Hybrid Estimation of State and Input for Linear Continuous Time-varying Systems: A Game Theory Approach[1)]

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Abstract The H_∞ hybrid estimation problem for linear continuous time-varying systems is investigated in this paper, where estimated signals are linear combination of state and input. Design objective requires the worst-case energy gain from disturbance to estimation error be less than a prescribed level. Optimal solution of the hybrid estimation problem is the saddle point of a two-player zero sum differential game. Based on the differential game approach, necessary and sufficient solvable conditions for the hybrid estimation problem are provided in terms of solutions to a Riccati differential equation. Moreover, one possible estimator is proposed if the solvable conditions are satisfied. The estimator is characterized by a gain matrix and an output mapping matrix that reflects the internal relations between the unknown input and output estimation error. Both state and unknown inputs estimation are realized by the proposed estimator. Thus, the results in this paper are also capable of dealing with fault diagnosis problems of linear time-varying systems. At last, a numerical example is provided to illustrate the proposed approach.

Key words Time-varying system, input estimation, game theory, Riccati equation

1 Introduction

When estimated signal includes both state and unknown input of the system, estimation problem is referred to as state and input hybrid estimation (in the following, only hybrid estimation will be used for brevity). Hybrid estimation is originated from need of practical application and theory[1]. One practical example is load current estimation of uninterruptible power supply (UPS), where load current is a linear function of capacitor voltage (state) and back electromotive force (unknown input)[2]. From theoretical view point, either state observation (including filtering, smoothing and prediction) or deconvolution (input estimation) is just a special case of hybrid estimation. Both of the former two can be treated in the framework of hybrid estimation. Therefore, research on hybrid estimation is more general. Fault diagnosis is another important related area of hybrid estimation. Scheme of fault diagnosis can be designed based on hybrid estimation approach since fault signal can be treated as unknown input.

For the past decade, H_∞ optimization-based estimation is an active research area[3~5]. Differential game-theory approach is one of the main time-domain approaches since H_∞ estimation is a min-max problem in essential. Differential game-theory approach can directly deduce estimator′s design method from the performance specification and therefore is a constructive approach. Moreover, the existence conditions of the proposed estimator are necessary and sufficient to achieve the least conservativeness. Differential game-theory approach is also capable of dealing with time-varying problems. Banavar and Speyer[6] first investigated H_∞ filtering and smoothing for continuous linear time-varying (LTV) systems using differential game-theory approach. Latter, discrete differential game-theory approach was applied to H_∞ filtering for discrete LTV systems[7]. Recently, in [8] differential game-theory approach was considered for discrete H_∞ deconvolution, where estimation error was $\boldsymbol{e} = \boldsymbol{Lx} - \boldsymbol{u}$, *i.e.*, $\boldsymbol{e}$ is the residual between linear combination of states and unknown input $\boldsymbol{u}$. Design method of matrix L, however, was not provided in that paper and their method is uncompleted. Other related research of H_∞ hybrid estimation is introduced in the following. Optimal performance was first presented for continuous LTV system by differential game-theory approach[9]. Khargonekar *et al.* gave results on H_2/H_∞ hybrid estimation for continuous linear time-invariant (LTI) systems[10]. In [11], H_∞ filtering was explored, where uncertain initial state was deemed as a fictitious extern input and the H_∞ filtering was converted to an equivalent hybrid estimation problem. At last, Cuzzola and Ferrante proposed LMI conditions for H_2 estimation for discrete LTI systems[1]. They also illustrated explicitly the theoretic and practical sense of hybrid estimation.

Above research on hybrid estimation mostly focused on LTI systems. Basar investigated continuous LTV H_∞ hybrid estimation, however, construction of the estimator was not discussed[9]. In view of

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most systems being time-varying in real world, a class of basic hybrid estimation problem – H_∞ hybrid estimation for continuous LTV systems is explored in this paper. Both necessary and sufficient existence conditions for the hybrid estimator and parameterization design approach was presented based on differential game-theory approach. Noting that hybrid estimation was also discussed in [9] using differential game-theory approach, we emphasize that their approach is different from our method since different quasi-performance is formulated from differential game to LQ control. Moreover, construction method for the estimator was not provided in [9] as mentioned previously.

