第30卷 第3期 自 动 化 学 报 Vol. 30, No. 3
2004年5月 ACTA AUTOMATICA SINICA May, 2004

# A Geometric Method of Singularity Analysis for Parallel Robots[1)]

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**Abstract** Using the language of differential geometry, this paper provides a fine classification of singularities of general parallel robots. Based on the relations between singularity manifolds and singularity distributions, these singularities are further subclassified into first-order singularities and second-order ones. Furthermore, the second-order singularities can be distinguished as degenerate or nondegenerate singularities by whether they form continuous curves on configuration manifolds. This paper also gives an insight into the degenerate singularities, which can sometimes be a source of danger not only to the mechanism itself but also to workers to operate the mechanism. Finally, a planar two degrees-of-freedom mechanism with one redundant actuator is given to illuminate the method.

**Key words** Parallel robots, singularity, differential geometry, distribution

## 1 Introduction

Compared with its serial counterparts, a parallel robot (or a closed-chain mechanism or manipulator) has much more complex singularities in its configuration space[1]. However, unlike its serial counterparts, for which mathematical tools have been well establishe, study on parallel robots, especially on their singularities is still relatively limited. Moreover, up to now, there is a lack of a unified theoretical framework of singularities analysis for general parallel robots. Some important works in this field are given as follows: Gosselin and Angeles[2] classified singularities of parallel manipulators into three types utilizing some derived Jacobian relations, but did not discuss the topological characteristics of configuration spaces at singularities. Park and Kim[3] used metric on various spaces to study singularities of general parallel mechanisms. Their results are fundamental to understand the geometric nature of singularities in parallel robots. Merlet[4], on the other hand, studied extensively singularities of the Stewart-Gough platform and several of its variants. Based on Taylor series expansion of trajectories at singularities, Kieffer[5] studied ordinary singularities, isolated singularities and their bifurcations, but his results were confined to serial mechanisms.

In this paper, we use distribution and differential forms on configuration manifold to study the geometric nature of singularities of general parallel robots, and based on topological and geometric properties of configuration space, present a fine classification method of singularities. We also study the topology of singularity manifold, which consists of all singularities on configuration space. Based on relations between singularity manifolds and singular motion directions, these singularities are further classified into first-order and second-order singularities. We analyze geometric structures of configuration spaces at degenerate singularities, which give us an insight on self-motion of parallel robots in some special configuration. Finally, we give an example of a planar two degrees-of-freedom mecha-

1) Supported by National Netural Science Foundation of P. R. China (50029501)
Received April 16, 2003; in revised form August 31, 2003
收稿日期 2003-04-16; 收修改稿日期 2003-08-31

nism with one reduntant actuator to validate the method proposed in this paper.

## 2 First-order classification of singularities

### 2.1 Configuration singularity

A general parallel robot can be descripted as the following constraint equation[1]

$$H: E \to R^m, \quad \theta \mapsto H(\theta) = [h_1(\theta), \cdots, h_m(\theta)]^T = 0 \tag{1}$$

By the Implicit Function Theorem[6], the configuration space $Q = H^{-1}(0)$ of the constraint system (1) is an n-m dimensional embedded submanifold in the ambient space $E$, if and only if the rank of $T_pH$ is equal to $m$ for all $p \in Q$. However, at a point $q \in Q$, if rank($T_q H$) $< m$, $Q$ can not be locally embedded into $E$. In this case, the parallel robot with the constraint Equation (1) is in some singular configuration. This type of singularity is called the configuration singularity (CS), since it is caused by the singularity of the configuration space.

**Definition 1.** For a general parallel robot with the constraint Equation (1), its configuration singularity occurs when the rank of the constraint equation is less than its normal rank, that is, rank($T_pH$) $< m$, where $p$ denotes the configuration singularity point.

