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Exponential Stabilization of Euler-Bernoulli Beam with Dissipative Boundary Feedback[1)]

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Abstract This paper studies the stabilization problem of Euler-Bernoulli beam with general dissipative boundary feedback controls. First, by virtue of semigroup theory, the considered system is converted into an evolution equation in abstract space and the uniqueness of the solution to the evolution equation is proved. Then the eigenvalues of the closed loop system is studied and the necessary and sufficient condition for the closed loop system to be asymptotically stable is derived. Finally, the condition for the closed loop system to be exponentially stable is worked out by estimating the corresponding eigenfunctions.

Key words Euler-Bernoulli beam, boundary feedback, C_0 semigroups, exponential stabilization.

1 Introduction

The purpose of this paper is to study the boundary stabilization problem of Euler-Bernoulli beam. To avoid the tedious formula and calculations, we consider the following boundary control system of the Euler-Bernoulli beam with the unit length and the normalized parameters[1]:

$$\begin{cases} \ddot{y}(x,t) + y''''(x,t) = 0, & 0 \leqslant x \leqslant 1,\ t > 0 \\ y(0,t) = y'(0,t) = 0, & t \geqslant 0 \\ y(x,0) = y_0(x), \quad y'(x,0) = y_1(x), & x \in (0,1) \\ y'''(1,t) = u_1(t), \quad -y''(1,t) = u_2(t), & t \geqslant 0 \end{cases} \tag{1}$$

Here and henceforth, the prime and the dot always denote the derivatives with respect to space and time variables, respectively.

We apply the following linear boundary feedback

$$\begin{cases} u_1(t) = \alpha\dot{y}(1,t) + \beta\dot{y}'(1,t) \\ u_2(t) = \tau\dot{y}(1,t) + \gamma\dot{y}'(1,t) \end{cases} \tag{2}$$

as the controls to the right end of the beam. Here $\alpha,\beta,\tau,\gamma \in \mathbb{R}$ are some feedback gain constants. Later we will use the notations:

$$F \triangleq \begin{bmatrix} \alpha & \beta \\ \tau & \gamma \end{bmatrix}, \quad B \triangleq \begin{bmatrix} \alpha & \dfrac{\beta+\tau}{2} \\ \dfrac{\beta+\tau}{2} & \gamma \end{bmatrix}$$

Up to now, a lot of interesting results on the boundary feedback stabilization of Euler-Bernoulli beam model have been obtained by many investigators[1~8]. It is well known that this type of controls, under the conditions of $\alpha,\gamma \geqslant 0$, $\alpha^2+\gamma^2 \neq 0$ and $\beta=\tau=0$, can stabilize the Euler-Bernoulli beam exponentially[1,2]. Recently, in [8], under the conditions of $\alpha,\gamma > 0$ and $\beta=\tau=0$, it is proven that the infinitesimal generator of the C_0 semigroup corresponding to the closed loop system has the Riesz basis property, and that the

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corresponding C_0 semigroup satisfies the spectral determined growth assumption and the energy of the closed loop system (1)~(2) is exponentially stable. Now, it is natural to ask what the asymptotic behavior will be like in the general cases of α,β,τ and γ. The significance of the investigation of this type of boundary feedback control problem can be found in [1,2]. In the following, we study the general case as described above, and particularly we consider the degenerate case of matrix B. It will be shown that, in the case of $\beta\neq\tau$ and $B\geqslant 0$, the closed loop system (1)~(2) is exponentially stable if and only if $\mathrm{rank}(B)\geqslant 1$, and that in the case of $\beta=\tau$, some sufficient and necessary conditions are also given for the closed loop system (1)~(2) to be exponentially stable.

This paper is organized as follows. In Section 2, the wellposedness of the corresponding closed loop system is considered. In Section 3, we outline some results about the asymptotic decay of the closed loop system. In Section 4, we prove that under the condition of $B\geqslant 0$ (which is equivalent to the dissipation of $\mathcal{A}$) and $\beta\neq\tau$, the closed loop system (1)~(2) is exponentially stable if and only if $\mathrm{rank}(B)\geqslant 1$. Moreover, in the case of $\beta=\tau$, we also derive some other necessary and sufficient conditions for the closed loop system of (1) and (2) to be exponentially stable.

