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Forward-Backward Stochastic Differential Equation and the Linear Quadratic Stochastic Optimal Control[1)]

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Abstract The uniqueness and existence of the solution are discussed for a special forward-backward stochastic differential equation and the linear quadratic stochastic optimal control problem. The explicit accurate formulas to the unique optimal control and the linear feedback are respectively obtained by the Riccati equation.

Key words Backward stochastic differential equation, optimal control, Riccati equation

1 Introduction

Let $(\Omega,\mathscr{F},P)$ be a probability space, and $(B_t)_{t\geqslant 0}$ be a 1-dimensional Brownian motion in this space. $(\mathscr{F}_t)$ is the natural filtration generated with the Brownian motion. We consider the following stochastic optimal control problems:

$$\begin{cases} \mathrm{d}\boldsymbol{x}_t = [A\boldsymbol{x}_t + B\boldsymbol{v}_t]\mathrm{d}t + [C\boldsymbol{x}_t + D\boldsymbol{v}_t]\mathrm{d}B_t \\ \boldsymbol{x}_0 = \boldsymbol{a} \end{cases} \tag{1}$$

where A and C are both $n\times n$ matrixes, B and D are both $n\times k$ matrixes; $\boldsymbol{v}_t$, $t\in[0,T]$ is the admissible control process valued in $U\subset R^k$ and $\mathscr{F}_t$-adapted square-integrable. $\boldsymbol{a}\in R^n$.

The cost function is

$$J(\boldsymbol{v}(\cdot)) = \frac{1}{2}\mathrm{E}\left[\int_0^T(\langle R\boldsymbol{x}_t,\boldsymbol{x}_t\rangle + \langle N\boldsymbol{v}_t,\boldsymbol{v}_t\rangle)\mathrm{d}t + \langle Q\boldsymbol{x}_T,\boldsymbol{x}_T\rangle\right] \tag{2}$$

where R, Q are $n\times n$ nonnegative symmetric bounded matrixes, N is a $k\times k$ positive symmetric bounded matrix and the inverse is N^{-1}.

In control theory[1,2], the existence and uniqueness of the optimal control was proved. In [3], the existence and uniqueness of optimal control were also proved by applying the maximum principle. In [4], the optimal control is obtained as

$$\boldsymbol{u}(t) = -N^{-1}(B^{\mathrm{T}}\boldsymbol{y}_t + D^{\mathrm{T}}\boldsymbol{z}_t)$$

where $(\boldsymbol{y}_t,\boldsymbol{z}_t)$ is the solution of the following backward stochastic differential equation:

$$\begin{cases} -\mathrm{d}\boldsymbol{y}_t = (A^{\mathrm{T}}\boldsymbol{y}_t + C^{\mathrm{T}}\boldsymbol{z}_t + R\boldsymbol{x}_t)\mathrm{d}t - \boldsymbol{z}_t\mathrm{d}B_t \\ \boldsymbol{y}_T = Q(\boldsymbol{x}_T) \end{cases} \tag{3}$$

In this paper, we will consider more general problems. Let

$$\begin{cases} \mathrm{d}\boldsymbol{x}_t = [A(\omega)\boldsymbol{x}_t + B(\omega)\boldsymbol{v}_t - L^{\mathrm{T}}(\omega)\boldsymbol{y}_t]\mathrm{d}t + [C(\omega)\boldsymbol{x}_t + D(\omega)\boldsymbol{v}_t]\mathrm{d}B_t \\ -\mathrm{d}\boldsymbol{y}_t = [A^{\mathrm{T}}(\omega)\boldsymbol{y}_t + C^{\mathrm{T}}(\omega)\boldsymbol{z}_t + R(\omega)\boldsymbol{x}_t]\mathrm{d}t - \boldsymbol{z}_t\mathrm{d}B_t \\ \boldsymbol{x}_0 = \boldsymbol{a},\ \boldsymbol{y}_T = Q(\omega)\boldsymbol{x}_T \end{cases} \tag{4}$$

