Robust H_2 Control with Regional Stability Constraints[1)]

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Abstract The problem of static state feedback robust H_2 control with regional stability constraints for the closed-loop system is considered. Both continuous and discrete-time systems with polytopic uncertainty are investigated. A new LMI-based sufficient condition for the existence of parameter-dependent Lyapunov functions is proposed. Static state feedback controllers required are not only guaranteed to satisfy all closed-loop poles to stay inside a specified region for all admissible parameter uncertainties, but also provide an upper bound for the H_2 cost function, which is minimized using LMI convex optimization approach.

Key words Robust control, polytopic uncertainty, stability, optimization, LMI

1 Introduction

Performance analysis of systems is obviously a very important problem in control theory. In particular, H_2 control is appealing since there is a well established connection between the performance index being optimized and performance requirements encountered in practical situations[1,2]. For example, if the input is zero-mean stationary white noise of unit covariance then, at steady state, the variance of the output is given by the square of the H_2 norm.

One of the practical concerns of control design is its time-domain performance. Many of these time-domain (transient as well as steady-state) performance specifications are determined or influenced by the closed-loop system zeros and poles[3,4]. While the standard H_2 design is primarily concerned with stability and closed-loop performance specifications, it says little about the transient performance. However, it is well known that the locations of poles (such as stability degree) not only serve as an indicator of response speed but also provide indirect tolerance against structured uncertainties. Therefore, it is often the case that a satisfying feedback design must impose constraints on the location of the closed-loop poles.

In this paper, we consider the problem of robust H_2 controller that satisfies additional constraints on the pole location of closed-loop with polytopic type uncertainties. Although some methods have been proposed before, the main drawback associated with these methods is that a single Lypunov matrix is used to guarantee the desired closed-loop multiobjective specification or must work for all matrices in the uncertain domain and ensure the poles of the closed-loop are clustered in some regions. This condition is often too conservative if used with time-invariant systems. The same reasoning can be drawn for the robust H_2 performance. In this paper, we explore a new LMI characterization of minimum H_2 with the pole constraint regions. This idea was first introduced in [5] for linear continuous-time uncertain systems, and for linear discrete-time uncertain systems in [6~8]. The results obtained in this paper go beyond the ones attainable by the quadratic approach for time-invariant parameters uncertainty. It is expressed as LMIs and exhibits a kind of separation property between the Lyapunov matrices and the uncertain dynamic matrices. In

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terms of the new LMI characterization, sufficient conditions are obtained for the existence of an upper bound on the H_2 norm of closed-loop system with pole clustering in specified regions of the complex plane. By minimizing this upper bound we obtain explicit state feedback gain expressions for a controller that places the closed-loop poles within the specified pole-constraint region. Finally, two numerical examples are presented to illustrate the theory.

2 Continuous-time linear systems

2.1 Problem statement

We consider the following class of uncertain LTI systems

$$\begin{aligned} \dot{x} &= Ax(t) + B_1 w(t) + B_2 u(t) \\ y &= Cx(t) + Du(t) \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^r$ is the control input, $w(t) \in R^m$ is disturbance input, A, B_2, C, D are uncertain matrices which are assumed to belong to a polytopic convex domain:

$$\begin{bmatrix} A & B_2 & B_1 \\ C & D & 0 \end{bmatrix} := \left\{ \begin{bmatrix} A(\xi) & B_2(\xi) & B_1(\xi) \\ C(\xi) & D(\xi) & 0 \end{bmatrix} = \sum_{i=1}^{p} \xi_i \begin{bmatrix} A_i & B_{2i} & B_{1i} \\ C_i & D_i & 0 \end{bmatrix}, \ \xi \in \Omega \right\} \tag{2}$$

where Ω is the unit simplex

$$\Omega := \left\{ (\xi_1, \xi_2, \cdots, \xi_p) : \sum_{i=1}^{p} \xi_i = 1, \ \xi_i \geq 0 \right\} \tag{3}$$