Consider the one-forms on the generalized force space $T_p^* E$ and differentiate the constraint Equation (1), and we have

$$\langle dh_i, \dot\theta \rangle_p = 0, \quad dh_i \in T_p^* E, \quad \dot\theta \in T_pE, \ p \in E, \ i = 1, \cdots, m \tag{2}$$

Here $\langle \cdot, \cdot \rangle$ denotes the pairing of the one-forms $dh_i$ with the generalized velocity vector $v \in T_pE$. Physically, these one-forms have the meaning of generalized constraint forces.

**Lemma 1**[6]. One-forms $\xi^1, \cdots, \xi^r \in V^*$ is linearly dependent if and only if their exterior product is equal to zero, that is,

$$\xi^1 \wedge \cdots \wedge \xi^r = 0 \tag{3}$$

**Theorem 1.** The configuration singularity occurs if and only if the one-forms $dh_i, i = 1, \cdots, m$ corresponding to the constraint forces satisfy

$$dh_1 \wedge \cdots \wedge dh_m = 0 \tag{4}$$

**Proof.** The result is straightforward. Since one-forms $dh_i$ can be given in local coordinates by

$$dh_i = \sum_{j=1}^{n} \frac{\partial h_i}{\partial \theta_j} d\theta_j, \ i = 1, \cdots, m \tag{5}$$

and

$$T_ph_i = \sum_{j=1}^{n} \frac{\partial h_i}{\partial \theta_j} \frac{\partial}{\partial \theta_j}, \quad T_pH = (T_ph_1, \cdots, T_ph_m)^T \tag{6}$$

It is obvious that the tangent mapping $T_pH$ has full rank if and only if $dh_1, \cdots, dh_m$ is linearly dependent. Consider $dh_i \in \Lambda^1(T_pE), i = 1, \cdots, m$. By Lemma 1, the tangent mapping $T_pH$ loses rank if and only if $dh_1, \cdots, dh_m$ satisfy (4). The theorem holds. □

### 2.2 Actuator singularity

Regardless of configuration singularities, the configuration spaces of parallel robots are the regular submanifolds of ambient spaces, which are called configuration manifolds. Normally, the configuration manifold can be locally parameterized by the actuator coordinates as

$$\psi: Q \to R^m: p \mapsto \psi(p) = \theta_a \tag{7}$$

where $\theta_a(p) = [\theta_a^1(p), \cdots, \theta_a^{n-m}(p)]^T$ is the local coordinates of the configuration manifold at point $p \in Q$.

**Definition 2.** When the locally parameterized equation (7) at point $p \in Q$ is not surjective, the parallel robot is in the actuator singular configuration. Here the point $p$ is called the actuator singularity (AS).

Differentiating Equation (7), we have one-forms $d\theta_a = (d\theta_a^1, \cdots d\theta_a^{n-m}) \in T_p^* E$, which physically has a meaning of generalized actuator forces. Along with $dh_i$, these one-forms span the generalized force space $T_p^* E$. Similar to Theorem 1, we have

**Theorem 2.** For general parallel robots with the constraint Equation (1), the actuator singularities occur if and only if

$$dh_1 \wedge \cdots \wedge dh_m \wedge d\theta_a^1 \wedge \cdots \wedge d\theta_a^{n-m} = 0 \tag{8}$$

**Proof.** Assume that parallel robots are in regular (non-singular) configurations. Then all actuator coordinates $(\theta_a^1, \cdots \theta_a^{n-m})$ form the local coordinate system of the configuration manifold. Evidently, we have $T_p^* Q = span\{d\theta_a^1, \cdots, d\theta_a^{n-m}\}$ and $T_p^* Q^\perp = span\{dh_1, \cdots, dh_m\}$. Since

$$T_p^* E = T_p^* Q \oplus T_p^* Q^\perp \tag{9}$$

we obtain

$$T_p^* E = span\{dh_1, \cdots, dh_m, d\theta_a^1, \cdots, d\theta_a^{n-m}\} \tag{10}$$

by Lemma 1, Theorem 2 holds. □

(10) is also named generalized force closure condition of mechanical system controllability in robotics literature[1]. Theorem 2 can be extended to the parallel robots with redundant actuators as follows.

