Highly Efficient Monte-Carlo for Estimating the Unavailability of Markov Dynamic System[1)]

XIAO Gang[1,2] DENG Li[1] ZHANG Ben-Ai[1] ZHU Jian-Shi[1]
[1](Computational Physics Laboratory, Institute of Applied Physics and Computational Mathematics, Beijing 100088)
[2](The Second Artillery Engineering College, Xi'an 710025)
(E-mail: xiaohengzhou@vip.sina.com)

Abstract Monte Carlo simulation has become an important tool for estimating the reliability and availability of dynamic system, since conventional numerical methods are no longer efficient when the size of the system to solve is large. However, evaluating by a simulation the probability of occurrence of very rare events means playing a very large number of histories of the system, which leads to unacceptable computing time. Highly efficient Monte Carlo should be worked out. In this paper, based on the integral equation describing state transitions of Markov dynamic system, a uniform Monte Carlo for estimating unavailability is presented. Using free-flight estimator, direct statistical estimation Monte Carlo is achieved. Using both free-flight estimator and biased probability space of sampling, weighted statistical estimation Monte Carlo is also achieved. Five Monte Carlo schemes, including crude simulation, analog simulation, statistical estimation based on crude and analog simulation, and weighted statistical estimation, are used for calculating the unavailability of a repairable Con/3/30 : F system. Their efficiencies are compared with each other. The results show the weighted statistical estimation Monte Carlo has the smallest variance and the highest efficiency in very rare events simulation.

Key words Unavailability, statistical estimation Monte Carlo, Markov dynamic system, variance reduction technique, uniform Monte Carlo

1 Introduction

Unavailability is the probability that the system is in failed states at time t. Conventionally, in order to obtain availability of Markov dynamic system, a set of 2^N coupled first-order differential equations must be solved, where N is the number of components. Thus even a system with only ten components will result in a system of over one thousand coupled equations with a transition matrix with over a million elements. Moreover, if some of the components are repairable, the equations would become quite stiff, requiring that very small time steps be used in the numerical integration. When the size of the system is too big to solve, or the system is non-Markov, Monte Carlo simulation appears to be the only tool likely. In very rare event simulation, crude Monte Carlo is too time-consuming or inaccurate, so researchers working on reliability have given more allowance to the design of highly efficient simulation algorithms.

In [1~10], several Monte Carlo techniques for estimating availability of static system were given. Fishman[11] compared the methods given in [1~5]. Cancela and Khadiri[8] compared the methods given in [2~6,8].

[12,13] gave the state transition Monte Carlo (also called analog simulation) to sample random walks of dynamic system. Compared with crude simulation, analog simulation saves time of sampling of random walks. In analogy to particle transport problem, [13] gave two new Monte Carlo techniques for estimating unreliability of Markov dynamic system, including forced transitions and failure biasing. Forced transitions and failure biasing

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are more efficient than crude simulation and analog simulation. In [14,15], free-flight estimator for the estimation of unreliability was conducted, and by combining with biased scheme a highly efficient Monte Carlo technique for the estimation of unreliability of Markov system is achieved. The estimation of unavailability by Monte Carlo is more difficult than that of unreliability, since, in spite of both unreliability and unavailability are conditional expectations, the condition of unavailability is much stricter than that of unreliability. There are not many papers that design highly efficient Monte Carlo for estimating unavailability of the dynamic system. [16] discussed the variance of several estimators of unavailability.

This paper is organized as follows. In Section 2, based on integral equation of state transitions of Markov dynamic system, the unavailability is expressed as the functional of probability of transition out of current state. In Section 3, a uniform Monte Carlo for estimating this functional is given, and based on this uniform Monte Carlo, statistical estimation and weighted statistical estimation Monte Carlo are achieved in Section 4. In Section 5, five Monte Carlo schemes, including weighted statistical estimation, direct statistical estimation based on crude and analog simulation, crude simulation and analog simulation, are used to estimate the unavailability of a repairable Con/3/30 : F system, and the efficiencies of five methods are compared with each other.