and the cost function be

$$J(\boldsymbol{v}(\cdot)) = \frac{1}{2}\mathrm{E}\left[\int_0^T(\langle R(\omega)\boldsymbol{x}_t,\boldsymbol{x}_t\rangle + \langle N(\omega)\boldsymbol{v}_t,\boldsymbol{v}_t\rangle + \langle L(\omega)\boldsymbol{y}_t,\boldsymbol{y}_t\rangle)\mathrm{d}t + \langle Q(\omega)\boldsymbol{x}_T,\boldsymbol{x}_T\rangle\right] \tag{5}$$

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where $R(\omega)$, $Q(\omega)$, $L(\omega)$ are all nonnegative bounded $n\times n$ matrixes, $N(\omega)$ is a positive bounded $k\times k$ matrix and the inverse is N^{-1}.

Note that (4) is a fully coupled forward backward stochastic differential equation (FBSDE for short). We will prove that there exists a unique optimal control and will give the explicit linear feedback form

$$\boldsymbol{u}(t) = -N^{-1}[B^{\mathrm{T}}(\omega)\boldsymbol{y}_t + D^{\mathrm{T}}(\omega)\boldsymbol{z}_t]$$

and the relative Riccati differential equation. In Section 2, we firstly give an existence and uniqueness theorem for a special forward backward differential equation. In Section 3, we prove the existence and uniqueness of the optimal control and give the explicit form of linear feedback while the related Riccati equation is obtained.

2 A special FBSDE

To our knowledge, there are two main methods to study the existence and uniqueness of solution of FBSDE. [5] gave the "four-step scheme" by using partial differential equation methods with the condition that the coefficients could not be random. [6] used probabilistic method under some monotone assumptions when $\boldsymbol{x},\boldsymbol{y}$ take same dimensions. [3] extended it to different dimensional FBSDE and weakened the monotone assumptions.

We consider the special FBSDE:

$$\begin{cases} \mathrm{d}\boldsymbol{x}_t = \boldsymbol{b}(t,\boldsymbol{x}_t,\boldsymbol{y}_t)\mathrm{d}t + \boldsymbol{\sigma}(t,\boldsymbol{x}_t,\boldsymbol{y}_t)\mathrm{d}B_t \\ -\mathrm{d}\boldsymbol{y}_t = \boldsymbol{f}(t,\boldsymbol{x}_t,\boldsymbol{y}_t,\boldsymbol{z}_t)\mathrm{d}t - \boldsymbol{z}_t\mathrm{d}B_t \\ \boldsymbol{x}_0 = \boldsymbol{a},\ \boldsymbol{y}_T = \Phi(\boldsymbol{x}_T) \end{cases} \tag{6}$$

where $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\in R^{n+n+n}$, for fixed $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$, $\boldsymbol{b},\boldsymbol{f},\boldsymbol{\sigma}$ are $(\mathscr{F}_t)$ progressively measurable. Let

$$\boldsymbol{u} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix},\ A(t,\boldsymbol{u}) = \begin{pmatrix} -\boldsymbol{f} \\ \boldsymbol{b} \\ \boldsymbol{\sigma} \end{pmatrix}$$

Let the assumptions be:

H1)

$$\begin{cases} \text{i) } A(t,\boldsymbol{u}) \text{ is uniformly } Lipschitz \text{ with respect to } \boldsymbol{u} \\ \text{ii) for each } \boldsymbol{u},\ A(t,\boldsymbol{u}) \in M^2(0,T) \\ \text{iii) } \Phi(\boldsymbol{x}) \text{ is uniformly } Lipschitz \text{ with respect to } \boldsymbol{x} \\ \text{iv) for each } \boldsymbol{x}, \Phi(\boldsymbol{x}) \in L^2(\Omega,\mathscr{F},\boldsymbol{P}) \end{cases}$$