**Theorem 3.** For a general parallel robot with $l$ redundant actuators $\theta_a^k, k=1, \cdots, l$, its actuator singularities occur if and only if the following condition is satisfied

$$\dim(span\{dh_1, \cdots, dh_m, d\theta_a^1, \cdots, d\theta_a^{n-m}, d\theta_a^{n-m+1}, \cdots, d\theta_a^{n-m+l}\}) < n \tag{11}$$

**Proof.** By the generalized force closure condition (10), we have

$$T_p^* E = span\{dh_1, \cdots, dh_m, d\theta_a^1, \cdots, d\theta_a^{n-m}, d\theta_a^{n-m+1}, \cdots, d\theta_a^{n-m+1}\} \tag{12}$$

Since $\dim(E) = \dim(T_p^* E) = n$, Theorem 3 holds. □

When a parallel robot is in actuator singular configuration, although all actuated joints are fixed, there still exists an instantaneous inner motion in the mechanism. (11) shows that actuator redundancy can remove the singularities of parallel robots and improve their performance[1].

### 2.3 End-effector singularity

Let $f: Q \rightarrow W$ be the forward kinematic mapping of parallel mechanisms, where $W$ is the workspace of the end-effector. If $T_* f$ is surjective, then $f$ is the diffeomorphism from the configuration space to the workspace, which shows that the configuration space also can be paramenterized by the end-effector coordinates.

**Definition 3.** End-effector singularity (ES) happens when the configuration space $Q$ can not be parameterized by the local coordinates corresponding to end-effector coordinates.

Physically, end-effector singularity means the end-effector loses degrees of freedom. Another equivalent explanation is that the motion of the actuated joints could result in no motion of the end-effector when the end-effector singularity happens.

Let $x = [x_1 \quad \cdots \quad x_{n-m}]^T$ denote the end-effector coordinates. Similar to (8), we have the sufficient and necessary condition of end-effector singularities as the following

$$dh_1 \wedge \cdots \wedge dh_k \wedge dx_1 \wedge \cdots \wedge dx_{n-m} = 0 \tag{13}$$

Since actuator singularities and end-effector singularities are caused by different parameterization coordinates in the configuration manifold, both of them are called the parameterization singularities of parallel robots.

## 3 Second-order singularity

The singularity sets of parallel robots consist of all the same type of singularities on the configuration space. These singularity sets can be subclassified into different subsets according to the degree of deficiency of singularities. Without loss of generality, consider the actuator singularities. Define the set of AS points

$$Q_s = \{p \in Q \mid dh_1 \wedge \cdots \wedge dh_m \wedge d\theta_a^1 \wedge \cdots \wedge d\theta_a^{n-m}|_p = 0\} \tag{14}$$

and the set annihilation vectors at $p \in Q_s$

$$T_pV = \{v \in T_pQ \mid \langle dh_i, v\rangle = \langle d\theta_j, v\rangle = 0, i = 1,\cdots,m, j = 1,\cdots,n-m\} \tag{15}$$

The dimension $d$ of $T_pV$ measures the degree of deficiency (DoD) of singularities. Let $d_0 (0 < d_0 \leqslant m)$ denote an upper bound on $d$ and let

$$Q_{sk} = \{p \in Q_s \mid \dim(T_pV) = k\} \tag{16}$$

be the set of actuator singularities of DoD $k(0 < k \leqslant d_0)$. We have $Q_s = \bigcup_{k=1}^{d} Q_{sk}$.

We introduce the notion of singularity distribution. Define

$$\Delta_{sk} = \bigcup_{p \in Q_{sk}} T_pV \Delta_s = \bigcup_{k=1}^{d} \Delta_{sk} \tag{17}$$

where $\Delta_{sk}$ is a distribution of annihilation spaces of dimension $k$ on $Q_{sk}$ and $\Delta_s$ is a distribution of annihilation spaces on $Q_s$, which are called singularity distribution of parallel robots. Note that the dimensions of $Q_{sk}$ and $\Delta_{sk}$ are not necessarily the same.

