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Unified Fast Suboptimal Fixed-Interval White Noise Wiener Smoothing Algorithm[1)]

DENG Zi-Li XU Yan SHI Ying SUN Shu-Li
(Institute of Applied Mathematics and Department of Automation, Heilongjiang University, Harbin 150080)
(E-mail: dzl@hlju.edu.cn)

Abstract For the discrete stochastic control systems with correlated noises, using the Kalman filtering method, based on the CARMA innovation model, a unified optimal fixed-interval white noise recursive Wiener smoother is derived. It contains high degree polynomial matrices with coefficient matrices exponentially decaying to zero. Further, by the truncation method, the corresponding fast suboptimal fixed-interval white noise Wiener smoothing algorithm is presented, which obviously reduces the computational burden. The error formula of the smoother and the formula of selecting the truncated index are given. A simulation example for Bernoulli-Gaussian white noise shows the effectiveness of the proposed results.

Key words Reflection seismology, white noise estimators, fixed-interval white noise Wiener smoother, fast suboptimal smoothing algorithm, Kalman filtering method

1 Introduction

In oil seismic exploration[1], an explosive is detonated below the earth's surface, so that the seismic waves are generated and reflected in different geological layers. Oil exploration is performed via the reflection coefficient sequence, which can be described by Bernoulli-Gaussian white noise. Estimating white noise reflection coefficient sequence has important application for discovering the oil field and determining the geometry shape of oil field. Since the reflection coefficient sequence is the input to the receiver, estimating the input white noise is called deconvolution or input estimation. The white noise estimation theory for systems with uncorrelated noises has been presented by Kalman filtering method in [2]. The fixed-interval optimal input white noise smoother for systems with correlated noises has been presented by Kalman filtering method in [3]. Further, the unified fixed-interval optimal input and measurement of white noise smoothers for systems with correlated noises have been presented in [4]. But the disadvantage of above optimal white smoothers is that to compute the Kalman filter gain, variance matrix of prediction error, and inverse of innovation variance matrix is required each time. This yields a large computational burden. In this paper, based on the Kalman filtering, a unified fast suboptimal fixed-interval white noise Wiener smoothing algorithm is presented by the truncation method, which obviously reduces the computational burden and has a satisfactory accuracy.

2 Problem formulation

Consider the linear discrete-time stochastic control system with correlated noises

$$\boldsymbol{x}(t+1) = \Phi\boldsymbol{x}(t) + B\boldsymbol{u}(t) + \Gamma\boldsymbol{w}(t) \tag{1}$$

$$\boldsymbol{y}(t) = H\boldsymbol{x}(t) + \boldsymbol{v}(t) \tag{2}$$

where the state $\boldsymbol{x}(t)\in R^n$, the measurement $\boldsymbol{y}(t)\in R^m$, $\boldsymbol{w}(t)\in R^r$ is the input white noise, $\boldsymbol{v}(t)\in R^m$ is the measurement white noise, $\boldsymbol{u}(t)\in R^p$ is known control input, Φ, B, Γ and H are constant matrices.

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Assumption 1. $w(t)$ and $v(t)$ are correlated white noises with zero mean, and

$$\mathrm{E}\left\{\begin{bmatrix} w(t) \\ v(t) \end{bmatrix}\begin{bmatrix} w^{\mathrm{T}}(j) & v^{\mathrm{T}}(j) \end{bmatrix}\right\} = \begin{bmatrix} Q_w & S \\ S^{\mathrm{T}} & Q_v \end{bmatrix}\delta_{tj} \tag{3}$$

where E denotes the mathematical expectation, T denotes the transpose, $\delta_{tt}=1, \delta_{tj}=0(t\neq j)$.

Assumption 2[5]. $(\bar{\Phi}, H)$ is a completely detectable pair, $(\bar{\Phi}, \Gamma\bar{Q}_w)$ is a completely stabilizable pair, where we define that

$$\bar{\Phi} = \Phi - JH, \quad J = \Gamma S Q_v^{-1}, \quad \bar{Q}_w \bar{Q}_w^{\mathrm{T}} = Q_w - S Q_v^{-1} S^{\mathrm{T}} \tag{4}$$

The problem is based on measurements $(y(N), y(N-1), \cdots)$ and controls $(u(N-1), u(N-2), \cdots)$, to find the steady-state optimal fixed-interval white noise Wiener smoothers $\hat{w}(t|N)$ and $\hat{v}(t|N)$ of $w(t)$ and $v(t)$, which are denoted by a unified symbol $\hat{\theta}(t|N)$, $\theta = w, v$, and to find the corresponding fast suboptimal smoothers $\hat{\theta}_s(t|N)$, $t=1,\cdots,N$, where the positive integer N is called the length of fixed interval.