and

H2)

$$\begin{cases} \langle A(t,\boldsymbol{u}) - A(t,\overline{\boldsymbol{u}}),\ \boldsymbol{u}-\overline{\boldsymbol{u}}\rangle \leqslant -\beta_1|\boldsymbol{x}-\overline{\boldsymbol{x}}|^2 - \beta_2|\boldsymbol{y}-\overline{\boldsymbol{y}}|^2 \\ \langle \Phi(\boldsymbol{x}) - \Phi(\overline{\boldsymbol{x}}),\ \boldsymbol{x}-\overline{\boldsymbol{x}}\rangle \geqslant \mu_1|\boldsymbol{x}-\overline{\boldsymbol{x}}|^2 \\ \forall \boldsymbol{u} = (\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}),\ \overline{\boldsymbol{u}} = (\overline{\boldsymbol{x}},\overline{\boldsymbol{y}},\overline{\boldsymbol{z}}) \end{cases}$$

where β_1, β_2 are nonnegative constants, $\beta_1+\beta_2>0$, $\mu_1+\beta_2>0$. Then we have the following theorems.

Theorem 1. Under H1) and H2) there exists a unique $\boldsymbol{u}_t=(\boldsymbol{x}_t,\boldsymbol{y}_t,\boldsymbol{z}_t)$ that solves the FBSDE (6).

For the uniqueness proof, refer to [3].

To prove the existence of the solutions, we consider the family of FBSDEs with $\alpha\in[0,1]$:

$$\begin{cases} \mathrm{d}X_t^\alpha = [(1-\alpha)\beta_2(-Y_t^\alpha) + \alpha b(t,X_t^\alpha,Y_t^\alpha) + \phi_t]\mathrm{d}t + [\alpha\sigma(t,X_t^\alpha,Y_t^\alpha) + \varphi_t]\mathrm{d}B_t \\ -\mathrm{d}Y_t^\alpha = [(1-\alpha)\beta_1 X_t^\alpha + \alpha f(t,U_t^\alpha) + \gamma_t]\mathrm{d}t - Z_t^\alpha\mathrm{d}B_t \\ \boldsymbol{x}_0^\alpha = \boldsymbol{a},\ Y_T^\alpha = \alpha\Phi(X_T^\alpha) + (1-\alpha) \end{cases} \tag{7}$$

where ϕ,φ,γ are the processes in $M^2(0,T)$. When $\alpha=1$, (7) becomes (6); when $\alpha=0$, the existence and uniqueness of solution for FBSDE can be obtained from Lemma 2.5 in [3]. The following lemma gives a prior estimate for the "existence interval" of (7) with respect to $\alpha\in[0,1]$.

Lemma 1. Under H1) and H2) there exists a positive constant $\delta_0\in[0,1]$ such that if (7) has a solution $(x^{\alpha_0}, y^{\alpha_0}, z^{\alpha_0})$, then for each $\delta\in[0,\delta_0]$ (7) also has the solution $(x^{\alpha_0+\delta}, y^{\alpha_0+\delta},$

$z^{\alpha_0+\delta}$).

Proof. For $\phi,\varphi,\gamma\in M^2(0,T)$, $\xi\in L^2(\Omega,\mathscr{F},P)$ and $\alpha_0\in[0,1]$, (7) has a unique solution, therefore, for each $x_T\in L^2(\Omega,\mathscr{F},P)$ and $u_s=(x_s,y_s,z_s)\in M^2(0,T;R^{n+n+n})$, (7) has the unique triple solution $U_s=(X_s,Y_s,Z_s)\in M^2(0,T;R^{n+n+n})$.