**Definition 4.** A point $p \in Q_{sk}$ is called a first-order singularity point if there does not exist a vector $v \in \Delta_{sk}$ that is also tangent to $Q_{sk}$. Otherwise, $p$ is called a second-order singularity point (see Fig. 1).

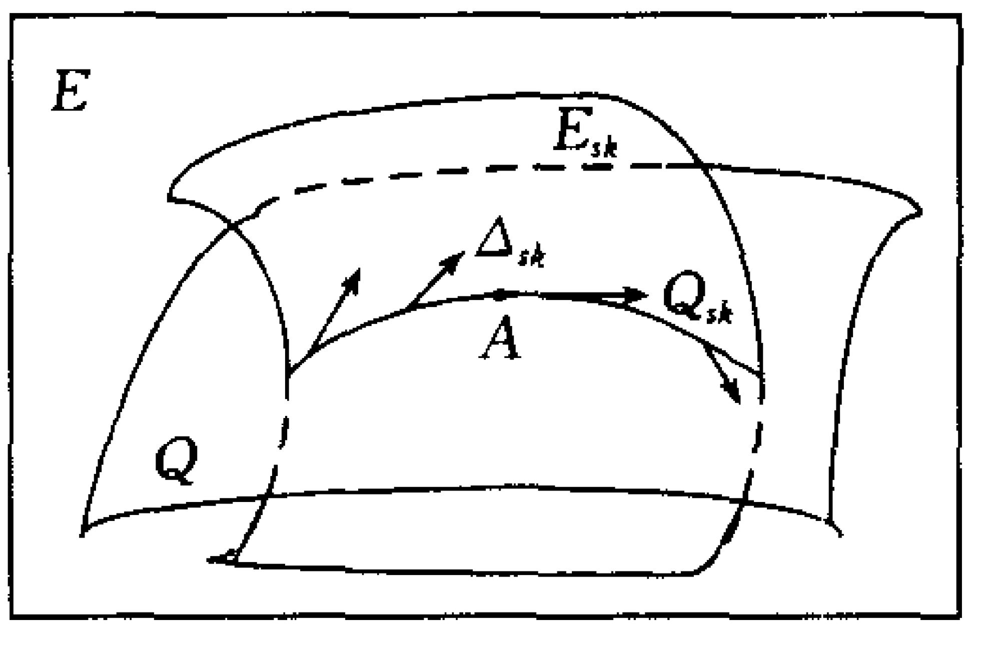


Fig. 1 Singularity distribution along singularity manifolds

## 4 Degeneration of second-order singularity

A second-order singularity can be further classified into a nondegenerate singularity if it is isolated or a degenerate one if it is continuous. All degenerate singularities form continuous curves (or surfaces) on the configuration manifold. When parallel robots drop into these degenerate singularities, some inner self-motions occur, which is dangerous not only to the mechanism itself but also to the workers to operate the mechanism. Hence these degenerate singularities in the workspace must be removed during the mechanism design.

In [3], Park defined degenerate singularities by degenerate Hessian matrixes of Morse functions on configuration spaces. However, a nondegenerate Hessian matrix is only a sufficient condition of a nondegenerate singularity. Here we give a better definition of degenerate singularities by the notion of singularity distribution.

**Definition 5.** A second-order singularity point is degenerate if and only if there exists an involutive sub-distribution $\tilde{\Delta}_{sk} \subseteq \Delta_{sk}$ with a constant dimension $k$ such that $\tilde{\Delta}_{sk} \subset T_pQ_{sk}$. Its integral manifold is called degenerate singularity manifold $Q_{dk}$.

A hierarchic diagram of our classification of singularities is given in Fig. 2.