3 Steady-state Kalman predictor and CARMA innovation model

Under Assumptions 1 and 2, from [6] there exists the steady-state Kalman predictor

$$\hat{x}(t+1 \mid t) = \Psi_P \hat{x}(t \mid t-1) + Bu(t) + K_P y(t) \tag{5}$$

$$y(t) = H\hat{x}(t \mid t-1) + \varepsilon(t) \tag{6}$$

$$\Psi_P = \bar{\Phi} - \bar{K}_P H, \quad \bar{K}_P = \bar{\Phi} K, \quad K_P = \bar{K}_p + J \tag{7}$$

where the innovation process $\varepsilon(t) \in R^m$ is white noise with zero mean and variance matrix Q_ε, and K is the steady-state Kalman filter gain, and

$$Q_\varepsilon = H\Sigma H^{\mathrm{T}} + Q_v, \quad K = \Sigma H^{\mathrm{T}} Q_\varepsilon^{-1} \tag{8}$$

and the prediction error variance matrix Σ satisfies the steady-state Riccati equation

$$\Sigma = \bar{\Phi}[\Sigma - \Sigma H^{\mathrm{T}}(H\Sigma H^{\mathrm{T}} + Q_v)^{-1} H\Sigma]\bar{\Phi}^{\mathrm{T}} + \Gamma(Q_w - SQ_v^{-1} S^{\mathrm{T}})\Gamma^{\mathrm{T}} \tag{9}$$

which can simply be solved by iteration [6]. It can be proved [5] that Ψ_p is a stable matrix, *i.e.*, all eigenvalues of Ψ_p lie inside the unite circle. Introducing the backward shift operator q^{-1}, (5) can be written as

$$\hat{x}(t \mid t-1) = (I_n - q^{-1}\Psi_p)^{-1}[Bu(t-1) + K_p y(t-1)] \tag{10}$$

where I_n is the $n\times n$ unity matrix. Substituting (10) into (6) yields the controlled autoregressive moving average (CARMA) innovation model

$$A(q^{-1})y(t) = B(q^{-1})u(t) + \psi(q^{-1})\varepsilon(t) \tag{11}$$

where we define

$$\psi(q^{-1}) = \det(I_n - \Psi_P q^{-1}), \quad B(q^{-1}) = H\mathrm{adj}(I_n - q^{-1}\Psi_p)Bq^{-1} \tag{12}$$

$$A(q^{-1}) = \psi(q^{-1})I_m - H\mathrm{adj}(I_n - q^{-1}\Psi_P)K_P q^{-1} \tag{13}$$

4 Optimal and suboptimal fixed-interval white noise Wiener smothers

In [6], it is proved that the unified steady-state optimal fixed-lag white noise innovation smoothers are given as

$$\hat{\theta}(t \mid t+N) = L_N^\theta(q^{-1})\varepsilon(t+N), \quad L_N^\theta(q^{-1}) = \sum_{i=0}^{N} M_\theta(i)q^{i-N}, \quad \theta = w, v \tag{14}$$

$$M_\theta(i) = D_\theta(1)(\Psi_p^{\mathrm{T}})^{i-1} H^{\mathrm{T}} Q_\varepsilon^{-1}, \quad M_w(0) = SQ_\varepsilon^{-1}, \quad M_v(0) = Q_v Q_\varepsilon^{-1} \tag{15}$$

$$D_w(1) = -SK^{\mathrm{T}}\bar{\Phi}^{\mathrm{T}} + Q_w^{\mathrm{T}}\Gamma^{\mathrm{T}} - SJ^{\mathrm{T}}, \quad D_v(1) = -Q_v K^{\mathrm{T}}\bar{\Phi}^{\mathrm{T}} \tag{16}$$

Equation (14) yields the unified steady-state optimal fixed-interval white noise innovation smoothers as

$$\hat{\theta}(t \mid N) = L_{N-t}^\theta(q^{-1})\varepsilon(N) \tag{17}$$

From (11) we have $\varepsilon(N) = \psi^{-1}(q^{-1})[A(q^{-1})y(N) - B(q^{-1})u(N)]$. Substituting it into (17) yields the following Theorems.

