non-minimum phase systems can not be stabilized by output feedback[5]. Although the design idea of the unlocking method is different from all the traditional approaches, the conclusion is still the same —— it has no robustness. This fact has been observed by Ishida et al. [6], who compared the state feedback with the H_∞ design (it is actually an output feedback design) for an inverted pendulum-cart system. Simulation showed that both designs were quite well. But experiments showed that only the state feedback could stabilize the real plant and had a similar result as in simulation. It showed that the real system with the H_∞ controller was unstable. The authors of [5] believed at that time that it might be caused by the strong stabilization problem. In fact, from today's point, it is really a robust problem——because of lack of robustness, the real system can not be stabilized.

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不稳定零动态解开设计的鲁棒性

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摘 要 对零动态解开法用于输出反馈控制进行了详细分析. 零动态解开法是一种回路变换法，通过变换将一个对不稳定对象进行镇定的控制器变换成对非最小相位系统的控制器. 变换中虽然系统的特征方程式不变，但由于回路的结构已经改变，二者的鲁棒性并不等价. 文中以一典型的非最小相位系统为例，详细分析了这种设计方法的鲁棒性.

关键词 不稳定零动态，非最小相位，鲁棒稳定性，输出反馈

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