2 Problem statement

Let us consider the following LTV system Σ_1

$$\begin{cases} \dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t) + B(t)\boldsymbol{u}(t), \quad \boldsymbol{x}(0) = 0 \\ \boldsymbol{y}(t) = C(t)\boldsymbol{x}(t) + D(t)\boldsymbol{u}(t) + \boldsymbol{v}(t) \end{cases} \tag{1}$$

where $\boldsymbol{x} \in \Re^n$ is state vector, $\boldsymbol{y} \in \Re^m$ is measured output, $\boldsymbol{u} \in \Re^p$ is unknown input, $\boldsymbol{v} \in \Re^q$ is measurement noise, matrices A, B, C, and D are time-varying parameter matrices with appropriate dimensions. Estimated signal $\boldsymbol{z} \in \Re^r$ is described as

$$\boldsymbol{z}(t) = L_{\boldsymbol{x}}(t)\boldsymbol{x}(t) + L_{\boldsymbol{u}}(t)\boldsymbol{u}(t) \tag{2}$$

where L_x and L_u are predefined parameter matrices. From the form of signal $\boldsymbol{z}$, we can define three classes of estimation problems as: 1) $L_x = 0$ corresponds to deconvolution; 2) $L_u = 0$ corresponds to state observation; 3) $L_x \neq 0$ and $L_u \neq 0$ corresponds to hybrid estimation. It is seen that either deconvolution or state observation is a special case of hybrid estimation. By (2) estimation of signal $\boldsymbol{z}$ has encompassed fault information if $\boldsymbol{u}$ is fault. Scheme of fault diagnosis can then be designed based on hybrid estimation.

Object of $\boldsymbol{H_\infty}$ hybrid estimation. Considering system (1), (2) and given by measured output y, design estimator $\Im$ to reconstruct signal $\boldsymbol{z}$ satisfying the following L_2 induced norm performance index

$$\sup_{\boldsymbol{u},\, \boldsymbol{v} \in L_2} \frac{\|\boldsymbol{z} - \hat{\boldsymbol{z}}\|^2_{[0,T]}}{\|\boldsymbol{u}\|^2_{[0,T]} + \|\boldsymbol{v}\|^2_{[0,T]}} < \gamma^2 \tag{3}$$

where $\|\boldsymbol{p}\|^2_{[0,T]}$ denotes L_2 norm. For a vector $\boldsymbol{p}$, the relation $\|\boldsymbol{p}\|^2_{[0,T]} = \int_0^{\mathrm{T}} \boldsymbol{p}^{\mathrm{T}}(t)\boldsymbol{p}(t)\mathrm{d}t$ holds, $\hat{\boldsymbol{z}}$ is the estimate, $\gamma > 0$ is a prescribed scalar.

3 Main results of $\boldsymbol{H_\infty}$ hybrid estimation

By (3) and output equation in (1), a new cost function is defined as

$$J(\boldsymbol{u}, \boldsymbol{y}, \hat{\boldsymbol{z}}) = \|\boldsymbol{z} - \hat{\boldsymbol{z}}\|^2_{[0,T]} - \gamma^2 \lfloor \|\boldsymbol{u}\|^2_{[0,T]} + \|\boldsymbol{y} - C\boldsymbol{x} - D\boldsymbol{u}\|^2_{[0,T]} \rfloor \tag{4}$$

H_∞ hybrid estimation is then equivalent to the following 'minmax' problem

$$\inf_{\hat{\boldsymbol{z}}} \sup_{\boldsymbol{y}} \sup_{\boldsymbol{u}} J(\boldsymbol{u}, \boldsymbol{y}, \hat{\boldsymbol{z}}) \tag{5}$$

This is a two-player zero sum differential game, where disturbances $\boldsymbol{u}, \boldsymbol{y}$ and estimator $\hat{\boldsymbol{z}}$ are the two opponents. The former try to make performance function J maximize and the latter acts counter. By differential game theory, the optimal solution to H_∞ hybrid estimation is exactly the saddle point of the differential game $(\boldsymbol{u}^*, y^*, \hat{\boldsymbol{z}}^*)$.