2 Wellposedness of the closed loop system

To begin with, we incorporate the closed loop system (1)~(2) into a certain function space. To this end, we define a product Hilbert space $\mathcal{H}$ as

$$\mathcal{H}=V_0^2\times L^2(0,1)$$

where $V_0^2=\{\varphi\in H^2(0,1)\mid\varphi(0)=\varphi'(0)=0\}$, and $H^k(0,1)$ is the usual Sobolev space of order k on the interval $(0,1)$. The inner product in $\mathcal{H}$ is defined as follows.

$$(Y_1,Y_2)_{\mathcal{H}}=\int_0^1(y_1''\bar{y}_2''+z_1\bar{z}_2)\mathrm{d}x$$

where $Y_k=[y_k,z_k]^\tau\in\mathcal{H}$ for $k=1,2$, and the superscript τ denotes the transposition of a matrix.

We then define a linear operator $\mathcal{A}$ in $\mathcal{H}$ by

$$\mathcal{A}\begin{bmatrix}y\\z\end{bmatrix}=\begin{bmatrix}z\\-y''''\end{bmatrix},\quad\begin{bmatrix}y\\z\end{bmatrix}\in\mathcal{D}(\mathcal{A})$$

$$\mathcal{D}(\mathcal{A})=\{[y,z]^\tau\in\mathcal{H}\mid y\in V_0^2\cap H^4(0,1),\ z\in V_0^2,\ [y'''(1),-y''(1)]^\tau=F[z(1),z'(1)]^\tau\}$$

Then the closed loop system (1)~(2) can be written as the following linear evolution equation in $\mathcal{H}$.

$$\frac{\mathrm{d}Y(t)}{\mathrm{d}t}=\mathcal{A}Y(t)\tag{3}$$

where $Y(t)=[y(\cdot,t),\dot{y}(\cdot,t)]^\tau$.

Lemma 1. If $B\geqslant 0$, then $\mathcal{A}$ generates a C_0 contraction semigroup $T(t)$ in $\mathcal{H}$.

The proof of this Lemma is similar to that of the related Lemma in [8], hence it is omitted here.

Thus, according to the C_0 semigroup theory, we get

Theorem 1. If $B\geqslant 0$, then for any $Y_0\in\mathcal{H}$, (3) has a unique weak solution $Y(t)=T(t)Y_0$, where $T(t)$ is the C_0 contraction semigroup generated by $\mathcal{A}$. Moreover, if $Y_0\in\mathcal{D}(\mathcal{A})$, $Y(t)=T(t)Y_0$ becomes a unique strong solution to (3).

3 Asymptotic stability

The energy corresponding to the solution of the closed loop system (3) is defined as

$$E(t)=\frac{1}{2}\int_0^1(|y''(x,t)|^2+|\dot{y}(x,t)|^2)\mathrm{d}x$$

where $Y(t)=[y(\cdot,t),\dot{y}(\cdot,t)]^\tau$ is the solution to (3). If $Y_0\in\mathcal{D}(\mathcal{A})$, then

$$\dot{E}(t) = -[\dot{y}(1,t),\dot{y}'(1,t)]B[\bar{\dot{y}}(1,t),\bar{\dot{y}}'(1,t)]^\tau \tag{4}$$

From the definition of $\mathcal{A}$, it is not difficult to prove the following lemmas.

Lemma 2. Assume that $B\geqslant 0$. Then $\mathcal{A}^{-1}$ exists and is a compact operator on $\mathcal{H}$. Therefore, $\sigma(\mathcal{A})$, the spectrum set of $\mathcal{A}$, consists of only isolated eigenvalues with finite multiplicity.

Lemma 3. Assume that $B\geqslant 0$. Then any nonzero eigenvalue λ of $\mathcal{A}$ satisfies

$$z(\cosh z+\cos z+2)+(\beta+\tau)z(\cosh z-\cos z)+|F|z(\cosh z+\cos z-2)+\gamma z^2(\sinh z+\sin z)+2\alpha(\sinh z-\sin z)=0 \tag{5}$$

where $z=\sqrt{2\lambda}$.

The proof is omitted.

Theorem 2. Assume that $B\geqslant 0$. Then the energy of the closed loop system (3) decays asymptotically to zero if and only if the following transcendental equations on $x\in\mathbb{R}$ have no nonzero solution x.

$$\begin{cases}(1+|F|)\cosh x\cos x+1-|F|=0\\ \gamma(\cosh x\sin x+\sinh x\cos x)x^2+(\beta+\tau)x\sinh x\sin x+\\ \qquad \alpha(\cosh x\sin x-\sinh x\cos x)=0\end{cases} \tag{6}$$

Corollary 1. Assume that $B\geqslant 0$ and $\beta=\tau=0$. Then the energy of the closed loop system (3) decays asymptotically to zero if and only if $\alpha^2-\gamma^2\neq 0$.