Consider the kind of FBSDEs with parameter $\delta\in[0,1]$:

$$\begin{cases} dX_t = [(1-\alpha_0)\beta_2(-Y_t)+\alpha_0 b(t,X_t,Y_t)+\delta(\beta_2 y_t+b(t,x_t,y_t))+\phi_t]dt+ \\ \qquad [\alpha_0\sigma(t,X_t,Y_t)+\delta\sigma(t,x_t,y_t)+\varphi_t]dB_t \\ -dY_t = [(1-\alpha_0)\beta_1 X_t+\alpha_0 f(t,U_t)+\delta(-\beta_1 x_t+f(t,u_t))+\gamma_t]dt - Z_t dB_t \\ x_0 = a,\ Y_T = \alpha_0\Phi(X_T)+(1-\alpha_0)X_T+\delta(\Phi(x_T)-x_T)+\xi \end{cases}$$

We need only to prove the mapping

$$I_{\alpha_0+\delta}(u\times x_T) = U\times X_T:$$

$$M^2(0,T;R^{n+n+n})\times L^2(\Omega,\mathscr{F}_T,P;R^n)\to M^2(0,T;R^{n+n+n})\times L^2(\Omega,\mathscr{F}_T,P;R^n)$$

is a contraction.

Let

$$\bar{u}=(\bar{x},\bar{y},\bar{z})\in M^2(0,T;R^{n+n+n})\quad \bar{U}\times\overline{X_T}=I_{\alpha_0+\delta}(\bar{u}\times\overline{x_T}),$$

$$\hat{u}=(\hat{x},\hat{y},\hat{z})=(x-\bar{x},y-\bar{y},z-\bar{z}),\ \hat{U}=(\hat{X},\hat{Y},\hat{Z})=(X-\bar{X},Y-\bar{Y},Z-\bar{Z}).$$

By using Itô formula, we have

$$(\alpha_0\mu_1+(1-\alpha_0))E|\hat{X}_T|^2+\beta_1 E\int_0^T|\hat{X}_s|^2ds+\beta_2 E\int_0^T|\hat{Y}_s|^2ds\leqslant$$

$$\delta K_1 E\int_0^T(|\hat{u}_s|^2+|\hat{U}_S|^2)ds+\delta K_1 E|\hat{X}_T|^2+\delta K_1 E|\hat{x}_T|^2$$

We can have the following estimates:

$$\sup_{0\leqslant s\leqslant T}E|\hat{X}_T|^2\leqslant\delta K_1 E\int_0^T|\hat{u}_s|^2ds+K_1E\int_0^T|\hat{Y}_s|^2ds,$$

$$E\int_0^T|\hat{X}_S|^2ds\leqslant K_1T\delta E\int_0^T|\hat{u}_s|^2ds+K_1TE\int_0^T|\hat{Y}_s|^2ds,$$

$$E\int_0^T(|\hat{Y}_S|^2+|\hat{Z}_S|^2)ds\leqslant K_1\delta E\int_0^T|\hat{u}_s|^2ds+K_1\delta E\int_0^T|\hat{Y}_s|^2ds+$$

$$K_1\delta E\int_0^T|\hat{X}_s|^2ds+K_1E|\hat{X}_T|^2$$

where constant K_1 depends on β_1,β_2,T and the Lipschitz constant. If $\mu_1>0$, then

$$\alpha_0\mu_1+(1-\alpha_0)\geqslant\mu=\min(1,\mu_1)>0$$

Combining the above estimates, there exists a constant K depending on $\beta_1,\mu_1,\beta_2,K_1,T$, such that

$$E\int_0^T|\hat{U}_S|^2ds+E|\hat{X}_T|\leqslant K\delta(E\int_0^T|\hat{u}_s|^2ds+E|\hat{x}_T|^2)\qquad\square$$

It is clear that, by taking $\delta_0=\frac{1}{2K}$, for $\delta\in[0,\delta_0]$, the mapping $I_{\alpha_0+\delta}$ is a contraction. The unique fixed point $U^{\alpha_0+\delta}=(X^{\alpha_0+\delta},Y^{\alpha_0+\delta},Z^{\alpha_0+\delta})$ of this mapping is the solution of (7).