2 Integral equation describing state transitions of Markov system and the functional of unavailability

Assume that we have a system with N components, each of which may be in an operational or a failed state, and the failure rate and the repair rate of component i are λ_i and μ_i, respectively. There are then 2^N states corresponding to the unique combinations of operational and failed components. We let $p_k(t)=$ probability that the system is in state k at time t, and $\sum_k p_k(t)=1$. We denote $k=0$ as the initial state in which all components are operational, and $p_k(0)=\delta_{k0}$. The integral equation describing state transitions of Markov dynamic system can be written as[13]

$$\chi_k(t) = \int_0^t \mathrm{d}t' f(t \mid t',k')[\delta_{k0}\delta(t') + \sum_{k'} q(k \mid t',k')\,\chi_{k'}(t')] \tag{1}$$

where $\chi_k(t) \equiv \gamma_k(t) p_k(t)$ is the probability density of transition out of state k, $\gamma_k(t)$ is the transition rate out of state k, and $\gamma_k(t)=\sum_{k'\neq k}\gamma_{k'k}(t)$, where $\gamma_{k'k}(t)$ are the transition rates between states. In general, $\gamma_k(t)$ can be expressed as

$$\gamma_k(t) = \sum_{l\in o_k}\lambda_{lk}(t) + \sum_{l\in \Gamma_k}\mu_{lk}(t) \tag{2}$$

and the o_k and Γ_k are the sets of operational and failed components, respectively, in state k.

$$f(t \mid t',k') = \gamma_k(t') \exp\left[-\int_{t'}^t \gamma_k(t'')\mathrm{d}t''\right], \quad t \geqslant t' \tag{3}$$

is the probability density that there will be a transition at t given that the system is in state k' at t'. The quantity $q(k|k',t)=\gamma_{kk'}(t)/\gamma_{k'}(t)$ is the conditional probability that given a transition out of state k' at time t, the new state will be k. The first term on the right of (1) is due to the convention that the problem is initialized by a transition into state $k=0$ at $t=0$.

Monte Carlo simulation of the integral equations may be used to estimate weighted integral of the term

$$I = \sum_k \int_0^T \mathrm{d}t\, \chi_k(t) a(t,k) \tag{4}$$

Since unavailability is the probability that the system is in a failed state at time t, it can be estimated by

$$\hat{U} = \sum_{k \in F} \int_0^T \mathrm{d}t\, \chi_k(t) a(t, k) \tag{5}$$

where F is the sets of failed state, and

$$a(t,k) = \begin{cases} \exp\left[-\int_t^T \sum_{l \in \Gamma_k} \mu_{lk}(t')\mathrm{d}t'\right], & t \leqslant T \\ 0, & t > T \end{cases} \tag{6}$$

3 Uniform Monte Carlo for estimating the unavailability of Markov dynamic system

The uniform Monte Carlo for estimating functional (5) covers the following steps.

In the first step, the probability space of sampling should be constructed. Let $s_l = (k_l, t_l)$ denote that the system is in state k_l at time t_l through l transitions. The probability density function of sampling can be determined as

$$\begin{aligned} & q(s_0, s_1, \cdots, s_m)\bar{f}(s_0, s_1, \cdots, s_m) = \\ & [\bar{q}_1(s_0 \to s_1)\bar{f}_1(s_0 \to s_1)] \times \cdots \times [\bar{q}_m(s_{m-1} \to s_m)\bar{f}_m(s_{m-1} \to s_m)] \end{aligned} \tag{7}$$

where $\bar{q}_l(s_{l-1} \to s_l)\bar{f}_l(s_{l-1} \to s_l)$ is a conditional probability density with s_{l-1} known, namely, $\bar{q}_l(s_l | s_0, s_1, \cdots, s_m)\bar{f}_l(s_l | s_0, s_1, \cdots, s_m)$, while

$$q_l(s_{l-1} \to s_l) f_l(s_{l-1} \to s_l) \neq 0, \quad \bar{q}_l(s_{l-1} \to s_l)\bar{f}_l(s_{l-1} \to s_l) \neq 0$$

Secondly, by the probability space of sampling, the history of random transition of Markov dynamic system is sampled.

$$\Gamma_{M+1}: s_0, s_1, \cdots, s_m, \cdots, s_M, s_{M+1} \tag{8}$$

where Γ_{M+1} is a history of random transition. At the same time, we need to calculate the weight of sampling

$$w_m = w_{m-1} \frac{q(s_{m-1} \to s_m) f(s_{m-1} \to s_m)}{\bar{q}(s_{m-1} \to s_m)\bar{f}(s_{m-1} \to s_m)} \tag{9}$$

and $w_0 = 1$. The simulation is stopped when $t_{M+1} > T$, where T is the mission time.