Let $u = Kx(t)$ and define the closed-loop matrices $\bar{A} = A + B_2 K$ and $\bar{C} = C + DK$. Supposing that a state feedback gain K is calculated in such a way that $\bar{A}$ is asymptotically stable, the closed-loop transfer function from w to z is given by

$$T_{zw}(s) = \bar{C}[sI - \bar{A}]^{-1} B_1 \tag{4}$$

The H_2 norm for a stable transfer matrix $T_{zw}(s)$ can be defined as

$$\| T_{zw} \|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} Tr\{T_{zw}(e^{-j\omega})\} d\omega \tag{5}$$

Lemma 1[9]. If there exists matrix K such that $\bar{A}$ is asymptotically stable, then the H_2-norm of $T_{zw}(s)$ is given by

$$\| T_{zw} \|_2^2 = Tr(\bar{C} L_c \bar{C}^T) = Tr(B_1^T L_o B_1) \tag{6}$$

where L_c and L_o are controllability and observability Gramians respectively, that is

$$\bar{A} L_c + L_c \bar{A}^T + B_1 B_1^T = 0 \tag{7}$$

$$\bar{A}^T L_o + L_o \bar{A} + \bar{C}^T \bar{C} = 0 \tag{8}$$

Definition 1. The uncertain system (1) is said to be robustly d-stable if all eigenvalues of uncertain system matrix A lie in the disk $D(-\alpha, r)$ with center $-\alpha + j0$ and radius r with respect to real polytopic uncertainty, where $\alpha > 0$ and $0 < r < \alpha$.

Similarly, the uncertain system (1) is said to be robustly d-stabilizable if there exists a linear state feedback control law $u(t) = Kx(t), K \in R^{n \times n}$, such that the resulting closed-loop system is robustly d-stable.

The problem to be addressed in this section is to determine the state feedback $u(t) = Kx(t)$ such that

1) the closed-loop system is robustly d-stable

2) an upper bound of the worst case performance J with respect to the system uncertainty is minimized, where

$$J = \max_{\xi \in \Omega} \{ \| T_{zw}(s) \|_2 \} \tag{9}$$

To motivate the technique used in this paper, the following Lemmas are introduced.

Lemma 2[3]. Let $A \in R^{n \times n}$ be a given matrix. The eigenvalues of A belong to $D(-\alpha, r)$ if and only if there exists a symmetric matrix $P \in R^{n \times n}$ such that

$$\begin{bmatrix} -P^{-1} & r^{-1}(A+\alpha I) \\ r^{-1}(A+\alpha I)^{T} & -P \end{bmatrix} < 0 \tag{10}$$

Lemma 3 (Reciprocal Projection Lemma)[10]. Let U be any given positive definite matrix. The following statements are equivalent.

i)
$$\Psi + S + S^{T} < 0 \tag{11}$$

ii) the LMI problem

$$\begin{bmatrix} \Psi + U - (W+W^{T}) & S^{T} + W^{T}(12) \\ S+W & -U \end{bmatrix} < 0 \tag{12}$$

is feasible with respect to W.

2.2 Main results

Theorem 1. Let $\gamma > 0$ be constant. The following matrix inequality conditions, with positive definite matrix variables Y, X, and general matrix variable V, are equivalent.

$$YA + A^{T}Y + A^{T}YA + Y + \gamma^{-1}C^{T}C < 0 \tag{13}$$

$$\begin{bmatrix} -(V+V^{T}) & V^{T}A^{T}+X & V^{T} & V^{T}A^{T} & V^{T} & V^{T}C^{T} \\ AV+X & -X & 0 & 0 & 0 & 0 \\ V & 0 & -X & 0 & 0 & 0 \\ AV & 0 & 0 & -X & 0 & 0 \\ V & 0 & 0 & 0 & -X & 0 \\ CV & 0 & 0 & 0 & 0 & -\gamma I \end{bmatrix} < 0 \tag{14}$$

Proof.