$$J(\boldsymbol{u}, \boldsymbol{y}, \hat{\boldsymbol{z}}^*) \leqslant J(\boldsymbol{u}^*, \boldsymbol{y}^*, \hat{\boldsymbol{z}}^*) \leqslant J(\boldsymbol{u}^*, \boldsymbol{y}^*, \hat{\boldsymbol{z}}) \tag{6}$$

For notation compact, define two coefficient matrices, the estimator′s gain matrix and output mapping matrix as follows:

$$\Delta = \gamma^2(I + D^{\mathrm{T}}D) - L_u^{\mathrm{T}}L_u, \quad F = \gamma^2 C^{\mathrm{T}}D - L_x^{\mathrm{T}}L_u$$
$$K = (\gamma^2 QC^{\mathrm{T}} + BD^{\mathrm{T}})(I + DD^{\mathrm{T}})^{-1}, \quad H = L_u D^{\mathrm{T}}(I + DD^{\mathrm{T}})^{-1}$$

Theorem 1. (Continuous LTV $\boldsymbol{H_\infty}$ hybrid estimation on finite horizon)

Consider continuous LTV system (1), (2) and performance (3) on finite horizon $[0, T]$. Given a scalar $\gamma > 0$, H_∞ hybrid estimation is solvable if and only if there exists symmetric matrix function $Q(t)$, $\forall t \in [0, T]$ such that the following conditions hold:

$$\gamma^2(I + D^{\mathrm{T}}D) - L_u^{\mathrm{T}}L_u > 0 \tag{7}$$

$$\dot{Q} = (A - B\Delta^{-1}F^{\mathrm{T}})Q + Q(A - B\Delta^{-1}F^{\mathrm{T}})^{\mathrm{T}} - Q(\gamma^2 C^{\mathrm{T}}C - L_x^{\mathrm{T}}L_x - F\Delta^{-1}F^{\mathrm{T}})Q + B\Delta^{-1}B^{\mathrm{T}}, \ \ Q(0) = 0 \quad (8)$$

If H_∞ hybrid estimation is solvable, one possible H_∞ suboptimal hybrid estimator satisfying performance (3) is given by

$$\begin{cases} \dot{\hat{\boldsymbol{x}}} = A\hat{\boldsymbol{x}} + K(\boldsymbol{y} - C\hat{\boldsymbol{x}}), \quad \hat{\boldsymbol{x}}(0) = 0 \\ \hat{\boldsymbol{z}} = L_x\hat{\boldsymbol{x}} + H(\boldsymbol{y} - C\hat{\boldsymbol{x}}) \end{cases} \quad (9)$$

Remark 1. Define signal $\hat{\boldsymbol{z}}_x = L_x\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{z}}_u = H(\boldsymbol{y} - C\hat{\boldsymbol{x}})$. From the output equation in (9), we can obtain

$$\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}_x + \hat{\boldsymbol{z}}_u$$

i.e., hybrid estimated signal $\hat{\boldsymbol{z}}$ is made up of two parts including state observation $\hat{\boldsymbol{z}}_x$ and unknown input estimation part $\hat{\boldsymbol{z}}_u$. Thus, the estimator (9) can realize state and unknown input estimation simultaneously.

In Theorem 1, the existence conditions of H_∞ hybrid estimator are equivalent to the solvability of Riccati differential equation (RDE) (8). The state equation of H_∞ hybrid estimator owns an observer structure and its gain matrix can be constructed from solution to RDE. On the other hand, state observation part $L_x\hat{\boldsymbol{x}}$ is included in the output equation of the estimator. All above conclusions are consistent with those on standard H_∞ filtering[3]. Additionally, inequality (7) is a new added constraint due to deconvolution part in H_∞ hybrid estimation. Parameter matrices to describe information on unknown input, D and L_u, restrict the rang of minimum which the disturbance attenuation level γ can attain. Dynamics of H_∞ hybrid estimator (9) is completely characterized by gain matrix K and output mapping matrix H which reflects the linear mapping relation from output estimation error (innovation) $\boldsymbol{y} - C\hat{\boldsymbol{x}}$ to unknown input. At every moment, the output estimation error is used to update the state of the estimator and to provide unknown input estimation through output mapping matrix simultaneously.