Thirdly, we choose an estimator of unavailability. Usually, there are two kinds of estimators of unavailability, one is called the last-event estimator $\hat{U}_1$, *i.e.*,

$$\hat{U}_1 = w(s_M) a(s_M) \tag{10}$$

another is the free-flight estimator $\hat{U}_2$, *i.e.*,

$$\hat{U}_2 = \sum_{m=0}^{\infty} w(s_m) a(s_m) \tag{11}$$

4 Statistical estimator and weighted statistical estimator of unavailability

Since free-flight estimator uses more information of history of random walks than the last-event estimator, Monte Carlo based on free-flight estimator should have smaller variance than that of the last-event estimator. Substituting expression (6) into (11), we have

$$\hat{U}_3 = \sum_{m=0}^{M} I(k_m \in F) w(s_m) \exp\left[-\int_t^T \sum_{l \in \Gamma_m} \mu_l(t')\mathrm{d}t'\right] \tag{12}$$

where $I(\cdot)$ is a binary function, which means $I(\cdot) = 1$ when $\cdot$ holds, else $I(\cdot) = 0$. Monte Carlo based on estimator $\hat{A}_3$ is called statistical estimation method.

If the original probability space of sampling is chosen, namely, $\bar{q}(s_0, s_1, \cdots, s_m)\bar{f}(s_0, s_1, \cdots, s_m) = q(s_0, s_1, \cdots, s_m) f(s_0, s_1, \cdots, s_m)$, then $w(s_m) = 1$ in expression (11). Statistical estimation method in such a way is called direct one, in which sampling of the history of state transitions may use the method of crude simulation or analog simulation [12, 13]. In analog simulation, the time intervals Δt between transitions is

$$\Delta t = -\frac{1}{\gamma_k}\ln(1-\xi) \tag{13}$$

where random number ξ is sampled from uniform distribution $(0,1]$. Determination of which component has failed or been repaired, and thereby the new state of the system is carried out as follows. A random number ξ' is first generated. If $\gamma_k\xi' < \sum_{i\in o_k}\lambda_{ik}(t)$, the failed component is determined by

$$\sum_{\substack{i'=1\\ i'\in O_k}}^{i}\lambda_{i'} \leqslant \gamma_k\xi' \leqslant \sum_{\substack{i'=1\\ i'\in \Gamma_k}}^{i+1}\lambda_{i'} \tag{14}$$

If $\gamma_k\xi' > \sum_{i\in o_k}\lambda_{ik}$, the repaired component is determined by

$$\sum_{i\in O_k}\lambda_{ik} + \sum_{\substack{i'=1\\ i'\in \Gamma_k}}^{i}\mu_{i'} \leqslant \gamma_k\xi' \leqslant \sum_{i\in O_k}\lambda_{ik} + \sum_{\substack{i'=1\\ i'\in \Gamma_k}}^{i+1}\mu_{i'} \tag{15}$$

If the biased probability density space of sampling is chosen, in a history of random walk more failed states which can contribute to the result would be sampled than that by using original probability space of sampling. Let

$$\bar{q}(s_0,s_1,\cdots,s_m)\bar{f}(s_0,s_1,\cdots,s_m) = q(s_0,s_1,\cdots,s_m)\bar{f}(s_0,s_1,\cdots,s_m) \tag{16}$$

where

$$\bar{f}(k \mid k',t') = \begin{cases} f(k \mid k',t')\Big/\Big\{1-\exp\Big[-\int_{t'}^{T}\gamma_{k'}(t'')\mathrm{d}t''\Big]\Big\}, & t' \leqslant t \leqslant T \\ 0, & \text{else} \end{cases} \tag{17}$$

Then,

$$w(s_m) = w(s_{m-1})\Big\{1-\exp\Big[-\int_{t'}^{T}\gamma_{k'}(t'')\mathrm{d}t''\Big]\Big\}$$

where $w_0 = 1$. Statistical estimation method based on biased probability space of sampling is usually called weighted statistical estimation method. By expression (13), the time interval Δt between transition is

$$\Delta t = -\frac{1}{\gamma_k}\ln(1-\xi(1-\exp(-\gamma_k(T-t')))),\quad 0 \leqslant \Delta t \leqslant T-t \tag{18}$$

where t' is the cumulating time of random transition of system. The new state of system is sampled from formula (14) or (15). Formula (18) shows the biased density function of sampling leads to a never-ending simulation, namely, cumulative time of random transition of system can not reach the mission time T forever. A Russian roulette[17] could then be added to this scheme, to kill the histories with very small weight.

