A PRECISION METHOD FOR OPTIMIZING THE RELIABILITY OF A REDUNDANT CONTROL SYSTEM AND ITS APPLICATION*

SHU Songgui

(*Institute of Automation, Chinese Academy of Sciences, Beijing, China*)

Abstract In this paper, the precision method for optimizing the reliability of a redundant control system with single or multiple constraints have been investigated.

This method is presented in an easy way. Starting from the physical concept for the least cost and the most benefit in engineering design, we obtain two criteria and a basic principle for optimizing the reliability of a control system with single constraint.

For a multiple-constraint problem we can find out only one active constraint in general or sometimes a few active constraints. All the other constraints are non-active and can be discarded without any trouble.

Once the ideal solution has been obtained, this solution should be rounded to integral numbers.

1 INTRODUCTION

Usually the optimal design of system reliability is the optimization of a system with unequal constraints. This kind of problems has been solved by some authors[1]. However, all these methods are rather complicated and not convenient to designers and engineers, especially when the constraints are of high order, e.g., more than three.

In this paper, starting from the physical concept of the least cost of resources and the most benefit to the customers in engineering design we get the idea of how to use every small amount of slacks on resources most efficiently to improve system reliability. This is the way for optimizing the reliability of a redundant control system with one or more constraints. Two criteria have been obtained for the single constraint problem. For the multiple-constraint problem we have to determine the number of active constraints first. If we have only one active constraint, then the solution corresponding to this constraint is the answer. Otherwise, we have to do some special treatment.

* Proceding of IEEE Int. Conf. on System Engineering, Dayton, Chio. USA. 1989.

2 OPTIMIZING THE RELIABILITY OF A SERIES PARALLEL SYSTEM SUBJECT TO A SINGLE UNEQUAL CONSTRAINT

Suppose we have an automatic control system with n stages in series and m_i identical parallel units for redundancy in ith stage. Assume that these units have only independent failures such as open circuit.

A single constraint is given by

$$w_i m_i \le W \tag{1}$$

where w_i denotes the amount of resource for each unit in the ith stage and W denotes the total amount of resource for the whole system.

The system reliability can be expressed by [2]

$$R = \prod_{i=1}^{n} \left(1 - q_i^{m_i}\right) \tag{2}$$

where q_i denotes the unreliability for each unit in ith stage.

Now the question is how to distribute the constrained resource among different stage such that the system reliability reaches its maximum. Starting from the physical concept in engineering design and supposing that m_i could be changed, we should continuously add any small amount of slacks of resources to that stage which would improve the system reliability most effectively.

Along this way the following criteria can be given for optimizing the reliability of a control system with single constraint:

1) In the ideal case the partial derivatives of the system reliability with respect to the given resource for every stage within the system should be equal each to each.

This criterion may be expressed by a formula which can be derived from equation (2) as

$$\frac{\partial(R/R_m)}{\partial(w_i m_i / W)} = -\frac{W q_i^{m_i} \ln q_i}{w_i\left(1 - q_i^{m_i}\right)} = \frac{1}{A_i\left(q_i^{-m_i}\right)} = c \tag{3}$$

where

$$A_i = w_i / W(-\ln q_i), \qquad R_m = \max R = \prod_{i=1}^{n} \left(1 - q_i^{m_i}\right)$$

Solving eqn.(3), we get

$$m_i = -\ln\left(1 + 1/cA_i\right)/\ln q_i \tag{4}$$

where the slope c is a constant to be determined.

2) The system reliability increases monotonically with the degree of redundancy m_i, so that the ideal optimal reliability must occur at the maximum limit of the given constraint.

Substituting eq(4) into eq(1) and taking the equal sign in it, we have

$$\sum_{i=1}^{n} \left[A_i \ln\left(1 + 1/cA_i\right)\right] = 1 \tag{5}$$

The above equation can be rearranged as

$$C = C_0 \exp\left\{\sum_{i=1}^{n} \left[A_i \ln(1 + cA_i)\right] \Big/ \sum_{i=1}^{n} A_i\right\} \tag{6a}$$

$$C_0 = \exp\left\{-\left[1 + \sum_{i=1}^{n} \left(A_i \ln A_i\right)\right] \Big/ \sum_{i=1}^{n} A_i\right\} \tag{6b}$$

where C_0 is the main part of C.

Substituting C_0 calculated from (6b) into the right side of eq(6a) instead of C, we get the 1st approximation C' of C, and so on. Finally we have

$$C^{(n)} = C^{(n-1)}$$

which is the true value of C. Thus eqn(4) gives the ideal answers of m_i.

