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Robust Stability Analysis for Dynamic Matrix Control

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Abstract This paper describes the system by means of finite impulse response (FIR) model, and defines the model uncertainty in the form of sum of squares of impulse response coefficient errors. Then the closed-loop system using DMC (dynamic matrix control) algorithm based on finite impulse response is given. Finally, the robust stability conditions for DMC algorithm are derived.

Key words Predictive control, robust stability, dynamic matrix control (DMC)

1 Introduction

Model predictive control (MPC), also known as moving horizon control or receding horizon control, is a popular technique for the control of slow dynamical systems, such as those encountered in chemical process control in petrochemical, pulp and paper industries, and in gas pipeline control. In using MPC, the stability is an important characteristic of controlled systems. In this paper the dynamic matrix control (DMC) algorithm based on finite impulse response is first discussed, and the closed-loop system is obtained. Then, the robust stability conditions of dynamic matrix control (DMC) algorithm are derived. The results proposed by this paper provide a theoretical foundation for analysis and design of predictive controllers.

2 System model and predictive law

Consider a single-input single-output system, which is described in terms of its truncated impulse response:

$$y(k) = \sum_{i=1}^{N} h(i)u(k-i) \tag{1}$$

where $y(k)$ and $u(k)$ are output and input variables respectively, $h(i)$ is the impulse response coefficient, while N is the order of the system. We use

$$y'(k) = \sum_{i=1}^{N} h'(i)u(k-i) \tag{2}$$

as the model of system (1). So we can get the predictive output as

$$y'(k+j) = \sum_{i=1}^{N} h'(i)u(k+j-i) + d(k) \tag{3}$$

where

$$d(k) = y(k) - y'(k) \tag{4}$$

Using the algorithm proposed in [1], we can get

$$u(k) = k_r r(k) + \sum_{i=1}^{N} (k_r \Delta h(i) + a_i) u(k-i) \tag{5}$$

where $a_1 = k_u + \sum_{i=1}^{P} k_{ej}[-h'(1+j)]$, $a_i = \sum_{i=1}^{P} k_{ej}[-h'(i+j)] (i=2,\cdots,N)$, $\Delta h(i) = h'(i) - h(i)$, k_r, k_{ej}, k_u are coefficients obtained during optimization.

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3 Model uncertainty and robust stability conditions for DMC

Using (2) as the model of system (1), we define the model error M as:

$$M = \sum_{i=1}^{N} | h(i) - h'(i) |^2 \tag{6}$$

From (5), we can rewrite the DMC closed-loop system as

$$\begin{cases} \boldsymbol{U}(k+1) = (A_H + \Delta A_H)\boldsymbol{U}(k) + \boldsymbol{B}r(k) \\ y(k) = C\boldsymbol{U}(k) \end{cases} \tag{7}$$

where

$$A_H = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a_N & \cdots & \cdots & -a_1 \end{bmatrix},\ C = \begin{bmatrix} h(N) & & \\ & \ddots & \\ & & h(1) \end{bmatrix},\ \Delta A_H = k_r \begin{pmatrix} & & \boldsymbol{0} & \\ \Delta h(N) & & \cdots & \Delta h(1) \end{pmatrix}$$

$$\boldsymbol{B} = (0 \quad \cdots \quad 0 \quad k_r)^{\mathrm{T}},\quad \boldsymbol{U}(k) = (u(k-N) \quad \cdots \quad u(k-1))^{\mathrm{T}}$$

Lemma 1[2]. Let $A, Q \in R^{n\times n}$ and $Q = Q^{\mathrm{T}} > 0$. If $|\lambda(A)| < 1$, then the equation

$$A^{\mathrm{T}}PA - P = -Q \tag{8}$$

has a unique solution $P \in R^{n\times n}$ and $P = P^{\mathrm{T}} > 0$. $\lambda(A)$ are all the eigenvalues of matrix A.

Lemma 2[3]. Let $A \in R^{n\times n}$. If there exist $P \in R^{n\times n}$ and $P = P^{\mathrm{T}} > 0$ satisfying

$$A^{\mathrm{T}}PA - P < 0 \tag{9}$$

then $|\lambda(A)| < 1$.