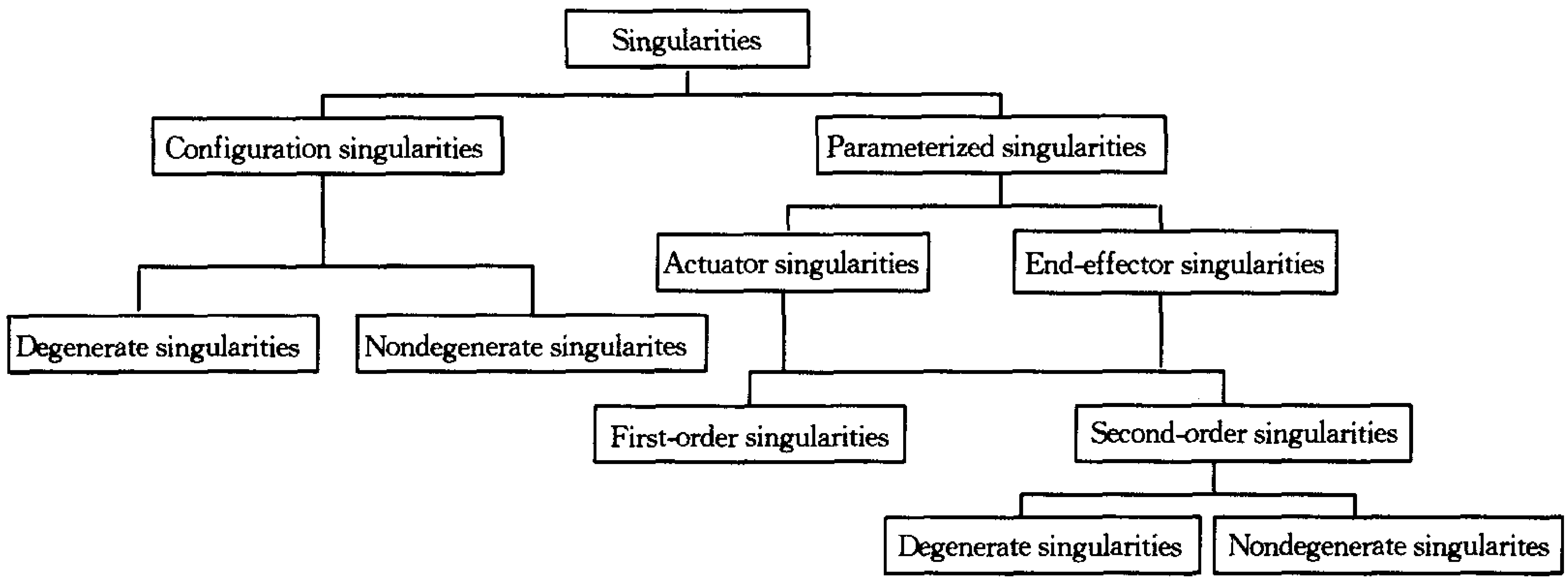


Fig. 2 A hierarchic diagram of singularities of general parallel robots

## 5 Mechanism analysis

Fig. 3 shows a planar five-bar mechanism. Let $(\theta_1, \phi_1, \theta_2, \phi_2)$ be the ambient coordinates, where $\theta_1, \theta_2$ are actuated joints. Its constraint equations are written as

$$h_1 = L_1 \sin\theta_1 + L_2 \sin\phi_1 - L_1 \sin\theta_2 - L_2 \sin\phi_2 = 0 \quad (18)$$

$$h_2 = L_1 \cos\theta_1 + L_2 \cos\phi_1 - L_1 \cos\theta_2 - L_2 \cos\phi_2 - c = 0 \quad (19)$$

1) CS points. By (4), all configuration singularities satisfy

$$\begin{aligned} dh_1 \wedge dh_2 = & L_1^2 \sin(\theta_2 - \theta_1) d\theta_1 \wedge d\theta_2 - L_1 L_2 \sin(\phi_1 - \theta_1) d\theta_1 \wedge d\phi_1 \\ & + L_1 L_2 \sin(\phi_2 - \theta_1) d\theta_1 \wedge d\phi_2 + L_1 L_2 \sin(\phi_1 - \theta_2) d\theta_2 \wedge d\phi_1 \\ & + L_2^2 \sin(\phi_2 - \phi_1) d\phi_1 \wedge d\phi_2 - L_1 L_2 \sin(\phi_2 - \theta_2) d\theta_2 \wedge d\phi_2 = 0 \end{aligned} \quad (20)$$