At last, we give the proof of Theorem 1.

From Lemma 2.5 in [3], if $\alpha=0$, (7) has a unique solution. According to Lemma 1, for each $\delta\in[0,\delta_0]$, when $\alpha=0+\delta$, (7) has a unique solution. Since δ_0 only depends on constant K, we can repeat this process for $N(1\leqslant N\delta_0<1+\delta_0)$-times. Specially, if $\alpha=1$, $\phi_s\equiv0,\gamma_s\equiv0$, then (7) has a unique solution. The proof is completed. $\square$

3 Linear quadratic stochastic optimal control problem

Consider the following linear stochastic control system coupled with a BSDE:

$$\begin{cases} dx_t = [A(\omega)x_t + B(\omega)u_t - L^{T}(\omega)y_t]dt + [C(\omega)x_t + D(\omega)u_t]dB_t \\ -dy_t = [A^{T}(\omega)y_t + C^{T}(\omega)z_t + R(\omega)x_t]dt - z_t dB_t \\ x_0 = a,\ y_T = Q(\omega)x_T \end{cases} \tag{8}$$

where $A(\omega)$ and $C(\omega)$ are both bounded $n\times n$ matrixes, $B(\omega)$ and $D(\omega)$ are both bounded $n\times k$ matrixes; $v_t, t\in[0,T]$ is an admissible control, i. e. , a square integrable process taking values in R^k, that is, the control is not restricted.

To minimize the cost function:

$$J(v(\cdot)) = \frac{1}{2}E\left[\int_0^T(\langle R(\omega)x_t, x_t\rangle + \langle N(\omega)v_t, v_t\rangle + \langle L(\omega)y_t, y_t\rangle)dt + \langle Q(\omega)x_T, x_T\rangle\right] \tag{9}$$

where $R(\omega), Q(\omega), L(\omega)$ are all nonnegative symmetric bounded $n\times n$ matrixes, $N(\omega)$ is a positive symmetric bounded $k\times k$ matrix and the inverse N^{-1}.

Theorem 2. There exists a unique optimal control:

$$u(t) = -N^{-1}[B^{T}(\omega)y_t + D^{T}(\omega)z_t]$$

Proof. It is easy to check that (8) satisfies assumptions H1) and H2) in Theorem 1. So, (8) has the unique solution (x_t, y_t, z_t). We need only to prove that $u(t)$ is the optimal control.

For any $v_t\in R^k$, we denote x_t^v as the trajectory. Then we have

$$J(v(\cdot)) - J(u(\cdot)) = \frac{1}{2}E\left[\int_0^T(\langle R(\omega)x_t^v, x_t^v\rangle - \langle R(\omega)x_t, x_t\rangle + \langle N(\omega)v_t, v_t\rangle - \langle N(\omega)v_t, v_t\rangle + \langle L(\omega)y_t^v, y_t^v\rangle - \langle L(\omega)y_t, y_t\rangle)dt + \langle Q(\omega)x_T^v, x_T^v\rangle - \langle Q(\omega)x_T, x_T\rangle\right]$$

Hence,

$$J(v(\cdot)) - J(u(\cdot)) = \frac{1}{2}E\Big[\int_0^T(\langle R(\omega)(x_t^v - x_t, x_t^v - x_t\rangle + \langle N(\omega)v_t - u_t, v_t - u_t\rangle + \langle L(\omega)y_t^v - y_t, y_t^v - y_t\rangle + 2\langle L(\omega)y_t, y_t^v - y_t\rangle + 2\langle R(\omega)x_t, x_t - x_t^v\rangle + 2\langle N(\omega)u_t, v_t - u_t\rangle)dt + \langle Q(\omega)x_T^v - x_T, x_T^v - x_T\rangle + 2\langle Q(\omega)x_T, x_T^v - x_T\rangle\Big]$$