5 Numerical example

We have studied a repairable linear Con/3/30 : F system, in which each component is of two states, operating or failed. Failure rate of each component is $\lambda=0.001(day^{-1})$, repair rate is $\mu=0.05(day^{-1})$. The linear con/3/30 : F system is defined as: 30 components linearly connected in such a way that the system fails iff at least 3 consecutive components fail. Repairable con/k/n : F system has broad application because of its high availability and low cost[18].

As a numerical example, we deal with efficiency of five types of Monte Carlo schemes for estimating the current unavailability of a con/3/30 : F system mentioned above, including weighted statistical estimation, direct statistical estimation based on analog simulation or crude simulation, crude simulation and analog simulation. Usually, efficiency of

Monte Carlo is defined as

$$efficiency = \frac{1}{\hat{\sigma}^2 \times t} \tag{19}$$

where $\hat{\sigma}^2$ is the variance of the score and t the mean time per a history.

In Fig. 1, five types of Monte Carlo simulation all give the same estimation of unavailability, which verifies the statistical estimation method presented in our paper.

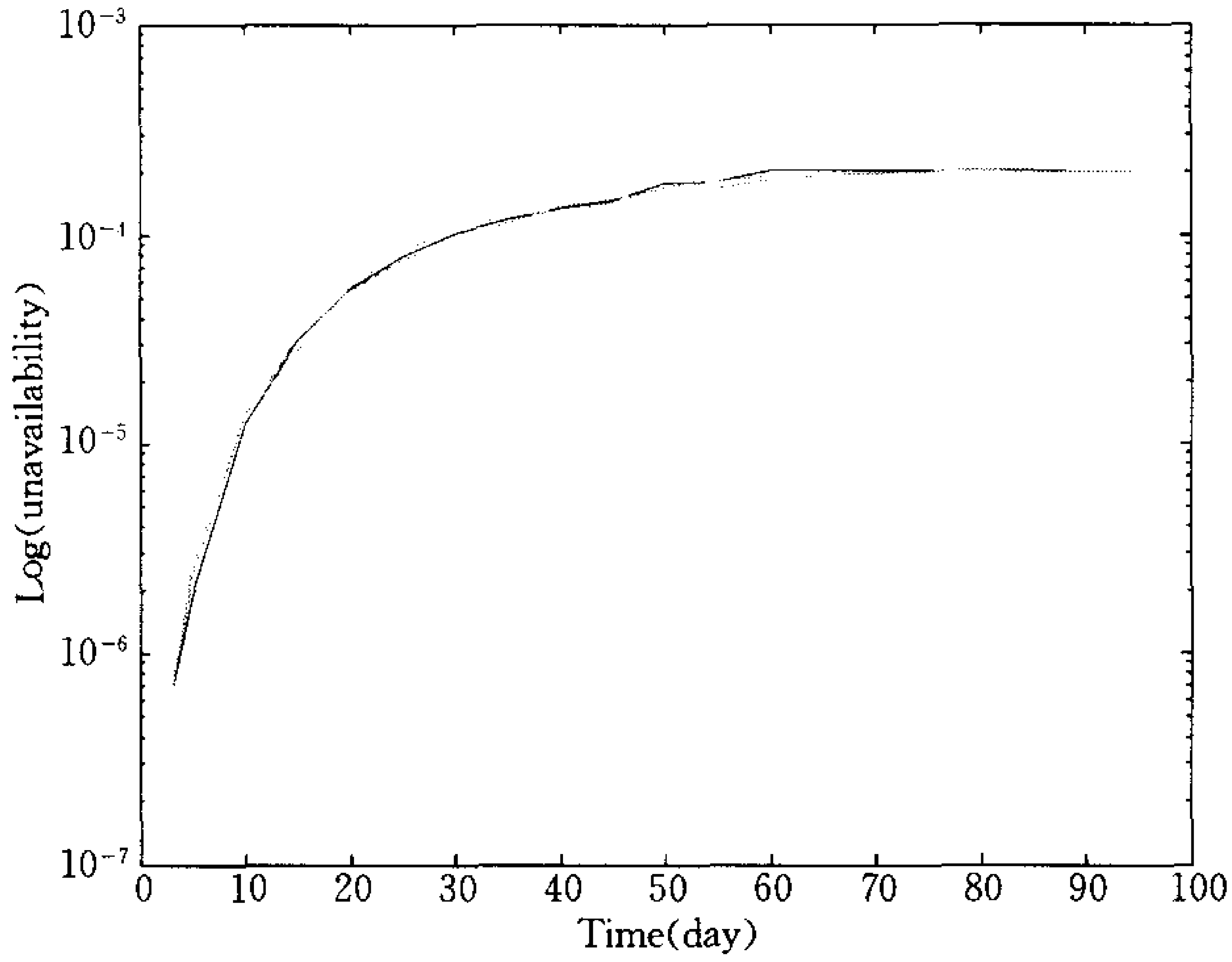

Fig. 1 Unavailability of a Con/3/30 : f system vs. Time

(Estimating by five types of Monte Carlo techniques: crude simulation, analog simulation, weighted statistical estimation, direct statistical estimation based on crude simulation, direct statistical estimation based on analog simulation)

Fig. 2 shows the methods using free-flight estimator (include direct statistical estimator and weighted statistical estimator) have less variance than the other methods. The reason is that for all the failed states in a history of random walks, free-flight estimator can contribute to the score. Fig. 2 also shows, when the unavailability is very low, the variances given by the methods except weighted statistical estimation are very close. The reason is, in this situation, the system is rarely under failed states. Weighted statistical estimation method, using free-flight estimator and biased sampling simultaneously, reduces variance markedly.