Lemma 3. Assume $A \in R^{n\times n}$ and $|\lambda(A)| < 1$. The sufficient condition of $|\lambda(A + \Delta A)| < 1$ is

$$\|\Delta A\| < 1 - \sqrt{1 - \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}} \tag{10}$$

where $P, Q \in R^{n\times n}$, and $P = P^{\mathrm{T}} > 0, Q = Q^{\mathrm{T}} > 0$ satisfying (8). $\lambda_{\max}(P) = \mathrm{Max}\{\lambda(P)\}$, $\lambda_{\min}(Q) = \mathrm{Min}\{\lambda(Q)\}$, $\|\Delta A\| = \lambda_{\max}^{1/2}(A^{\mathrm{T}}A)$.

Proof. See appendix.

Theorem 1. Assume $|\lambda(A_H)| < 1$. The closed-loop system obtained by using the DMC algorithm and Model (2) is asymptotically stable if Inequality (11) is true.

$$| k_r | \sqrt{M} < 1 - \sqrt{1 - \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}} \tag{11}$$

where $P, Q \in R^{n\times n}$, and $P > 0, Q > 0$ satisfying $A_H^{\mathrm{T}}PA_H - P = Q$.

Proof. From (7), we can get

$$\|\Delta A_H\| = | k_r | \sqrt{M} \tag{12}$$

Using (11), we can further get

$$\|\Delta A_H\| < 1 - \sqrt{1 - \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}} \tag{13}$$

By using Lemma 3, we get $|\lambda(A_H + \Delta A_H)| < 1$. So we can get that system (7) is asymptotically stable. □

Corollary 1. Assume $|\lambda(A_H)| < 1$. The closed-loop system obtained by using the DMC algorithm and Model (2) is asymptotically stable if Inequality (14) is true.

$$| k_r | \sqrt{M} < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} \tag{14}$$

where $P, Q \in R^{n\times n}$, and $P = P^{\mathrm{T}} > 0, Q = Q^{\mathrm{T}} > 0$ satisfying $A_H^{\mathrm{T}}PA_H - P = Q$.

Proof. Obviously, we can get

$$1 - \sqrt{1 - \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}} = \frac{1}{1 + \sqrt{1 - \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}}} \times \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} > \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} \tag{15}$$

So, if (14) is true, then (11) is also true. □

Theorem 2. If $|\lambda(A_H)|<1$ is true and there exist $\bar{P}\in R^{n\times n}, \bar{P}=\bar{P}^{\mathrm{T}}>0$, $\alpha\geqslant|k_r|\sqrt{M}$ satisfying (16), then the closed-loop system obtained by using the DMC algorithm and Model (2) is asymptotically stable.

$$\begin{bmatrix} -\bar{P} & A_H^{\mathrm{T}}\bar{P} & A_H^{\mathrm{T}}\bar{P} & I \\ \bar{P}A_H & \bar{P}-I & 0 & 0 \\ \bar{P}A_H & 0 & -\bar{P} & 0 \\ I & 0 & 0 & -\alpha^{-2}I \end{bmatrix}<0 \tag{16}$$

Proof. By using Lemma 2, we can get that if there exist $P\in R^{n\times n}$, $P=P^{\mathrm{T}}>0$ satisfying

$$(A_H+\Delta A_H)^{\mathrm{T}}P(A_H+\Delta A_H)-P<0 \tag{17}$$

then $|\lambda(A_H+\Delta A_H)|<1$ and system (7) is asymptotically stable. If $\alpha\geqslant|k_r|\sqrt{M}=\|\Delta A_H\|$, we can obtain that Inequality (18) is true for all $\boldsymbol{x}\in R^n$.

$$\begin{bmatrix} \boldsymbol{x} \\ \Delta A_H\boldsymbol{x} \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} \alpha^2 I & 0 \\ 0 & -I \end{bmatrix}\begin{bmatrix} \boldsymbol{x} \\ \Delta A_H\boldsymbol{x} \end{bmatrix}\geqslant 0 \tag{18}$$

By the property of matrix, we can get that Inequality (17) is equivalent to

$$\begin{bmatrix} \boldsymbol{x} \\ \Delta A_H\boldsymbol{x} \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} A_H^{\mathrm{T}}PA_H-P & A_H^{\mathrm{T}}P \\ PA_H & P \end{bmatrix}\begin{bmatrix} \boldsymbol{x} \\ \Delta A_H\boldsymbol{x} \end{bmatrix}<0 \tag{19}$$

for all $\boldsymbol{x}\in R^n$ and $\boldsymbol{x}\neq 0$. We further rewrite (19) using (18) as

$$\begin{bmatrix} A_H^{\mathrm{T}}PA_H-P+\tau\alpha^2 I & A_H^{\mathrm{T}}P \\ PA_H & P-\tau I \end{bmatrix}<0 \tag{20}$$