Since $y_T = Q(\omega)x_T$, by applying the Ito formula to $\langle x_t^v - x_t, y_t\rangle$, we have

$$E[\langle x_T^v - x_T, y_T\rangle] = E\left[\int_0^T(-\langle R(\omega)x_t, x_t^v - x_t\rangle + \langle B^{T}(\omega)y_t, v_t - u_t\rangle - \langle L(\omega)y_t, y_t^v - y_t\rangle + \langle D^{T}(\omega)z_t, v_t - u_t\rangle)dt\right]$$

As R, Q, L, N are nonnegative, we also have

$$J(v(\cdot)) - J(u(\cdot)) \geqslant E\left[\int_0^T(\langle R(\omega)x_t, x_t^v - x_t\rangle + \langle N(\omega)u_t, v_t - u_t\rangle + \langle L(\omega)y_t, y_t^v - y_t\rangle)dt + \langle y_T, x_T^v - x_T\rangle\right] =$$

$$E\int_0^T(\langle(B^{T}(\omega)y_t + D^{T}_{(\omega)}z_t), v_t - u_t\rangle + \langle N(\omega)u_t, v_t - u_t\rangle)dt = 0$$

So, $u(t) = -N^{-1}[B^{T}(\omega)y_t + D^{T}(\omega)z_t]$ is the optimal control.

To prove the uniqueness of the optimal control, the method is classical and can be seen in Pontryagin et al. (1962) and Bensoussan(1981). Let $u^1(\cdot)\neq u^2(\cdot)$ be both optimal control, $(x_t^1, y_t^1), (x_t^2, y_t^2)$ be the trajectories corresponding to the optimal control. It is easy to know that the trajectories corresponding to $\frac{u^1(\cdot)+u^2(\cdot)}{2}, \frac{u^1(\cdot)-u^2(\cdot)}{2}$ are $\left(\frac{x_t^1+x_t^2}{2}, \frac{y_t^1+y_t^2}{2}\right)$, $\left(\frac{x_t^1-x_t^2}{2}, \frac{y_t^1-y_t^2}{2}\right)$ respectively. Since N is positive, we have

$$J(u^1(\cdot)) = J(u^2(\cdot)) = \alpha \geqslant 0$$

$$2\alpha = J(u^1(\omega)) + J(u^2(\omega)) =$$
$$2J\left(\frac{u^1(\omega)+u^2(\omega)}{2}\right) + \mathrm{E}\left[\int_0^T\left(\left\langle R\frac{x_t^1-x_t^2}{2}, \frac{x_t^1-x_t^2}{2}\right\rangle + \left\langle N\frac{u_t^1-u_t^2}{2}, \frac{u_t^1-u_t^2}{2}\right\rangle + \right.\right.$$
$$\left.\left.\left\langle L\frac{y_t^1-y_t^2}{2}, \frac{y_t^1-y_t^2}{2}\right\rangle\right)\mathrm{d}t + \left\langle Q\frac{x_T^1-x_T^2}{2}, \frac{x_T^1-x_T^2}{2}\right\rangle\right] \geqslant$$
$$2J\left(\frac{u^1(\omega)+u^2(\omega)}{2}\right) + \mathrm{E}\int_0^T\left\langle N\frac{u_t^1-u_t^2}{2}, \frac{u_t^1-u_t^2}{2}\right\rangle\mathrm{d}t \geqslant$$
$$2\alpha + \frac{\delta}{4}\mathrm{E}\int_0^T |u_t^1 - u_t^2|^2\mathrm{d}t$$