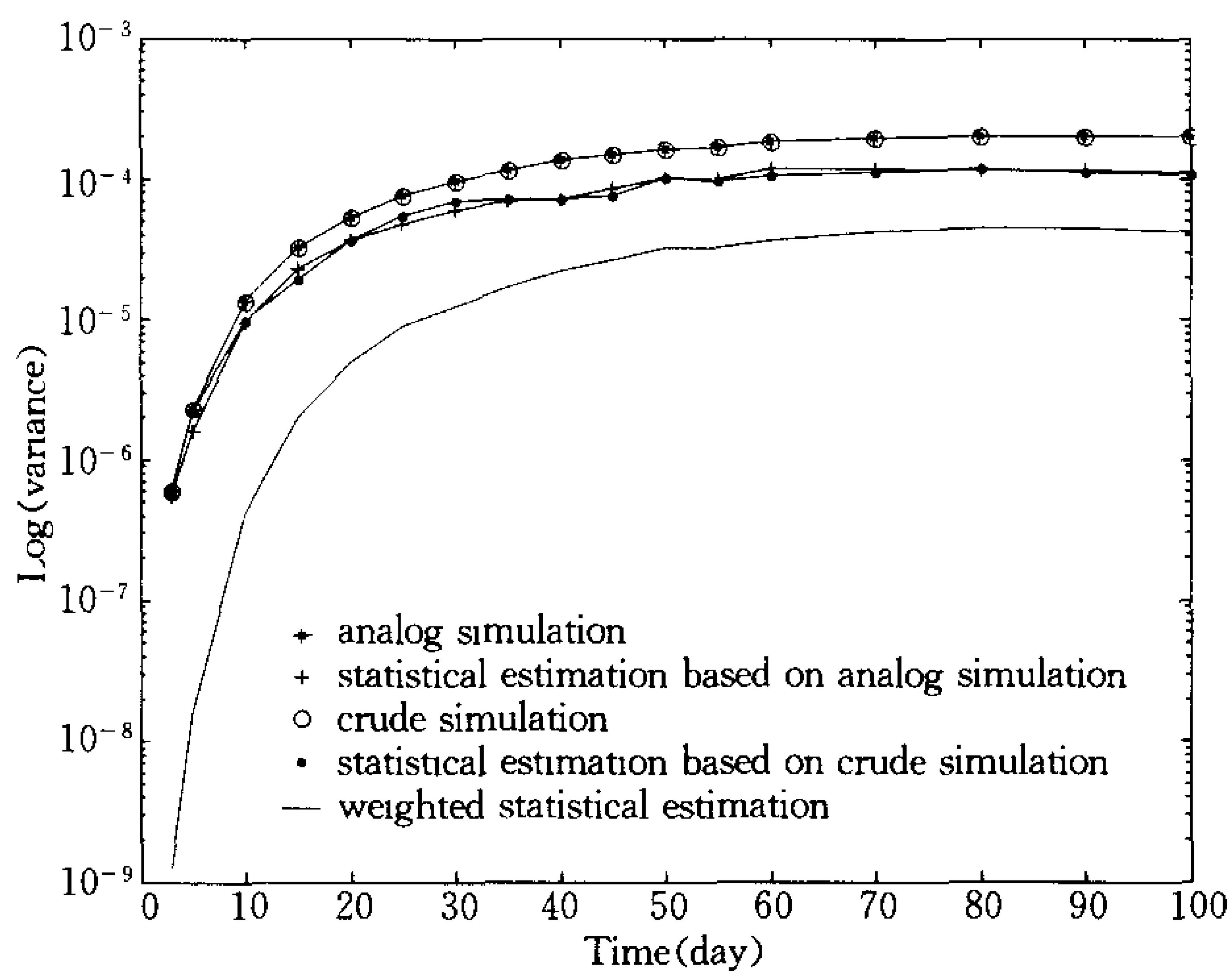

Fig. 2 Variance of unavailability vs. time

Fig. 3 shows weighted estimation is the most time-consuming. The reason is that biased sampling is used, which induces more state transitions than in using original one.