(13)⇒(14). From (13), using Lemma 3 with $\Psi := A^{T}YA + Y + \lambda^{-1}C^{T}C$, $S = AY$ and any given positive definite matrix U yields

$$\begin{bmatrix} A^{T}YA + Y + \gamma^{-1}C^{T}C + U - (W+W^{T}) & A^{T}Y + W^{T} \\ YA + W & -U \end{bmatrix} < 0 \tag{15}$$

By Shur complement operation with respect to the term Y and the congruence transformation

$$\begin{bmatrix} V & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & X \end{bmatrix}, \text{ with } X := Y^{-1}, V := W^{-1}$$

the inequality above in turn becomes

$$\begin{bmatrix} V^{T}A^{T}X^{-1}AV + \gamma^{-1}V^{T}C^{T}CV + V^{T}UV - (V+V^{T}) & V^{T}A^{T}+X & V^{T} \\ AV+X & -XUX & 0 \\ V & 0 & -X \end{bmatrix} < 0$$

By Shur complement operation with respect to the terms $V^{T}A^{T}X^{-1}AV$, $\gamma^{-1}V^{T}C^{T}CV$ and $V^{T}UV$, the above inequality becomes

$$\begin{bmatrix} -(V+V^{T}) & V^{T}A^{T}+X & V^{T} & V^{T}A^{T} & V^{T} & V^{T}C^{T} \\ AV+X & -XUX & 0 & 0 & 0 & 0 \\ V & 0 & -X & 0 & 0 & 0 \\ AV & 0 & 0 & -X & 0 & 0 \\ V & 0 & 0 & 0 & -U^{-1} & 0 \\ CV & 0 & 0 & 0 & 0 & -\gamma^{-1}I \end{bmatrix} < 0$$

The above inequality implies (14) with $U := X^{-1}$.

(14)⇒(13). If (14) holds, using Lemma 3 and Shur complement formula we can get that (13) holds with $U := X^{-1}$ and $Y = X^{-1}$. The proof is completed. □

Theorem 2. Let $\beta = \alpha(r^{2} - \alpha^{2})^{-1}$. If the following (vertex) conditions hold simultaneously for all $i = 1, 2, \cdots, p$:

$$Tr(Z) < 1 \tag{16}$$

$$\begin{bmatrix} X_i & B_{1i} \\ B_{1i}^{\mathrm{T}} & Z \end{bmatrix} > 0 \tag{17}$$

$$\begin{bmatrix} -(V+V^{\mathrm{T}}) & V^{\mathrm{T}}A_i^{\mathrm{T}}+X_i & V^{\mathrm{T}} & V^{\mathrm{T}}A_i^{\mathrm{T}} & V^{\mathrm{T}} & V^{\mathrm{T}}C_i^{\mathrm{T}} \\ A_iV+X_i & -X_i & 0 & 0 & 0 & 0 \\ V & 0 & -X_i & 0 & 0 & 0 \\ A_iV & 0 & 0 & -\alpha X_i & 0 & 0 \\ V & 0 & 0 & 0 & -\beta X_i & 0 \\ C_iV & 0 & 0 & 0 & 0 & -\gamma I \end{bmatrix} < 0 \tag{18}$$

then system (1) is robustly d-stable, and $\| C(sI-A)^{-1}B_1 \|_2 < \gamma$.

Proof. Suppose that (17) and (18) hold for all $i=1,2,\cdots,p$. We have

$$\sum_i^p \xi_i \begin{bmatrix} X_i & B_{1i} \\ B_{1i}^{\mathrm{T}} & Z \end{bmatrix} > 0$$

$$\sum_i^p \xi_i \begin{bmatrix} -(V+V^{\mathrm{T}}) & V^{\mathrm{T}}A_i^{\mathrm{T}}+X_i & V^{\mathrm{T}} & V^{\mathrm{T}}A_i^{\mathrm{T}} & V^{\mathrm{T}} & V^{\mathrm{T}}C_i^{\mathrm{T}} \\ A_iV+X_i & -X_i & 0 & 0 & 0 & 0 \\ V & 0 & -X_i & 0 & 0 & 0 \\ A_iV & 0 & 0 & -\alpha X_i & 0 & 0 \\ V & 0 & 0 & 0 & -\beta X_i & 0 \\ C_iV & 0 & 0 & 0 & 0 & -\gamma I \end{bmatrix} < 0.$$

which impies

$$\begin{bmatrix} X & B_1 \\ B_1^{\mathrm{T}} & Z \end{bmatrix} > 0 \tag{19}$$

$$\begin{bmatrix} -(V+V^{\mathrm{T}}) & V^{\mathrm{T}}A^{\mathrm{T}}+X & V^{\mathrm{T}} & V^{\mathrm{T}}A^{\mathrm{T}} & V^{\mathrm{T}} & V^{\mathrm{T}}C^{\mathrm{T}} \\ AV+X & -X & 0 & 0 & 0 & 0 \\ V & 0 & -X & 0 & 0 & 0 \\ AV & 0 & 0 & -\alpha X & 0 & 0 \\ V & 0 & 0 & 0 & -\beta X & 0 \\ CV & 0 & 0 & 0 & 0 & -\gamma I \end{bmatrix} < 0 \tag{20}$$