The results obtained in [1,2] imply the corollary. Here we can also use Theorem 2 to prove the desired assertion.

To limit this paper within certain pages, we omit the details of the proof here.

Corollary 2. Assume that $B\geqslant 0$ and $\beta+\tau=0$. Then the energy of the closed loop system (3) decays asymptotically to zero if and only if $\mathrm{rank}(B)\geqslant 1$.

4 Exponential stability

The following lemma is vital to proving the main result in this paper.

Lemma 4. Assume that $B\geqslant 0$, $\mathrm{rank}(B)>0$ and $\sigma(\mathcal{A})\cap i\mathbb{R}=\phi$. Then there exists a positive constant M such that

$$\| R(i\eta,\mathcal{A})\| \leqslant M,\quad \forall \eta\in\mathbb{R}$$

where $R(\lambda,\mathcal{A})=(\lambda-\mathcal{A})^{-1}$ is the resolvent of $\mathcal{A}$.

Proof. By using the continuity of the resolvent, we need only to prove Lemma 4. for sufficiently large positive η. For $\lambda=i\omega^2$ and $[f(x),g(x)]^\tau\in\mathcal{H}$, let $[y,z]^\tau\in D(\mathcal{A})$ such that $(\lambda-\mathcal{A})[y,z]^\tau=[f,g]^\tau$, *i.e.*,

$$\begin{cases}\lambda y-z=f\\ \lambda z+y''''=g\end{cases} \tag{7}$$

Eliminating z from (7), we have

$$\begin{cases}y''''-\omega^4 y=i\omega^2 f+g\\ y(0)=y'(0)=0\\ y'''(1)=\alpha(i\omega^2 y(1)-f(1))+\beta(i\omega^2 y'(1)-f'(1))\\ -y''(1)=\tau(i\omega^2 y(1)-f(1))+\gamma(i\omega^2 y'(1)-f'(1))\end{cases} \tag{8}$$

The general solution to the first equation of (8) is

$$Y(x)=Y_1(x)+Y_2(x) \tag{9}$$

where

$$Y(x)=[y,y',y'',y''']^\tau,\quad Y_1=\Phi(x)A$$

$$Y_k(x)=[y_k,y_k',y_k'',y_k''']^\tau,\quad k=1,2$$

$$Y_2(x)=\int_0^x\Phi(x-s)[0,0,0,1]^\tau(i\omega^2 f(s)+g(s))\mathrm{d}s$$

$$A = [a_1, a_2, a_3, a_4]^\tau, \quad a_k \in \mathbb{C}, \quad k = 1,2,3,4$$

with $\Phi(x)$, the state transition matrix to the first equation of (8), as

$$\Phi(x) \triangleq$$

$$\begin{bmatrix} \frac{1}{2}(\cosh\omega x+\cos\omega x) & \frac{1}{2\omega}(\sinh\omega x+\sin\omega x) & \frac{1}{2\omega^2}(\cosh\omega x-\cos\omega x) & \frac{1}{2\omega^3}(\sinh\omega x-\sin\omega x) \\ \frac{\omega}{2}(\sinh\omega x-\sin\omega x) & \frac{1}{2}(\cosh\omega x+\cos\omega x) & \frac{1}{2\omega}(\sinh\omega x+\sin\omega x) & \frac{1}{2\omega^2}(\cosh\omega x-\cos\omega x) \\ \frac{\omega^2}{2}(\cosh\omega x-\cos\omega x) & \frac{\omega}{2}(\sinh\omega x-\sin\omega x) & \frac{1}{2}(\cosh\omega x+\cos\omega x) & \frac{1}{2\omega}(\sinh\omega x+\sin\omega x) \\ \frac{\omega^3}{2}(\sinh\omega x+\sin\omega x) & \frac{\omega^2}{2}(\cosh\omega x-\cos\omega x) & \frac{\omega}{2}(\sinh\omega x-\sin\omega x) & \frac{1}{2}(\cosh\omega x+\cos\omega x) \end{bmatrix}$$