From the above analysis we can conclude the basic principle: For a control system with a single constraint, the system reliability increases monotonically with the increase in the given resource. Thus in the ideal case, the partial derivatives of the system reliability with respect to the given resource consumed at every stage should be equal each to each and the optimal reliability must occur at the upper limit of the given constraint.

The true integral solution for the optimal reliability can be obtained by several trials and checked with eqn(1).

This method can also be used directly to the problems of continuously constrained functions. Here the ideal solutions are also the required answers.

3 OPTIMIZING THE RELIABILITY OF A SERIES PARALLEL SYSTEM SUBJECT TO SOME MULTIPLE UNEQUAL CONSTRAINTS

If we have k constraints instead of one, then eqn(1) becomes

$$\sum_{i=1}^{n} w_{li} m_i \le W_l \qquad (l = 1, 2, \cdots, k) \tag{7}$$

wher w_{li} denotes the amount of lth resource for each unit in the ith stage and W_l the total amount of resource for the whole system.

First of all, we solve the problem with each constraint separately in the same way as before and obtain k sets of m_{li} and R_l. Then we can treat the problem in different cases in the following procedures:

1) If R_j is the only smallest R_1 and all

$$\sum_{i=1}^{n} \left\{\left(w_{li} m_{ji}\right) \big/ W_l\right\} \le 1 \qquad (l \ne j) \tag{8}$$

then W_j is the only strictest constraint and the set m_{ji} and R_j are the solutions of the problem. All other constraints are non-active (which may be called false constraints) and can be discarded. In other words all $W_l (l = j)$ are given too large without any limiting meaning so far as the constraints are considered.

2) If we have a few (say a) R_j which are the same smallest reliability (but m_{ji} may different

with different j) and a few (say b) R_1 with

$$\sum_{i=l}^{n} \left\{\left(w_{li} m_{ji}\right)/W_l\right\} > 1 \qquad \left[l=(a+1) \text{ to } (a+b)=d\right] \tag{9}$$

in addition to

$$\sum_{i=l}^{n} \left\{\left(w_{li} m_{ji}\right)/W_l\right\} \le 1 \qquad (l=d+1 \text{ to } k)$$

which have been discarded.

Then all d constraints are active, which contribute the solutions of the problem together. This problem can be solved as follows:

Recalling equation (3) for any one of the multiple constraints we have

$$\frac{\partial\left(R/R_m\right)}{\partial\left(m_{li}\right)} = \frac{\ln q_i}{1-q_i^{-m_i}} = \frac{C_l w_{li}}{W_l} \tag{10}$$

Since m_{li} should be equal to the same m_i at the same stage i in practical system, then substitute M_i for m_i in the above equation and sum up for all d constraints, we have

$$\frac{\partial\left(R/R_m\right)}{\partial m_i} = \frac{q_i^{m_i} \ln q_i}{1-q_i^{-m_i}} = \frac{1}{d}\sum_{l=1}^{d} \frac{C_l w_{li}}{W_l} \qquad (i=1,\cdots,n) \tag{11}$$

where C_1, may be somewhat different from C_1 and can be determined in the following way.

From eqn(11), and let

$$A_{li} = w_{li}/W_l\left(-\ln q_i\right)$$

we have

$$m_i = \ln\left(\frac{\sum_{l=1}^{d} A_{li} C_1'}{\sum_{l=1}^{d} A_{li} C_{l+d}'}\right) \Bigg/ \ln q_i \tag{12}$$

Substituting eqn(12) into eqn (7), we have

$$\sum_{i=1}^{n} W_{li} \ln\left(\frac{\sum_{l=1}^{d} A_{li} C_l'}{\sum_{l=1}^{d} A_{li} C_{l+d}'}\right) \Bigg/ \ln q_i = W_l$$

$$\sum_{i=1}^{n} A_{li} \ln\left(\frac{\sum_{l=1}^{d} A_{li} C_l'}{\sum_{l=1}^{d} A_{li} C_{l+d}'}\right) = -1$$

$$\ln\prod_{i=1}^{n}\left(\frac{\sum_{i=1}^{d} A_{li} C_l'}{\sum_{l=1}^{d} A_{li} C_{l+d}'}\right)^{A_{li}} = -1$$

$$\prod_{i=1}^{n}\left(\frac{\sum_{l=1}^{d} A_{li}C'_{l+d}}{\sum_{l=1}^{d} A_{li}C'_{l}}\right)^{A_{li}} = e \quad (l = 1,2,\cdots,d) \tag{13}$$

Solving the above equation for C_l $(l = 1,2,\cdots,d)$ and substituting C_l into eqn (12). We get a set of m_i $(i = 1,2,\cdots,)$. Of course, the ideal values of m_i should be rounded to integral numbers for practical use.