From Theorem 1, (20) is equivalent to the following inequality with positive definite matrix variable Y

$$YA + A^{\mathrm{T}}Y + \alpha^{-1}A^{\mathrm{T}}YA + \beta^{-1}Y + \gamma^{-1}C^{\mathrm{T}}C < 0 \tag{21}$$

By (21), we have

$$YA + A^{\mathrm{T}}Y + \alpha^{-1}A^{\mathrm{T}}YA + \beta^{-1}Y < 0 \tag{22}$$

$$YA + A^{\mathrm{T}}Y + \gamma^{-1}C^{\mathrm{T}}C < 0 \tag{23}$$

In view of Lemma 2 and (22), we obtain $\delta(A) \in D(\alpha, r)$.

Using Lemma 1, we obtain

$$\| C(sI-A)^{-1}B_1 \|_2^2 = Tr(B_1^{\mathrm{T}}PB_1) \tag{24}$$

where P is observability Gramians, that is,

$$A^{\mathrm{T}}P + PA + C^{\mathrm{T}}C = 0 \tag{25}$$

Comparing (25) with (23), we have

$$P < \gamma Y$$

Noting (16) and (19), we have

$$\| C(sI-A)^{-1}B_1 \|_2^2 = Tr(B_1^{\mathrm{T}}PB_1) < \gamma Tr(B_1^{\mathrm{T}}YB_1) < \gamma Tr(Z) = \gamma$$

Hence, system (1) is robustly d-stable, and $\| C(sI-A)^{-1}B_1 \|_2 < \gamma$. The proof is completed. □

Now, with the result of Theorem 2, it is possible to determine a matrix gain K to minimize the upper bound of H_2 norm of the closed-loop system (4). The next theorem gives the optimal characteristics to the gain K.

Theorem 3. If the convex optimization problem

$$J = \min Tr(Z) \tag{26}$$

$$\begin{bmatrix} X_i & B_{1i} \\ B_{1i}^{\mathrm{T}} & Z \end{bmatrix} > 0 \tag{27}$$

$$\begin{bmatrix} -(V+V^{\mathrm{T}}) & \Psi_{12}^{\mathrm{T}} & V^{\mathrm{T}} & \Psi_{14}^{\mathrm{T}} & V^{\mathrm{T}} & \Psi_{16}^{\mathrm{T}} \\ \Psi_{12} & -X_i & 0 & 0 & 0 & 0 \\ V & 0 & -X_i & 0 & 0 & 0 \\ \Psi_{14} & 0 & 0 & -\alpha X_i & 0 & 0 \\ V & 0 & 0 & 0 & -\beta X_i & 0 \\ \Psi_{16} & 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0 \tag{28}$$

where $i=1,2,\cdots,p$, $\Psi_{12}=A_iV+B_{2i}R+X_i$, $\Psi_{14}=A_iV+B_{2i}R$, $\Psi_{16}=C_iV+D_iR$, has a solution in the matrix variables $Z>0$, $X_i>0, i=1,2,\cdots,p$, R and V for all $i=1,2,\cdots,p$, then system (1) is robustly d-stabilizable with state feedback gain $K=RV^{-1}$, and $\| \overline{C}(sI-\overline{A})^{-1}\overline{B}_1 \|_2 < J$.