Theorem 1. For the system by (1) and (2) with Assumptions 1 and 2, the unifield asymptotically stable optimal fixed-interval white noise Wiener smoothers are given as

$$\psi(q^{-1})\hat{\boldsymbol{\theta}}(t \mid N) = B^{\theta}_{N,t}(q^{-1})\boldsymbol{u}(N) + K^{\theta}_{N,t}(q^{-1})\boldsymbol{y}(N) \tag{18}$$

where $\boldsymbol{\theta}=\boldsymbol{w},\boldsymbol{v}$, $\psi(q^{-1})$ only operates on time t of $\hat{\boldsymbol{\theta}}(t|N)$, and

$$B^{\theta}_{N,t}(q^{-1}) = -L^{\theta}_{N-t}(q^{-1})B(q^{-1}),\quad K^{\theta}_{N,t}(q^{-1}) = L^{\theta}_{N-t}(q^{-1})A(q^{-1}) \tag{19}$$

Noting that Ψ_p is a stable matrix[5], from (12) we have that $\psi(q^{-1})$ is a state polynomial. Hence, Wiener smoothers (18) are asymptotically stable. Also note that the feature of Wiener smoothers (18) is that the degrees of polynomial matrices $L^{\theta}_{N-t}(q^{-1})$, $B^{\theta}_{N,t}(q^{-1})$, and $K^{\theta}_{N,t}(q^{-1})$ are time-varying, and $L^{\theta}_{N-t}(q^{-1})$ are polynomial matrices in q^{-1} with the degree of $N-t$. Hence, when N is larger and t is smaller, $B^{\theta}_{N,t}(q^{-1})$ and $K^{\theta}_{N,t}(q^{-1})$ are polynomial matrices with higher degrees. This may create a larger computational burden. Noting (15), because Ψ_p is a stable matrix [5], we have that $(\Psi_p^{\mathrm{T}})^{i-1}\rightarrow 0$, further $M_{\theta}(i)\rightarrow 0$, as $i\rightarrow\infty$. Therefore, if i is sufficiently large, for example, $i>m_0$, then we have that $M_{\theta}(i)\approx 0$. In order to reduce the computational burden, we truncate the terms of $L^{\theta}_{N-t}(q^{-1})$, that is, terms with coefficient matrices $M_{\theta}(i)\approx 0$ are omitted and m_0+1 terms are remained. m_0 is called the truncated index. From Theorem 1 we obtain the following fast suboptimal fixed-interval white noise Wiener smoothers.

Theorem 2. Under the conditions of Theorem 1, the unified asymptotically stable fast suboptimal fixed-interval white noise Wiener smoothers $\hat{\boldsymbol{\theta}}_s(t|N)$ are given as

$$\psi(q^{-1})\hat{\boldsymbol{\theta}}_s(t \mid N) = B^{\theta,m_0}_{N,t}(q^{-1})\boldsymbol{u}(N) + K^{\theta,m_0}_{N,t}(q^{-1})\boldsymbol{y}(N) \tag{20}$$

where $\boldsymbol{\theta}=\boldsymbol{w},\boldsymbol{v}$, and we define the truncated polynomial matrices $L^{\theta}_{N-t}(q^{-1})$ with truncated index m_0 as

$$L^{\theta,m_0}_{N-t}(q^{-1}) = \sum_{i=0}^{\min(N-t,m_0)} M_{\theta}(i)q^{i-N+t} \tag{21}$$

and the corresponding truncated polynomial matrices are defined as

$$B^{\theta,m_0}_{N,t}(q^{-1}) = -L^{\theta,m_0}_{N-t}(q^{-1})B(q^{-1}),\quad K^{\theta,m_0}_{N,t}(q^{-1}) = L^{\theta,m_0}_{N-t}(q^{-1})A(q^{-1}) \tag{22}$$

Since the number of terms of $L^{\theta,m_0}_{N-t}(q^{-1})$ is not larger than m_0+1, taking a smaller m_0 produced that $B^{\theta,m_0}_{N-t}(q^{-1})$ and $K^{\theta,m_0}_{N-t}(q^{-1})$ are polynomial matrices with lower degrees, so that the smoothers (20) obviously reduce the computational burden. Compared to the optimal smoothers (18) and those in [3,4], the suboptimal Wiener smoothers (20) constitute a fast smoothing algorithm.