From the boundary condition of $y(x)$ at $x=0$, we get immediately that $a_1 = a_2 = 0$. Similarly, from the boundary condition of $y(x)$ at $x=1$, it follows that

$$\begin{cases} y_1'''(1) - i\omega^2 \alpha y_1(1) - i\omega^2 \beta y_1'(1) = h_1(1) \\ y_1''(1) + i\omega^2 \tau y_1(1) + i\omega^2 \gamma y_1'(1) = h_2(1) \end{cases} \tag{10}$$

where

$$h_1(1) \triangleq - y_2'''(1) + i\omega^2 \alpha y_2(1) + i\omega^2 \beta y_2'(1) - \alpha f(1) - \beta f'(1)$$

$$h_2(1) \triangleq - y_2''(1) - i\omega^2 \tau y_2(1) - i\omega^2 \gamma y_2'(1) + \tau f(1) + \gamma f'(1)$$

By the definitions of $y_2(x)$, $h_1(1)$, $h_2(1)$, $\mathcal{D}(\mathcal{A})$ and $\mathcal{H}$, and integrating by parts, it follows that

$$y_2''(1) = \frac{1}{4\omega} e^{\omega} \int_0^1 e^{-\omega s} [if''(s) + g(s)] ds + O(\omega^{-1}(\| f'' \| + \| g \|)) \tag{11}$$

$$h_1(1) = -\frac{1}{4\omega} e^{\omega} (\omega - i\alpha - i\omega\beta) \int_0^1 e^{-\omega s} [if''(s) + g(s)] ds + O(\| f'' \| + \| g \|) \tag{12}$$

$$h_2(1) = -\frac{1}{4\omega} e^{\omega} (1 + i\tau + i\omega\gamma) \int_0^1 e^{-\omega s} [if''(s) + g(s)] ds + O(\| f'' \| + \| g \|) \tag{13}$$

From (10)

$$\begin{cases} a_3 = G^{-1}(g_{11} h_1(1) + g_{12} h_2(1)) \\ a_4 = G^{-1}(g_{21} h_1(1) + g_{22} h_2(1)) \end{cases} \tag{14}$$

where

$$G = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}$$

$$g_{11} = \frac{1}{2\omega}(\sinh\omega + \sin\omega) + \frac{i\tau}{2\omega}(\sinh\omega - \sin\omega) + \frac{i\gamma}{2}(\cosh\omega - \cos\omega)$$

$$g_{12} = -\frac{1}{2}(\cosh\omega + \cos\omega) + \frac{i\alpha}{2\omega}(\sinh\omega - \sin\omega) + \frac{i\beta}{2}(\cosh\omega - \cos\omega)$$

$$g_{21} = -\frac{1}{2}(\cosh\omega + \cos\omega) - \frac{i\tau}{2}(\cosh\omega - \cos\omega) - \frac{i\gamma\omega}{2}(\sinh\omega + \sin\omega)$$

$$g_{22} = \frac{\omega}{2}(\sinh\omega - \sin\omega) - \frac{i\alpha}{2}(\cosh\omega - \cos\omega) - \frac{i\omega\beta}{2}(\sinh\omega + \sin\omega)$$

When ω is sufficiently large, it is not difficult to turn out that

$$G = -\frac{1}{2}(1 + \cosh\omega\cos\omega) + \frac{|F|}{2}(1 - \cosh\omega\cos\omega) -$$

$$\frac{i\omega\gamma}{2}(\cosh\omega\sin\omega + \sinh\omega\cos\omega) -$$

$$\frac{i\alpha}{2\omega}(\cosh\omega\sin\omega - \sinh\omega\cos\omega) - \frac{i(\beta + \tau)}{2}\sinh\omega\sin\omega =$$

$$\Big(-\Big(\frac{|F|}{4} + \frac{1}{4}\Big)\cos\omega + \frac{i\alpha}{4\omega}(\cos\omega - \sin\omega) -$$

$$\frac{i\omega\gamma}{4}(\cos\omega + \sin\omega) - \frac{i(\beta+\tau)}{4}\sin\omega\Big)e^{\omega} + O(\omega) \tag{15}$$

By calculations, we obtain

$$|Ge^{-\omega}|^2 = \Big(\frac{\alpha}{4\omega}(\cos\omega - \sin\omega) - \frac{\omega\gamma}{4}(\cos\omega + \sin\omega) - \frac{(\beta+\tau)}{4}\sin\omega\Big)^2 + \Big(-\frac{|F|}{4}\cos\omega - \frac{1}{4}\cos\omega\Big)^2 + \begin{cases} O(\omega^2 e^{-\omega}), & \gamma \neq 0, \\ O(\omega e^{-\omega}), & \gamma = 0. \end{cases} \tag{16}$$