Proof. Since $-V-V^{\mathrm{T}}<0$, V is nonsingular. Taking $K=RV^{-1}$, $\overline{A}_i=A_i+B_{2i}K$, $\overline{C}_i=C_i+D_iK$, the above condition can be rewritten as

$$Tr(Z) < 1$$

$$\begin{bmatrix} X_i & B_{1i}(4) \\ B_{1i}^{\mathrm{T}} & Z \end{bmatrix} > 0$$

$$\begin{bmatrix} -(V+V^{\mathrm{T}}) & V^{\mathrm{T}}\overline{A}_i^{\mathrm{T}}+X_i & V^{\mathrm{T}} & V^{\mathrm{T}}\overline{A}_i^{\mathrm{T}} & V^{\mathrm{T}} & V^{\mathrm{T}}\overline{C}_i^{\mathrm{T}}(5) \\ \overline{A}_iV+X_i & -X_i & 0 & 0 & 0 & 0(6) \\ V & 0 & -X_i & 0 & 0 & 0(7) \\ \overline{A}_iV & 0 & 0 & -\alpha X_i & 0 & 0(8) \\ V & 0 & 0 & 0 & -\beta X_i & 0(9) \\ \overline{C}_iV & 0 & 0 & 0 & 0 & -\gamma I \end{bmatrix} < 0.$$

which is a transposed version of Theorem 2 by linear combination. Thus, (26)~(28) imply that system (1) is robustly d-stabilizable with state feedback gain $K=RV^{-1}$, and $\| \overline{C}(sI-\overline{A})^{-1}\overline{B}_1 \|_2 < J$. □

3 Discrete-time case

3.1 Problem statement

Let us consider a discrete-time linear system whose dynamic behavior is given by the following difference equations

$$\begin{aligned} \boldsymbol{x}_{k+1} &= A\boldsymbol{x}_k + B_1\boldsymbol{w}_k + B_2\boldsymbol{u}_k \\ \boldsymbol{z}_k &= C_k\boldsymbol{x}_k + D\boldsymbol{u}_k \end{aligned} \tag{29}$$

where $\boldsymbol{x}_k \in R^n$ is the state variable, $\boldsymbol{u}_k \in R^r$ is the control variable, $\boldsymbol{w}_k \in R^m$ is the external disturbance input and A, B_2, C, D are uncertain matrices which are assumed to belong to a polytopic convex domain:

$$\begin{bmatrix} A & B_1 & B_2 \\ C & D & 0 \end{bmatrix} := \left\{ \begin{bmatrix} A(\boldsymbol{\xi}) & B_2(\boldsymbol{\xi}) & B_1(\boldsymbol{\xi}) \\ C(\boldsymbol{\xi}) & D(\boldsymbol{\xi}) & 0 \end{bmatrix} = \sum_{i=1}^{p} \xi_i \begin{bmatrix} A_i & B_{2i} & B_{1i} \\ C_i & D_i & 0 \end{bmatrix}, \boldsymbol{\xi} \in \Omega \right\} \tag{30}$$

where Ω is the unit simplex

$$\Omega := \{(\xi_1, \xi_2, \cdots, \xi_p) : \sum_{i=1}^{p} \xi_i = 1, \xi_i \geqslant 0\} \tag{31}$$

Let $u_k = Kx_k$ and define the closed-loop matrices $\overline{A}=A+B_2K$ and $\overline{C}=C+DK$. Supposing that a state feedback gain K is calculated in such a way that $\overline{A}$ is asymptotically stable, the closed-loop transfer function from w to z is given by

$$T_{zw}(s) = \overline{C}\,[zI - \overline{A}]^{-1} B_1 \tag{32}$$

The H_2 norm for a stable transfer matrix $T_{zw}(s)$ can be defined as

$$\| T_{zw} \|_2^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} Tr\{T_{zw}(e^{-j\omega})^T T_{zw}(e^{j\omega})\}d\omega \tag{33}$$

Lemma 4[9]. If there exists matrix K such that $\overline{A}$ is asymptotically stable, then the H_2-norm of $T_{zw}(s)$ is given by

$$\| T_{zw} \|_2^2 = Tr(\overline{C}L_c\overline{C}^T) = Tr(B_1^T L_o B_1) \tag{34}$$

where L_c and L_o are controllability and observability Gramians respectively, that is,

$$\overline{A}L_c\overline{A}^T - L_c + B_1B_1^T = 0 \tag{35}$$

$$\overline{A}^T L_o \overline{A} - L_o + \overline{C}^T\overline{C} = 0 \tag{36}$$

Definition 2. The uncertain system (10) is said to be robustly d-stable if all eigenvalues of uncertain system matrix A lie in the disk $D(\alpha, r)$ with center $\alpha + j0$ and radius r with respect to real polytopic uncertainty, where $|\alpha| + r < 1$.