*i.e.*, $\sin(a_1 - a_2) = 0,\ a_1 \neq a_2,\ a_1, a_2 \in \{\theta_1, \phi_1, \theta_2, \phi_2\}$ (21)

2) AS points. By (8), the actuator singularities occur if and only if

$$dh_1 \wedge dh_2 \wedge d\theta_1 \wedge d\theta_2 = L_2^2 \sin(\phi_2 - \phi_1) d\phi_1 \wedge d\phi_2 \wedge d\theta_1 \wedge d\theta_2 = 0 \quad (22)$$

*i.e.*, $\sin(\phi_1 - \phi_2) = 0$ or $\phi_1 - \phi_2 = k\pi, \quad k = 0, \pm 1$ (23)

3) ES points. Let $(x_B, y_B)$ be end-effector coordinates, which are written as

$$x_B = x_{A_1} + L_1 \cos\theta_1 + L_2 \cos\phi_1, \quad y_B = y_{A_1} + L_1 \sin\theta_1 + L_2 \sin\phi_1$$

By (13), we have $dh_1 \wedge dh_2 \wedge dx_B \wedge dy_B = L_1^2 L_2^2 \sin(\phi_1 - \theta_1) \sin(\phi_2 - \theta_2) d\theta_1 \wedge d\phi_1 \wedge d\theta_2 \wedge d\phi_2 = 0$. Hence, the condition of ES points is $\phi_1 - \theta_1 = k\pi$ or $\phi_2 - \theta_2 = k\pi, \quad k = 0, \pm 1$.

4) Second-order AS points. The set $Q_{s1}$ of actuator singularities is a curve defined by (18), (19) and (23) in a four-dimensional Euclidean space. A basis of its tangent bundles is

$$Y = \gamma_1 \frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\phi_1} + \gamma_2 \frac{\partial}{\partial\theta_2} + \frac{\partial}{\partial\phi_2} \quad (24)$$

where $\gamma_1 = \dfrac{L_2(\sin(\theta_2 - \phi_2) - \sin(\theta_2 - \phi_1))}{L_1 \sin(\theta_2 - \theta_1)}, \gamma_2 = \dfrac{L_2(\sin(\phi_1 - \theta_1) + \sin(\theta_1 - \phi_2))}{L_1 \sin(\theta_2 - \theta_1)}$.

On the other hand, there exists a singularity distribution $\Delta_{s1} = span\{V^1\}$ along the curve $Q_{s1}$, where $V^1$ is an annihilation vector field written as

$$V^1 = \frac{\partial}{\partial\phi_1} + \frac{\partial}{\partial\phi_2} \quad (k = 0), \quad V^1 = -\frac{\partial}{\partial\phi_1} + \frac{\partial}{\partial\phi_2} \quad (k = \pm 1) \quad (25)$$

Second-order singularities happen if and only if $V^1$ is tangent to curve $Q_{s1}$. Hence, all points satisfying $\phi_1 = \phi_2$ are second-order AS points on the actuator singularity manifold, while the others are first-order AS points (see Fig. 3).