where $\tau>0$. From (20), we can further get

$$\begin{bmatrix} A_H^{\mathrm{T}}\bar{P}A_H-\bar{P}+\alpha^2 I & A_H^{\mathrm{T}}\bar{P} \\ \bar{P}A_H & \bar{P}-I \end{bmatrix}<0 \tag{21}$$

where $\bar{P}=P/\tau$. By using Schur complement formula we can obtain (16). □

4 Conclusions

The sufficient stability conditions for DMC algorithm have been presented in this paper, which can assure the closed-loop system using DMC algorithm to be asymptotically stable, when the coefficients of characteristic polynomial don't satisfy Jury's dominant coefficient lemma.

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Appendix A

Proof of Lemma 3. If $|\lambda(A)|<1$, we can get that there exist $P\in R^{n\times n}$, $P=P^{\mathrm{T}}>0$, and $Q\in R^{n\times n}, Q=Q^{\mathrm{T}}>0$ satisfying (8). For all $\boldsymbol{x}\in R^n$, we have $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}PA\boldsymbol{x}=\boldsymbol{x}^{\mathrm{T}}(P-Q)\boldsymbol{x}=\boldsymbol{x}^{\mathrm{T}}P\boldsymbol{x}-\boldsymbol{x}^{\mathrm{T}}Q\boldsymbol{x}\leqslant\lambda_{\max}(P)\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}-\lambda_{\min}(Q)\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$. So we can obtain $\|A^{\mathrm{T}}PA\|\leqslant\lambda_{\max}(P)-\lambda_{\min}(Q)$. Obviously, $\|\Delta A^{\mathrm{T}}P\Delta A\|\leqslant\|\Delta A\|^2\|P\|$ is always true. Since $P=P^{\mathrm{T}}>0$, we can rewrite P as $P=DD^H$. So we can get

$$\|A^{\mathrm{T}}P\Delta A\|=\|A^{\mathrm{T}}D^HD\Delta A\|\leqslant\|A^{\mathrm{T}}D^H\|\times\|D\Delta A\|=\|A^{\mathrm{T}}PA\|^{1/2}\|\Delta A^{\mathrm{T}}P\Delta A\|^{1/2}\leqslant \sqrt{\lambda_{\max}(P)(\lambda_{\max}(P)-\lambda_{\min}(Q))}\|\Delta A\| \tag{A1}$$

Similarly, we can get

$$\|\Delta A^{\mathrm{T}}PA\|\leqslant\sqrt{\|P\|(\|P\|-\lambda_{\min}(Q))}\|\Delta A\| \tag{A2}$$

From these, we have

$$(A+\Delta A)^{\mathrm{T}}P(A+\Delta A)-P=A^{\mathrm{T}}PA-P+A^{\mathrm{T}}P\Delta A+\Delta A^{\mathrm{T}}PA+\Delta A^{\mathrm{T}}P\Delta A=$$

$$-Q+A^{\mathrm{T}}P\Delta A+\Delta A^{\mathrm{T}}PA+\Delta A^{\mathrm{T}}P\Delta A\leqslant-\lambda_{\min}(Q)+2\sqrt{\lambda_{\max}(P)(\lambda_{\max}(P)-\lambda_{\min}(Q)}\|\Delta A\|+\|\Delta A\|^2\lambda_{\max}(P) \tag{A3}$$

Using the formula of solving the one-place quadratic equation $ax^2+bx+c=0$, we can obtain

$$-\lambda_{\min}(Q)+2\sqrt{\|P\|(\|P\|-\lambda_{\min}(Q))}\|\Delta A\|+\|\Delta A\|^2\|P\|<0 \tag{A4}$$

when Inequality (10) is true, where $a=\lambda_{\max}(P)$, $c=-\lambda_{\min}(Q)$, $b=2\sqrt{\lambda_{\max}(P)(\lambda_{\max}(P)-\lambda_{\min}(Q))}$. From Lemma 2 and Inequality (A4), we can get $|\lambda(A+\Delta A)|<1$. □

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DMC 鲁棒稳定性分析

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摘 要 以脉冲响应模型(FIR)描述系统,采用脉冲响应系数误差平方和的形式定义了模型的不确定性,分析得到了基于脉冲响应模型的动态矩阵控制(DMC)算法的闭环系统,在此基础上,推导得出了DMC闭环系统的鲁棒稳定性条件.

关键词 预测控制,鲁棒稳定性, 动态矩阵控制

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