Similarly, the uncertain system (1) is said to be robustly d-stabilizable if there exists a linear state feedback control law $\boldsymbol{u}_k(t) = K\boldsymbol{x}_k, K \in R^{n\times n}$, such that the resulting closed-loop system is robustly d-stable.

The problem to be addressed in this section is to determine the state feedback $\boldsymbol{u}(t) = K\boldsymbol{x}(t)$ such that

1) the closed-loop system is robustly d-stable

2) an upper bound of the worst case performance J with respect to the system uncertainty is minimized, where

$$J = \max_{\xi\in\Omega}\{ \| T_{zw}(z) \|_2 \} \tag{37}$$

The following Lemmas are needed to derive our results.

Lemma 5[8]. For any given $\mu > 0$ the inequality $\| C(zI - A)^{-1}B_1 \|_2^2 < \mu$ holds for all A_i, B_{1i}, B_{2i}, C_i and $D_i (i = 1, 2, \cdots, p)$ if there exist symmetric matrices P_i, W_i, $i = 1, 2, \cdots, p$ and a matrix G of compatible dimensions satisfying the following LMIs for all $i = 1, 2, \cdots, p$

$$Tr(W_i) < \mu \tag{38}$$

$$\begin{bmatrix} W_i & C_iG \\ G^TC_i^T & G + G^T - P_i \end{bmatrix} > 0 \tag{39}$$

$$\begin{bmatrix} P_i & A_iG & B_{1i} \\ G^TA_i & G + G^T - P_i & 0 \\ B_{1i}^T & 0 & I \end{bmatrix} > 0 \tag{40}$$

Lemma 6. Let $A \in R^{n\times n}$ be a given matrix. The eigenvalues of A belong to $D(-\alpha, r)$ if and only if there exist a symmetric matrix $P \in R^{n\times n}$ and matrix $G \in R^{n\times n}$ such that

$$\begin{bmatrix} r^2P & AG + \alpha G \\ G^TA^T + \alpha G^T & G + G^T - P \end{bmatrix} > 0 \tag{41}$$

Proof. Sufficiency: Since the matrix $[-I]$ has full rank, (41) implies that

$$[-I \quad A + \alpha I]\begin{bmatrix} r^2P & AG + \alpha G \\ G^TA^T + \alpha G^T & G + G^T - P \end{bmatrix}\begin{bmatrix} -I \\ A^T + \alpha I \end{bmatrix} > 0 \tag{42}$$

which implies

$$P - (A + \alpha I)P(A + \alpha)^T > 0 \tag{43}$$

Using Shur complement formula, we obtain

$$\begin{bmatrix} -P^{-1} & r^{-1}(A + \alpha I) \\ r^{-1}(A + \alpha I)^T & -P \end{bmatrix} < 0 \tag{44}$$

(44) is the dual of (10) in the transformation $A \to A^T$.

Necessity: Assuming (10) is satisfied, (44) holds. Using Shur complement, (44) is equivalent to

$$\begin{bmatrix} r^2P & AP + \alpha P \\ PA^T + \alpha P & P \end{bmatrix} > 0 \tag{45}$$

Choosing $G=G^{\mathrm{T}}=P$, we obtain (41). □

3.2 Main result

Theorem 4. Let $\beta=\alpha^2-r^2$. If there exist positive definite matrices P_i, Q_i, W_i, $i=1,2,\cdots,p$ and a matrix G of compatible dimensions satisfying (38),(48),(40) and the following LMIs for all $i=1,2,\cdots,p$

$$\begin{bmatrix} r^2 Q_i & A_i G+\alpha G \\ G^{\mathrm{T}} A_i^{\mathrm{T}}+\alpha G^{\mathrm{T}} & G+G^{\mathrm{T}}-Q_i \end{bmatrix}>0 \tag{46}$$

then system (29) is robustly d-stable, and $\| C(zI-A)^{-1}B_1 \|_2<\gamma$.