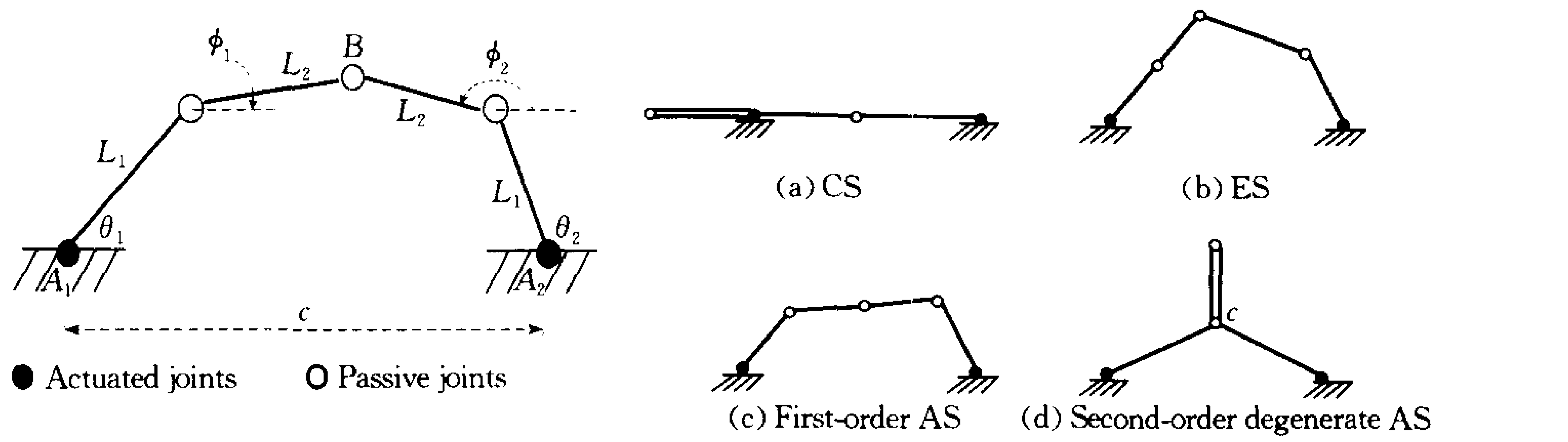


Fig. 3 A planar five-bar mechanism and its different types of singular configurations

5) Degenerate AS points. Since the singularity distribution $\Delta_{s1}$ has a constant dimension 1, it is an integrable subdistribution in the tangent bundle $TQ_{s1}$. Hence, the above second-order singularities are degenerate. The integral manifold of $\Delta_{s1}$ is a one-dimension degenerate actuator singularity submanifold in the configuration space. In this case, although all actuated joints are fixed, there exists a self-motion around the joint C in the five-bar mechanism (see Fig. 3(d)).

6) Our method can also be applied to parallel robots with redundant actuators. Consider the five-bar linkage with a redundant link (see Fig. 4), and two additional constraint equations are