We now prove that there exists a positive constant c such that

$$\begin{cases} |e^{-\omega}G| \geqslant c, & \gamma > 0, \\ |\omega e^{-\omega}G| \geqslant c, & \gamma = 0, \alpha > 0 \end{cases} \tag{17}$$

for ω large enough. In fact, for $\gamma \neq 0$, if $|\cos\omega + \sin\omega| > \frac{1}{400(\|F\|+1)}$, then it is trivial to prove assertion (17) for large ω; if $|\cos\omega + \sin\omega| \leqslant \frac{1}{400(\|F\|+1)}$, then for ω large enough,

$$|\omega G e^{-\omega}| \geqslant \frac{1}{8}(1+|F|)|\cos\omega| \geqslant \frac{1}{8}|\cos\omega| \geqslant \frac{1}{40} \triangleq c$$

which means that (17) is valid. For the case $\gamma = 0$, the proof is similar.

Therefore, when ω is large, by a lengthy calculation and analysis, it follows from (11)～(14) and (17) that

$$y_1''(x) = a_3\frac{\cosh\omega x + \cos\omega x}{2} + a_4\frac{\sinh\omega x + \sin\omega x}{2} = \frac{g_{11}h_1 + g_{12}h_2}{2G}(\cosh\omega x + \cos\omega x) + \frac{g_{21}h_1 + g_{22}h_2}{2\omega G}(\sinh\omega x + \sin\omega x) = -y_2''(x) + O(\|f''\| + \|g\|)$$

and

$$a_3,\ a_4/\omega = O(\|f''\| + |g\|)$$

Thus for ω large enough, we get

$$\|y''(x)\| = O(\|f''\| + \|g\|) \tag{18}$$

By (9) and the embedding theorem, we obtain

$$\omega^2 y(x) = y''(x) - a_3\cos\omega x - \frac{a_4\sin\omega x}{\omega} + O(\|f''\| + \|g\|)$$

Then from (18) we get

$$\|\lambda y\| = \|\omega^2 y\| = O(\|f''\| + \|g\|)$$

for large ω. Hence, by using the embedding theorem again, it follows from (7) that, for sufficiently large ω,

$$\|z\| = O(\|f''\| + \|g\|) \tag{19}$$

Thus from (18) and (19), the proof is finished. □

The main result of this paper is as follows.

Theorem 3. Assume that $B \geqslant 0$. Then

i) in the case of $\beta \neq \tau$, the energy of the closed loop system (3) decays exponentially to zero as time $t \to \infty$ if and only if $\mathrm{rank}(B) > 0$;

ii) in the case of $\beta = \tau$ and $B > 0$, the energy of the closed loop system (3) decays exponentially to zero as time $t \to \infty$;

iii) in the case of $\beta = \tau = 0$, the energy of the closed loop system (3) decays exponentially to zero as time $t \to \infty$ if and only if $\mathrm{rank}(B) \geqslant 1$;

iv) in the case of $\beta = \tau \neq 0$ and $\mathrm{rank}(B) = 1$, the energy of the closed loop system (3) decays exponentially to zero as time $t \to \infty$ if and only if

$$\frac{\gamma}{\beta} \notin \Delta =: \left\{ \frac{-\sinh x \sin x}{(\cosh x \sin x + \sinh x \cos x)x} \middle| \cosh x \cos x = -1 \right\} \tag{20}$$

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具有一般耗散边界反馈的 Euler-Bernoulli 梁的指数镇定问题

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摘　要　讨论具有一般线性耗散边界反馈的 Euler-Bernoulli 梁的指数镇定问题. 首先将所讨论的系统化为抽象空间中的发展方程，并利用 C_0 半群理论给出闭环系统解的存在唯一性. 其次，对相应的闭环系统特征方程进行详尽的讨论计算，得到了系统本征值的分布特性，从而利用 Lassel 不变原理得到了闭环系统渐近稳定的充分必要条件. 最后通过对闭环系统的本征值及其相应的本征函数进行估计，导出了相应的闭环系统指数稳定的充分必要条件.

关键词　Euler-Bernoulli 梁，耦合线性边界反馈，C_0 半群，指数稳定

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