Now, with the result of Theorem 1, it is possible to determine a matrix gain K to minimize the upper bound of H_2 norm of the closed-loop system (32). The next theorem gives the optimal characteristics to the gain K.

Theorem 5. If the convex optimization problem

$$\min J = Tr(W) \tag{47}$$

$$\begin{bmatrix} W & C_i G+D_i V \\ G^{\mathrm{T}} C_i^{\mathrm{T}}+V^{\mathrm{T}} D_i^{\mathrm{T}} & G+G^{\mathrm{T}}-P_i \end{bmatrix}>0 \tag{48}$$

$$\begin{bmatrix} P_i & A_i G+B_{2i} V & B_{1i} \\ G^{\mathrm{T}} A_i^{\mathrm{T}}+V^{\mathrm{T}} B_{2i}^{\mathrm{T}} & G+G^{\mathrm{T}}-P_i & 0 \\ B_{1i}^{\mathrm{T}} & 0 & I \end{bmatrix}>0 \tag{49}$$

$$\begin{bmatrix} r^2 Q_i & A_i G+B_{2i} V-\alpha G \\ G^{\mathrm{T}} A_i^{\mathrm{T}}+V^{\mathrm{T}} B_{2i}^{\mathrm{T}}-\alpha G^{\mathrm{T}} & G+G^{\mathrm{T}}-Q_i \end{bmatrix}>0 \tag{50}$$

has a solution in the matrix variables $P_i>0$, $Q_i>0$, $i=1,2,\cdots,p$, matrices G and V for all $i=1,2,\cdots,p$, then system (29) is robustly d-stabilizable with state feedback gain $K=VG^{-1}$, and $\| \overline{C}(zI-\overline{A})^{-1}\overline{B}_1 \|_2<J$.

Proof. Since P_i and Q_i are positive definite, $G+G^{\mathrm{T}}>0$, G is nonsingular. Take $K=VG^{-1}$, $\overline{A}_i=A_i+B_{2i}K$, $\overline{C}_i=C_i+D_i K$. The conditions can be written as

$$\min J = Tr(W)$$

$$\begin{bmatrix} W & \overline{C}_i G \\ G^{\mathrm{T}} \overline{C}_i^{\mathrm{T}} & G+G^{\mathrm{T}}-P_i \end{bmatrix}>0$$

$$\begin{bmatrix} P_i & \overline{A}_i G \\ G^{\mathrm{T}} \overline{A}_i^{\mathrm{T}} & G+G^{\mathrm{T}}-P_i \\ B_{1i}^{\mathrm{T}} & 0 \end{bmatrix}>0$$

$$\begin{bmatrix} r^2 Q_i & \overline{A}_i G-\alpha G \\ G^{\mathrm{T}} \overline{A}_i^{\mathrm{T}}-\alpha G^{\mathrm{T}} & G+G^{\mathrm{T}}-Q_i \end{bmatrix}>0$$

which is a transposed version of Theorem 4. Thus, (47)~(50) imply that system (29) is robustly d-stabilizable with state feedback gain $K=VG^{-1}$, and $\| \overline{C}(zI-\overline{A})^{-1}\overline{B}_1 \|_2<J$. □

4 Examples

Example 1 (continues-time case). Consider the linear continuous-time parameter uncertain system (1) with

$$A_1=\begin{bmatrix} -1 & 2 \\ -1 & -2 \end{bmatrix},\ B_{11}=\begin{bmatrix} 1 \\ 1 \end{bmatrix},\ B_{21}=\begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C_1=[1 \quad 3],\ D_1=1$$

$$A_2=\begin{bmatrix} -1.5 & 1.2 \\ -0.7 & -1.5 \end{bmatrix},\ B_{12}=\begin{bmatrix} 0.6 \\ 1 \end{bmatrix},\ B_{22}=\begin{bmatrix} 0 \\ 1.5 \end{bmatrix},\ C_2=[0.8 \quad 2],\ D_2=0.7$$

$$A_3=\begin{bmatrix} -1.2 & 2.3 \\ -1.2 & -2.3 \end{bmatrix},\ B_{13}=\begin{bmatrix} 1.2 \\ 1 \end{bmatrix},\ B_{23}=\begin{bmatrix} 0 \\ 3.3 \end{bmatrix},\ C_3=[1.2 \quad 4],\ D_3=2$$