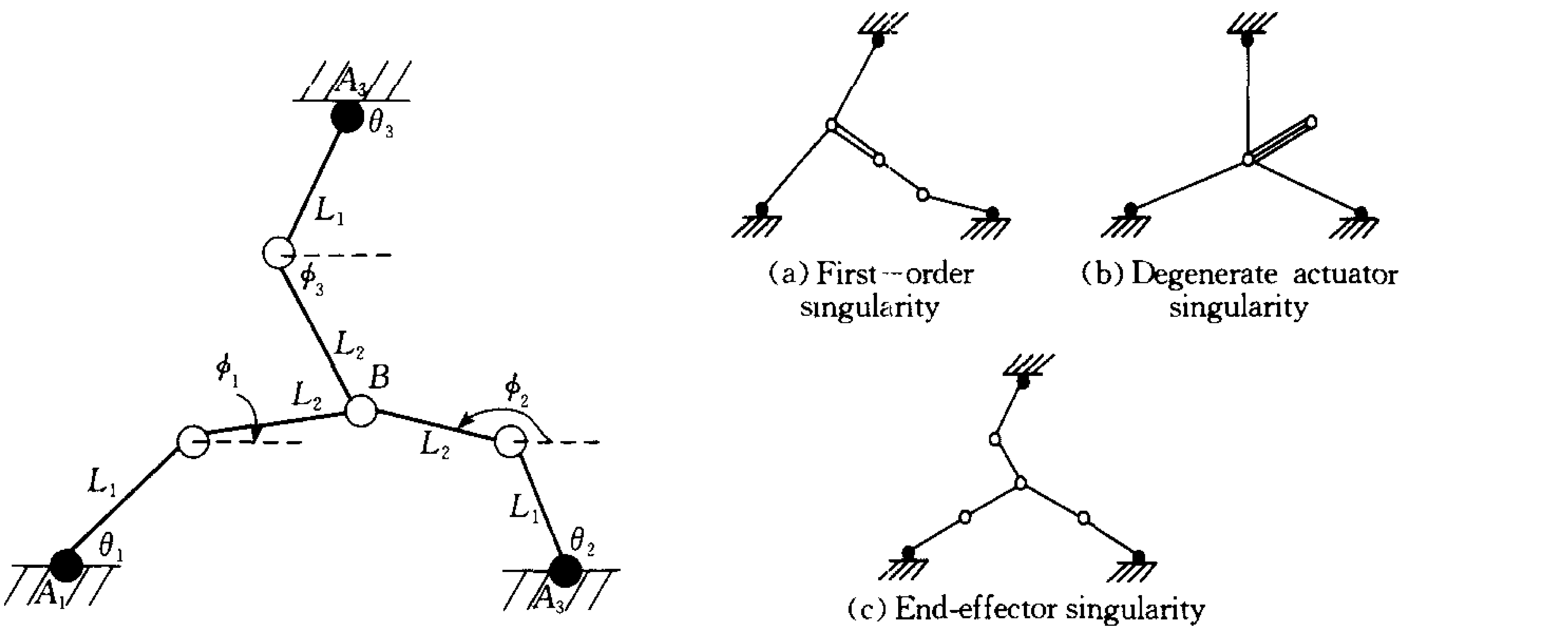


Fig. 4 Different types of singularities in a five-bar mechanism with a redundant actuated branch

$$h_3 = x_{A_1} + L_1\cos\theta_1 + L_2\cos\phi_1 - x_{A_3} - L_1\cos\theta_3 - L_2\cos\phi_3 = 0 \tag{26}$$

$$h_4 = y_{A_1} + L_1\sin\theta_1 + L_2\sin\phi_1 - y_{A_3} + L_1\sin\theta_3 + L_2\sin\phi_3 = 0 \tag{27}$$

By Theorem 3, the sufficient and necessary condition of actuator singularities is

$$\mathrm{d}h_1 \wedge \mathrm{d}h_2 \wedge \mathrm{d}h_3 \wedge \mathrm{d}h_4 \wedge \mathrm{d}\theta_i \wedge \mathrm{d}\theta_j = 0 \quad (0 < i < j < 4) \tag{28}$$

Computing the above equation, we have

$$\tan\phi_1 = \tan\phi_2 = \tan\phi_3 \tag{29}$$

In the same manner, we can analyse its second-order singularities, degenerate singularities and end-effector singularities. Some of the analysis results are shown in Fig. 4.

## 6 Conclusion

This paper provides a unified geometric framework for singularity analysis of general parallel robots and give a fine classification of these singularities, which is significant for us to understand the nature of singularities in parallel robots. The classification method proposed can be applied to general parallel robots including redundantly actuated mechanisms. Results of a typical planar five-bar linkage show that this kind of classification of singularities is effective and has obvious geometric and physical meanings.

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# 并联机器人奇异位形分析的几何方法

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**摘 要** 采用微分几何方法,提出一种针对一般并联机器人的奇异性的分类方法.依据奇异流形与奇异运动方向的关系,将奇异性进一步区分为一阶奇异性和二阶奇异性,基于二阶奇异点分布的连续性属性将其中的二阶奇异性进一步分为退化和非退化奇异性,并对退化奇异性的物理含义进行了分析,指出退化奇异给机构带来的危险性.最后针对冗余驱动的平面二自由度并联机构进行了分析.

**关键词** 并联机器人,奇异点,微分几何,分布

**中图分类号** TH112