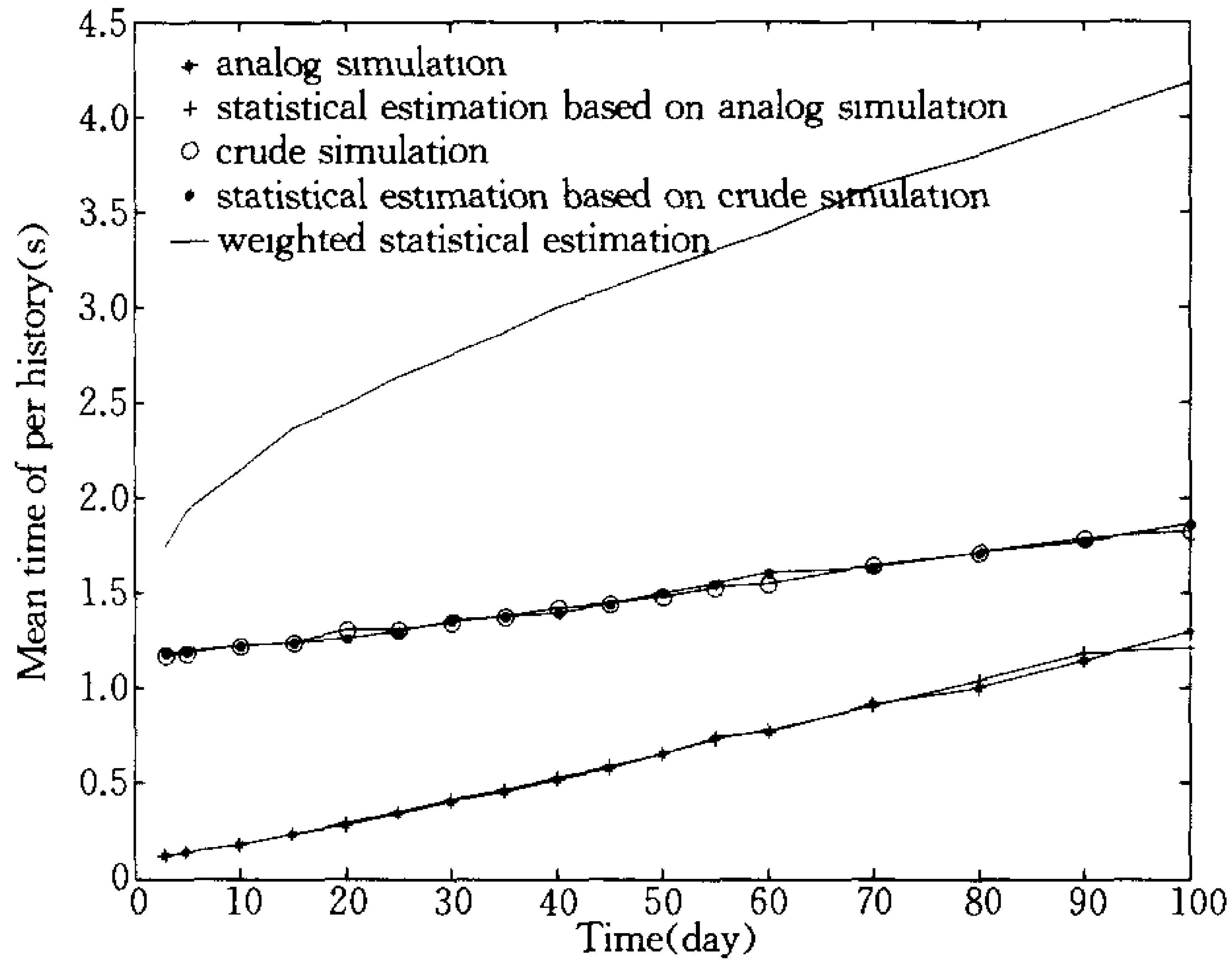

Fig. 3 The mean time of per history vs. time

Fig. 4 shows that weighted statistical estimation Monte Carlo has the highest efficiency when the unavailability is very low. For example, when unavailability of the system is 5.997×10^{-7}, the efficiency of weighted statistical estimation is 4.863×10^{7}, the statistical estimation based on crude simulation is 1.623×10^{5}, and the statistical estimation based on analog simulation is 1.545×10^{6}. This illuminates the weighted statistical estimation method is valuable for estimating unavailability of highly reliable system, which is very difficult for crude simulation. Fig. 4 also shows, in this example, when the unavailability is greater than 3.223×10^{-5}, efficiency of statistical estimation based on analog simulation is greater than that of weighted statistical estimation. The reason for this is that the latter is more time-consuming.