5 Truncated error analysis and selection of truncated index

The requirement of selecting the truncated index m_0 is that m_0 is a sufficiently small positive integer, so that the suboptimal smoothers (20) become a fast smoothing algorithm, and m_0 must ensure the accuracy of suboptimal smoothers (20). The truncated index m_0 can be selected according to the locations of eigenvalues of Ψ_p. In the following theorem, we reveal that the coefficient matrices $M_{\theta}(i)$ of $L^{\theta}_{N-t}(q^{-1})$ exponentially decay to zero, and the decaying rate is related to the spectral radius of Ψ_p.

Theorem 3. For the system by (1) and (2) with Assumptions 1 and 2, the coefficient matrices $M_{\theta}(i)$ given in (15) exponentially decay to zero, as $i\rightarrow\infty$, *i.e.*,

$$M_{\theta}(i) = O(\lambda^{i-1}) \tag{23}$$

where $\lambda_1,\cdots,\lambda_n$ are the eigenvalues of Ψ_p, and $\lambda=\max(|\lambda_1|,\cdots,|\lambda_n|)$ is the spectral radius of Ψ_p.

Proof. Let $\boldsymbol{\alpha}_1,\cdots,\boldsymbol{\alpha}_n$ be linearly independent $n\times 1$ eigenvectors corresponding to eigenvalues $\lambda_1,\cdots,\lambda_n$ of Ψ_p^{T}, *i.e.*, $\Psi_p^{\mathrm{T}}\boldsymbol{\alpha}_i=\lambda_i\boldsymbol{\alpha}_i$, $i=1,2,\cdots,n$. Setting $P=[\boldsymbol{\alpha}_1,\cdots,\boldsymbol{\alpha}_n]$, we have the relation $\Psi_p^{\mathrm{T}}P=P\mathrm{diag}(\lambda_1,\cdots,\lambda_n)$, so that $\Psi_p^{\mathrm{T}}=P\mathrm{diag}(\lambda_1,\cdots,\lambda_n)P^{-1}$. Hence (15) becomes

$$M_{\theta}(i) = D_{\theta}(1)P\mathrm{diag}(\lambda_1^{i-1},\cdots,\lambda_n^{i-1})P^{-1}H^{\mathrm{T}}Q_{\varepsilon}^{-1} \tag{24}$$

which yields

$$\| M_\theta(i) \| \leqslant c\lambda^{i-1} \tag{25}$$

in the sense of the spectral norm of matrix. Since Ψ_p^{T} is a stable matrix, we have $0<\lambda<1$, i. e., (23) holds. □

Now, we analyse the accuracy of suboptimal smoothers (20). The smoothers (20) are equivalent to fast suboptimal fixed-interval white noise innovation smoothers

$$\hat{\boldsymbol{\theta}}_s(t \mid N) = L_{N-t}^{\theta, m_0}(q^{-1})\varepsilon(N), \quad \boldsymbol{\theta} = \boldsymbol{w}, \boldsymbol{v} \tag{26}$$

When $N-t>m_0$, subtracting (26) from (17) yields the suboptimal smoothing error

$$\boldsymbol{e}_\theta(t \mid N) = \hat{\boldsymbol{\theta}}(t \mid N) - \hat{\boldsymbol{\theta}}_s(t \mid N) = \sum_{i=m_0+1}^{N-t} M_\theta(i)\varepsilon(t+i) \tag{27}$$

Theorem 4. For the system by (1) and (2) with Assumptions 1 and 2, if $\| \varepsilon(t) \| < \beta$, then the suboptimal smoothing error formula is

$$\| \boldsymbol{e}_\theta(t \mid N) \| < \rho\lambda^{m_0} \tag{28}$$

with $\rho=c\beta/(1-\lambda)$. In order to ensure the required accuracy of the suboptimal smoothers

$$\| \boldsymbol{e}_\theta(t \mid N) \| < \delta \tag{29}$$

where δ is the given accuracy index, the formula of selecting the truncated index m_0 is

$$m_0 = [\ln(\delta/\rho)/\ln\lambda] + 1 \tag{30}$$

Proof. From (25) and (27) we have

$$\| \boldsymbol{e}_\theta(t \mid N) \| < \sum_{i=m_0+1}^{N-t} c\lambda^{i-1}\beta = c\beta\lambda^{m_0}\frac{1-\lambda^{N-t-m_0}}{1-\lambda} < \frac{c\beta\lambda^{m_0}}{1-\lambda} = \rho\lambda^{m_0} \tag{31}$$