Applying the method proposed in [1] where a single Lyapunov matrix is used, the convex optimization problem is infeasible, while in the light of Theorem 3 it has been obtained using the software package LMI Lab that system (1) is robustly d-stabilizable with center $-2+j0$ and radius 1, the optimal value of H_2 of the closed-loop system is 28.3314 with state feedback $u=[0.2370\quad -0.2824]x$, the Lyapunov matrices at three vertices of the ploytope are

$$X_1=\begin{bmatrix}0.5487 & -0.1757\\ -0.1757 & 0.1307\end{bmatrix},\quad X_2=\begin{bmatrix}2.4848 & -0.0622(21)\\ -0.0622 & 2.3822\end{bmatrix}\times 10^4,$$
$$X_3=\begin{bmatrix}0.7962 & -0.3137(22)\\ -0.3137 & 0.2054\end{bmatrix}$$

respectively. Observe that X_2 is very different from X_1 and X_3 in the sense of norm of a matrix. Obviously, it will give more conservative result if a single Lyapunov matrix X that enforces multiple constraints is used.

Example 2(discrete-time case). Consider the linear discrete-time parameter uncertain system (29) with

$$A_1=\begin{bmatrix}-0.50 & 0.80\\ 0.35 & 0.25\end{bmatrix},\ B_{11}=\begin{bmatrix}0.10\\ 0.10\end{bmatrix},\ B_{21}=\begin{bmatrix}1\\ -1\end{bmatrix},\ C_1=[1\quad 0.3],\ D_1=0.1$$
$$A_2=\begin{bmatrix}-0.65 & 0.85\\ 0.40 & 0.35\end{bmatrix},\ B_{12}=\begin{bmatrix}0.26\\ 0.32\end{bmatrix},\ B_{22}=\begin{bmatrix}1\\ -0.8\end{bmatrix},\ C_2=[0.8\quad 0.2],\ D_2=0.2$$
$$A_3=\begin{bmatrix}-0.55 & 0.13\\ 0.28 & 0.10\end{bmatrix},\ B_{13}=\begin{bmatrix}0.20\\ 0.19\end{bmatrix},\ B_{23}=\begin{bmatrix}1\\ -0.86\end{bmatrix},\ C_3=[1.2\quad 0.4],\ D_3=0.2$$

Applying the method proposed in [1] where a single Lyapunov matrix is used, using the software package LMI Lab, the convex optimization value of J is 0.4040. Applying the formula for regional constraints derived in [4], the convex optimization problem is infeasible. While in the light of Theorem 1 it has been obtained that system (29) is robustly d-stabilizable with center $0.5+j0$ and radius 0.5, the optimal value of H_2 of the closed-loop system is 0.1735 with state feedback $u_k=[0.7926\quad -0.4314]x_k$. Hence, for these two examples, the robust performance with regional stability constraints of this paper gives less conservative results than those obtained by the methods of [1].

5 Conclusion

We have addressed the problem of robust H_2 control for linear systems with polytopic type uncertainties and d-stability constraints. Both linear continous- and discrete-time uncertain systems are considered. The results are based on a new extended LMI characterization that does not involve the product of the Lyapunov and the system dynamic matrices. State-feedback controller parameterizations that are able to linearize the extended H_2 controller synthesis problem have been applied to the multiobjective problem. The approach presented here provides a way to reduce the conservativeness of the existing conditions by decoupling the control parameterization from the Lyapunov matrix.

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基于区域极点约束的鲁棒 H_2 控制

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摘　要　考虑了在区域极点约束下状态反馈的鲁棒 H_2 控制问题.分别对含多面体不确定性的连续和离散系统进行了讨论.基于 LMI,给出了存在参数相关的 Lyapunov 矩阵的充分条件.利用 LMI 凸优化方法的解,所得静态反馈控制器,不仅保证闭环系统的极点在一给定区域内,而且还使性能指标 H_2 的一上界达到最小.

关键词　多面体不确定性,鲁棒控制,区域稳定性,矩阵不等式,优化

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