Table 1 Given Data

Stages	i	1	2	3	4	5	W_l
Weight	w_{1i}	11.0	2.4	13.0	18.4	16.0	130
Volume	w_{2i}	17.7	4.5	20.2	21.6	17.0	180
Power consumed	w_{3i}	3.0	2.2	20.7	18.0	14.0	120
	q_i	0.256	0.161	0.153	0.147	0.133	

Table 2 Calculated Results

	i /l	1	2	3	Remarks
m_{li}	1	2.777778708	2.699180889	3.772100376	By eqn.4
	2	3.055881085	2.912680849	3.142999434	
	3	2.096137819	2.057106979	1.892027599	
	4	1.886126508	1.990747670	1.934215595	
	5	1.884737514	2.033160228	1.984874866	
C_l		0.3742055381	0.3593454516	0.3212144442	up to 5 substitution
R_l		0.9081854455	0.9133037928	0.9218172536	By eqn.2
check $w_{li}m_{li}$		130(exact)	180.0000001=180	119.9999997=120	

The above analysis for a multiple constraint problem of a redundant system can be outlined as follows: All the given constraints can be classified into two groups, the active constraints and non-active constraints. Only the active constraints contribute the solutions of the problem and the non-active constraints can be discarded without any effect to the solution of the problem. The procedures for solving the multiple constraint reliability problem may be given in the following steps:

1) Solve the problem with each constraint separately and obtain k sets of M_{li} and R_l.

2) Classify the given constraints into active and non-active with equation (9) and (8).

3) If W_j (corresponding to the smallest one of R_i i.e. R_j) is the only strictest constraint, then the corresponding set of M_{ji} and R_j are the solutions of the problem.

4) If we have some active constraints (including the a strictest one (or ones) and some other active constraints), then we can solve equation (13) for C_1, and substitute C_1 into eqn.(12), we get

a set of m_i.

5) Round the ideal solution of m_i to integral numbers and substitute them into equation (2) to get the answer of the problem.

6) However, if the exact integral solution is required then the result of the above step should be checked by integral program or other means.

APPLICATION WITH NUMERICAL EXAMPLES

Example. Optimize the reliability of an attitude control system for a scientific satellite in 5 stages in series by means of parallel redundancy at the stages. The data of this problem are given in Table 1.

Solution. First of all we solve the problem for each single constraint separately and obtain the results as shown in Table 2.

From the above results, $R_1 < R_2 < R_3$, so constraint 1 (W_1) looks like the strictest one.

Check constraints 2 and 3 by substituting m_{li} into eqn (7), we have

$$\sum_{i=1}^{5} w_{2i} m_{li} = 178.0410023 < 180 \qquad \text{OK}$$

and

$$\sum_{i=1}^{5} w_{3i} m_{li} = 118.7829297 < 120 \qquad \text{OK}$$

The above checked results show that constrains 2 and 3 are non-active and can be discharged. That means constraint 1 is the only strictest one, so the corresponding set of m_{li} and R_1 are the ideal solutions of the problem.

Finally we have to round m_i to integral numbers. By a few trials, we get a set of m_i as 2, 3, 2, 2, 2.

Check

$$\sum_{i=1}^{5} w_{li} m_i = 124 < 130 \qquad \text{OK}$$

$$\sum_{i=1}^{5} w_{2i} m_i = 166.5 < 180 \quad \text{OK}$$

$$\sum_{i=1}^{5} w_{3i} m_i = 118 < 120 \qquad \text{OK}$$

$$R_1 = \prod_{i=1}^{5} \left(1 - q_i^{m_i}\right) = 0.8734147535 \quad \text{(the answer)}$$

REFERENCES

1. F. A. Tillman, C. L. Huang, W. Kuo, "Optimization Techniques for System Reliability with Redundancy — A review", *IEEE Transaction Reliability*, R-26, (3), 1977: 148 — 155

2. Shu Song-gui, The Synthesis of the Most Reliable Control Systems, *ACTA Automatica Sinica*, 6(1), 1980