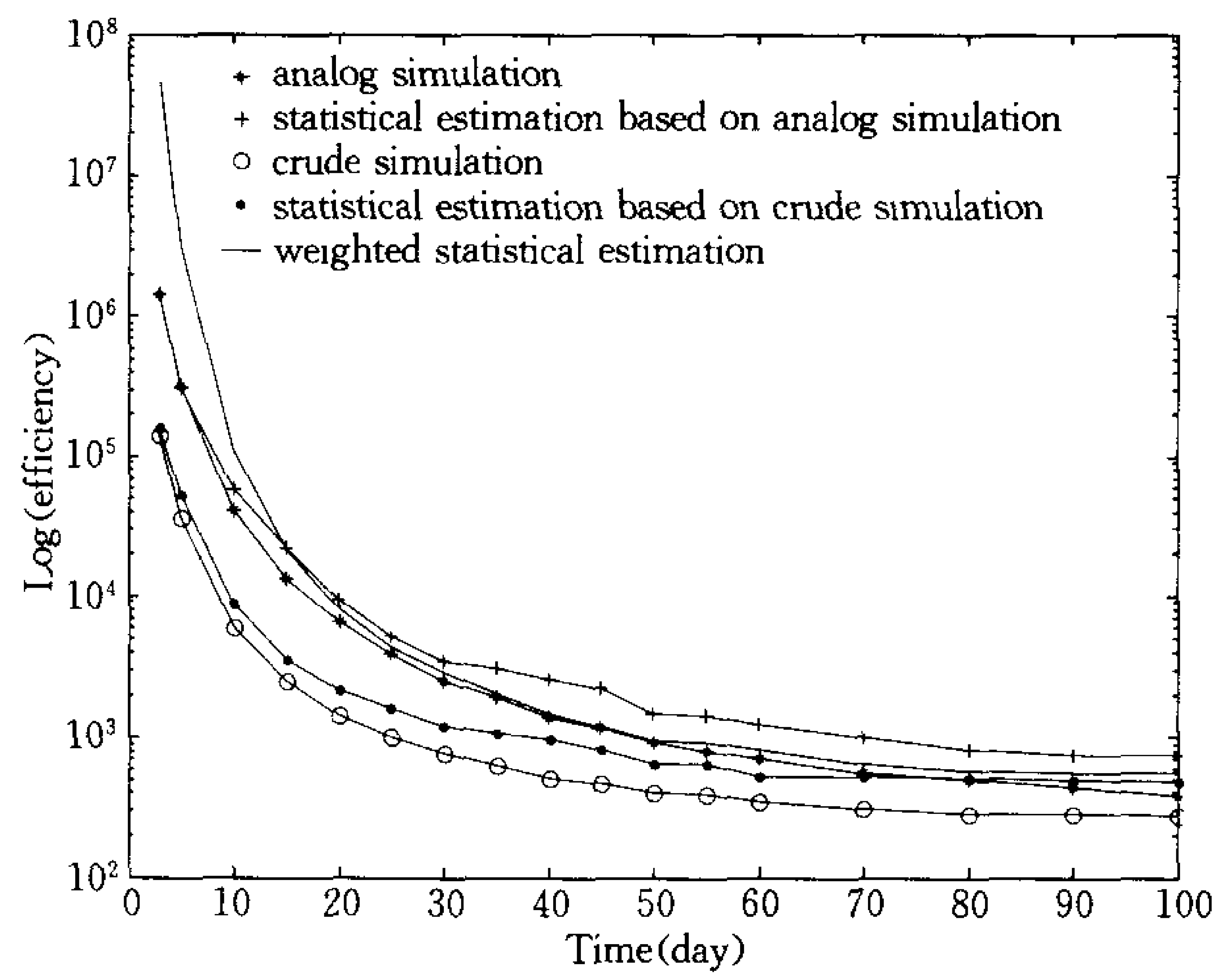

Fig. 4 Efficiencies of 5 types of Monte Carlo methods vs. time

6 Conclusion

Direct statistical estimation and weighted statistical estimation Monte Carlo for estimating unavailability of Markov dynamic system are propounded in this paper. Numerical example shows these methods can give less variance and higher efficiency than traditional methods. When the unavailability is very low, weighted statistical estimation is the most efficient, which shows this method is very valuable for rare events simulation.

For control system, electric power system, communication system and Internet system all can be described by continue-time Markov process, so methods presented in this paper can be used for estimating unavailability of these systems.

Of course, our example of system is so simple that Monte Carlo is not necessary for studying it. But we hope that what we have observed here will stay true for more complicated cases.

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XIAO Gang Received his Ph. D. degree from Xi'an Jiaotong university in 2001. His research interests include reliability analysis of complex system, stochastic modeling and simulation, particle transport, and information analysis of nuclear radiation.