with $\rho=c\beta/(1-\lambda)$. Hence (28) holds. From (28), a sufficient condistion to satisfy (29) is that the following inequality holds

$$\rho\lambda^{m_0} < \delta \tag{32}$$

Taking the logarithm of (32), and noting that $0<\lambda<1$, $\ln\lambda<0$, we obtain the inequality $m_0>\ln(\delta/\rho)/\ln\lambda$, which yields (30). □

6 Simulation example —— Bernoulli-Gaussian suboptimal white noise Wiener smoother

The estimation prolem of Bernoulli-Gaussian input white noise has the important application background in oil seismic exploration [1]. Consider the control system with correlated noises

$$\boldsymbol{x}(t+1) = \begin{bmatrix} 1 & 0 \\ 0.6 & -0.5 \end{bmatrix}\boldsymbol{x}(t) + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}u(t) + \begin{bmatrix} -1 \\ 1 \end{bmatrix}w(t) \tag{33}$$

$$y(t) = [1 \quad 1]\boldsymbol{x}(t) + v(t) \tag{34}$$

$$v(t) = 0.3w(t) + \xi(t), \quad u(t) = \sin(2\pi t/N) \tag{35}$$

where $w(t)$ is Bernoulli-Gaussian white noise, $w(t)=b(t)g(t)$, where $b(t)$ is a Bernoulli white noise taking values 1 and 0 with probabilities $P(b(t)=1)=0.3$, $P(b(t)=0)=0.7$, $g(t)$ is a Gaussian white noise with zero mean and variance $\sigma_g^2=1$, and is independent of $b(t)$, $\xi(t)$ is a Gaussian white noise with zero mean and variance $\sigma_\xi^2=0.05$, and is independent of $w(t)$. Taking $N=200$, $m_0=10$, applying Theorem 2, the simulation result of the fast suboptimal fixed-interval white noise Wiener smoother $\hat{w}_s(t \mid 200)$, $t=1, 2, \cdots, 200$, is shown in Fig. 1, where the points denote $\hat{w}_s(t \mid 200)$, and the lines denote $w(t)$. We see that