So,

$$\mathrm{E}\int_0^T |u_t^1 - u_t^2|^2\mathrm{d}t \leqslant 0$$

Hence, $u_t^1 = u_t^2$. The proof is completed. □

Now we assume A, B, C, D, R, L, Q, N are all deterministic. Introducing the matrixes $K(t), M(t)$ and the following generalized matrix Riccati equation:

$$\begin{cases} -\dot{K}(t) = A^{\mathrm{T}}K(t) + K(t)A - K(t)[L^{\mathrm{T}} + BN^{-1}B^{\mathrm{T}}]K(t) - \\ \qquad KBN^{-1}D^{\mathrm{T}}M(t) + C^{\mathrm{T}}M(t) + R \\ M(t) = K(t)C - K(t)DN^{-1}B^{\mathrm{T}}K(t) - K(t)DN^{-1}D^{\mathrm{T}}M(t) \\ K(T) = Q,\ t \in [0,T] \end{cases} \tag{10}$$

we have the following theorem.

Theorem 3. Suppose there exist matrices $(K(t), M(t))$ satisfying the generalized matrix Riccati equation (10). Then the optimal linear feedback regulator for the linear quadratic optimal problem is

$$u(t) = -N^{-1}[(B^{\mathrm{T}}K(t) + D^{\mathrm{T}}M(t)]x_t, \quad t \in [0,T]$$

the optimal value function is

$$J(u(\cdot)) = \frac{1}{2}\langle K(0)a, a\rangle \tag{11}$$

Proof. It is easy to check that if $(K(t), M(t))$ is the solution of Riccati equation (10) then the solution (x_t, y_t, z_t) of FBSDE(8) satisfies

$$y_t = K(t)x_t,\ z_t = M(t)x_t$$

So the optimal control is

$$u(t) = -N^{-1}[(B^{\mathrm{T}}K(t) + D^{\mathrm{T}}M(t)]x_t, \quad t \in [0,T]$$

The value of equation (11) can be calculated by using Itô formula. The proof is completed.

We discuss a special case: $L(\omega) \equiv 0$.

Equation

$$\begin{cases} \mathrm{d}x_t = [A(\omega)x_t + B(\omega)v_t]\mathrm{d}t + [C(\omega)x_t + D(\omega)v_t]\mathrm{d}B_t \\ x_0 = a \end{cases} \tag{12}$$

the cost function is

$$J(v(\cdot)) = \frac{1}{2}\mathrm{E}\left[\int_0^T(\langle R(\omega)x_t, x_t\rangle + \langle N(\omega)v_t, v_t\rangle)\mathrm{d}t + \langle Q(\omega)x_T, x_T\rangle\right] \tag{13}$$

From Theorem 2, it is easy to obtain the following corollary. □

Corollary 1. For the optimal control problems (12) and(13), there exists a unique optimal control

$$u(t) = -N^{-1}[B^{T}(\omega)y_t + D^{T}(\omega)z_t] \tag{14}$$

For the solvability of generalized matrix Riccati equation (10), we only discuss a special case: $D=0$.

In this case, (10) becomes

$$\begin{cases} -\dot{K}(t) = A^{T}K(t) + K(t)A - K(t)[L^{T} + BN^{-1}B^{T}]K(t) + C^{T}K(t)C + R \\ M(t) = K(t)C \\ K(T) = Q, \qquad t \in [0,T] \end{cases} \tag{15}$$

From the theory of Riccati equation, (15) has a unique solution $K(t) \in C(0,T;S_{+}^{n})$. Here S_{+}^{n} is the set of nonnegative symmetric matrixes.

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正倒向随机微分方程与一类线性二次随机最优控制问题

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摘　要　讨论一类正倒向随机微分方程解的存在唯一性及其对应的一类线性二次随机最优控制问题,利用单调性方法证明了一类特殊的正倒向随机微分方程解的存在唯一性定理,利用该结果研究一类耦合了一个倒向随机微分方程的线性随机控制系统广义最优指标随机控制问题,得到由正倒向随机微分方程的解所表示的唯一最优控制的显式表达式,并得到精确的线性反馈及其对应的 Riccati 方程.

关键词　倒向随机微分方程,最优控制,Riccati 方程

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