DENG Li Receive his Ph. D. degree from Xi'an Jiaotong university in 2001. His research interests include numerical simulation of particle transport problems and Monte Carlo methods.

ZHANG Ben-Ai Graduated from Peking university in 1958. He is the professor at Beijing institute of applied physics and computational mathematics. His research interests include nuclear physics.

ZHU Jian-Shi Graduated from Peking university in 1958. He is an academician of Chinese Academy of Engineering. His research interests include hydrodynamics and physics.

马尔可夫系统瞬态不可用度计算的高效率蒙特卡罗方法

肖 刚[1,2] 邓 力[1] 张本爱[1] 朱建士[1]

[1](北京计算物理与应用数学研究所 北京 100088)

[2](第二炮兵工程学院 西安 710025)

(E-mail: xiaohengzhou@vip.sina.com)

摘 要 当马尔可夫系统规模较大时,需要采用蒙特卡罗方法计算其瞬态不可用度.如果系统的不可用度很小,则需要采用高效率的蒙特卡罗方法.本文在马尔可夫系统寿命过程的积分方程的基础上,给出了系统瞬态不可用度计算的蒙特卡罗方法的统一描述,由此设计了马尔可夫系统瞬态不可用度计算的直接统计估计方法和加权统计估计方法.用直接仿真方法、拟仿真方法、基于直接仿真的统计估计方法、基于拟方仿真的统计估计方法和加权统计估计方法计算了一可修Con/3/30:F系统的瞬态不可用度.结果表明,由于同时采用了偏倚的抽样空间和逐次事件估计量,加权统计估计方法的方差最小,当系统不可用度很小时,该方法效率最高.

关键词 不可用度,统计估计方法,马尔可夫动态系统,减小方差方法,统一的蒙特卡罗方法

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