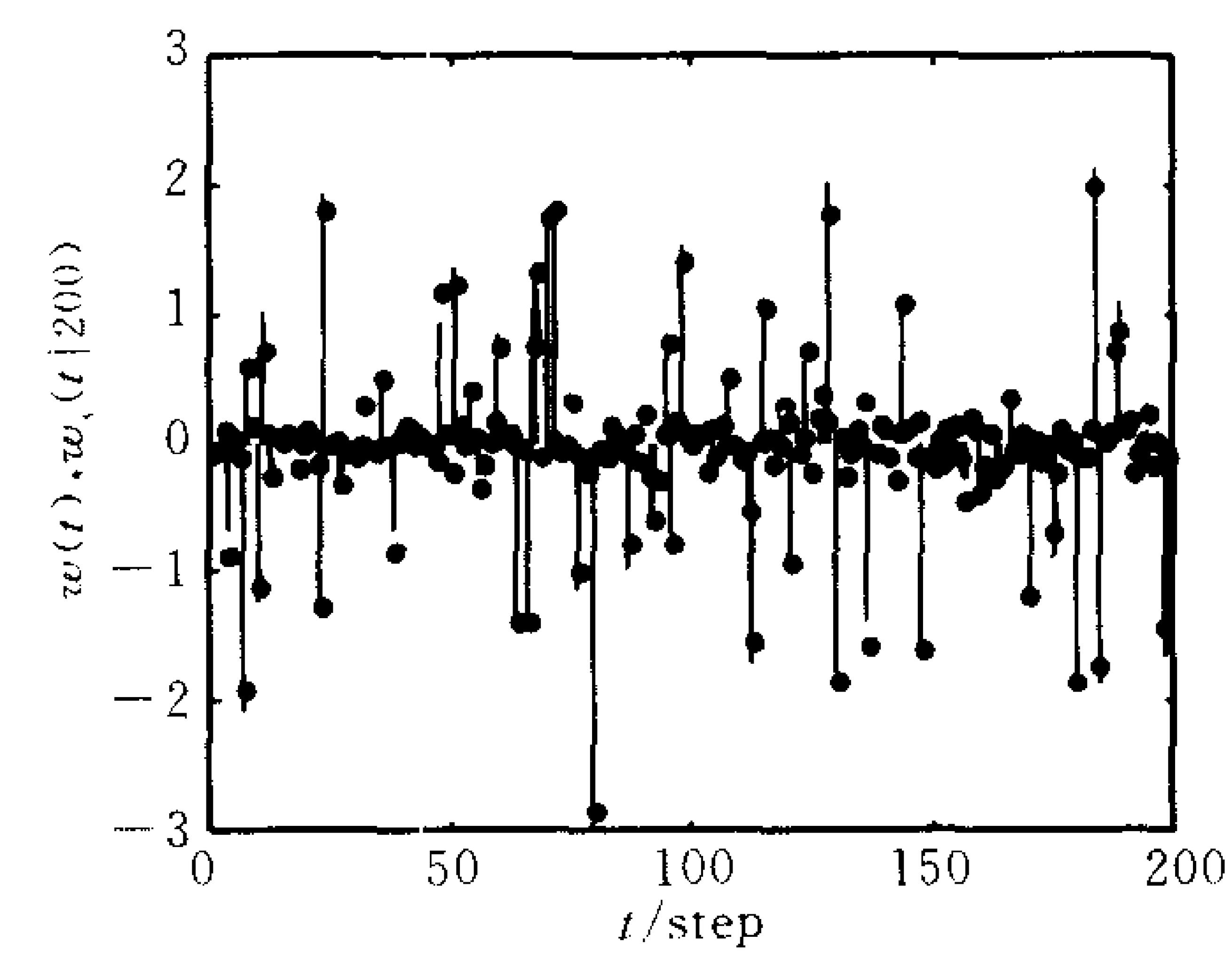

Fig. 1 Bernoulli-Gaussian white noise and fast suboptimal fixed-interval white noise Wiener smoother

$\hat{w}_s(t|200)$ has a higher accuracy. Noting that Ψ_p has the eigenvalues $\lambda_1=-0.4069$ and $\lambda_2=0.3516$, Ψ_p has the smaller spectral radius $\lambda=0.3516$. Hence taking $m_0=10$, $\hat{w}_s(t|200)$ has the satisfactory accuracy.

7 Conclusion

Based on the Kalman filtering and CARMA innovation model, for the systems with correlated noises, a fast unified fixed-interval white noise Wiener smoothing algorithm has been presented by a truncated method. The truncated error formula and the formula of selecting the truncated index are given. The proposed fast smoothing algorithm can be applied to signal processing in oil seismic exploration [1] and state estimation[6].

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Deng Zi-Li Professor in the Department of Automation at Heilongjiang University. His research interests include state estimation, optimal filtering, and signal processing.

Xu Yan Received her master degree from Heilongjiang University in 1998. She is currently an associate professor in the Department of Automation at Heilongjiang University. Her research interests include Kalman filtering and Wiener filtering.

Shi Ying Received her master degree from Heilongjiang University in 1996. She is a Ph. D. candidate at Harbin Institute of Technology. Her research interests include state estimation and deconvolution.

Sun Shu-Li Received his master degree from Heilongjiang University in 1999. He is a Ph. D. candidate at Harbin Institute of Technology. His research interests include optimal filtering and state estimation.

统一的快速次优固定区间白噪声 Wiener 平滑算法

邓自立　许　燕　石　莹　孙书利

（黑龙江大学应用数学研究所，黑龙江大学自动化系　哈尔滨　150080）

（E-mail：dzl@hlju.edu.cn）

摘　要　对带相关噪声的线性离散随机控制系统，应用 Kalman 滤波方法，基于 CARMA 新息模型导出了统一的最优固定区间白噪声递推 Wiener 平滑器，它带有系数阵指数衰减到零的高阶多项式矩阵. 进一步用截断方法提出了相应的快速次优固定区间白噪声 Wiener 平滑算法，它显著地减小了计算负担. 给出了平滑误差公式和选择截断指数的公式. 一个 Bernoulli-Gaussian 白噪声的仿真例子说明了所提出的结果的有效性.

关键词　反射地震学，白噪声估值器，固定区间白噪声 Wiener 平滑器，快速次优平滑算法，Kalman 滤波方法

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