From Theorem 1, it is easy to draw conclusions on H_∞ filtering and deconvolution. In case of standard H_∞ filtering ($L_u = 0$ and $D = 0$), let $P = \gamma^2 Q$ hold. It is easy to verify that Theorem 1 is consistent with standard H_∞ filtering theorem[3]. In case of H_∞ deconvolution ($L_x = 0$), condition (7) in Theorem 1 is kept, but RDE (8) is reduced to

$$\dot{Q} = (A - \gamma^2 B\Delta^{-1}D^{\mathrm{T}}C)Q + Q(A - \gamma^2 B\Delta^{-1}D^{\mathrm{T}}C)^{\mathrm{T}} - Q(\gamma^2 C^{\mathrm{T}}C - \gamma^4 C^{\mathrm{T}}D\Delta^{-1}D^{\mathrm{T}}C)Q + B\Delta^{-1}B^{\mathrm{T}} \quad (10)$$

H_∞ suboptimal deconvolution filter is given by

$$\begin{cases} \dot{\hat{\boldsymbol{x}}} = A\hat{\boldsymbol{x}} + K(\boldsymbol{y} - C\hat{\boldsymbol{x}}), \quad \hat{\boldsymbol{x}}(0) = 0 \\ \hat{\boldsymbol{z}} = H(\boldsymbol{y} - C\hat{\boldsymbol{x}}) \end{cases} \quad (11)$$

The physical meaning of output mapping matrix H is clearer in deconvolution filter (11), *i.e.*, it just reflects the linear mapping relation from output estimation error $\boldsymbol{y} - C\hat{\boldsymbol{x}}$ to the unknown input.

4 Differential game theory solution to H_∞ hybrid estimation

Proof of Theorem 1.

Step 1. To seek optimal solution $\boldsymbol{u}^*$ of $\boldsymbol{u}$.

According to performance (4), define a new cost function as

$$L(\boldsymbol{u}, \boldsymbol{y}, \hat{\boldsymbol{z}}) = \|\boldsymbol{z} - \hat{\boldsymbol{z}}\|^2 - \gamma^2\lfloor\|\boldsymbol{u}\|^2 + \|\boldsymbol{y} - C\boldsymbol{x} - D\boldsymbol{u}\|^2\rfloor \quad (12)$$

where $\| \ \|$ stands for Euclidian norm. For any vector $\boldsymbol{p}$, relation $\|\boldsymbol{p}\| = (\boldsymbol{p}^{\mathrm{T}}\boldsymbol{p})^{1/2}$ holds.

The Hamilton function is

$$H(\boldsymbol{u}, \boldsymbol{y}, \hat{\boldsymbol{z}}, \boldsymbol{\lambda}) = \frac{1}{2}L(\boldsymbol{u}, \boldsymbol{y}, \hat{\boldsymbol{z}}) + \boldsymbol{\lambda}^{\mathrm{T}}(A\boldsymbol{x} + B\boldsymbol{u}) \quad (13)$$

The first order necessary conditions are

$$\begin{cases} \dot{\boldsymbol{\lambda}} = -H_x \\ 0 = H_u \end{cases} \Rightarrow \begin{cases} \dot{\boldsymbol{\lambda}} = (\gamma^2 C^{\mathrm{T}}C - L_x^{\mathrm{T}}L_x)\boldsymbol{x} + F\boldsymbol{u} + L_x^{\mathrm{T}}\hat{\boldsymbol{z}} - \gamma^2 C^{\mathrm{T}}\boldsymbol{y} - A^{\mathrm{T}}\boldsymbol{\lambda} \\ 0 = -F^{\mathrm{T}}\boldsymbol{x} - \Delta\boldsymbol{u} + \gamma^2 D^{\mathrm{T}}\boldsymbol{y} - L_u^{\mathrm{T}}\hat{\boldsymbol{z}} + B^{\mathrm{T}}\boldsymbol{\lambda} \end{cases} \quad (14)$$

The Border condition is

$$\boldsymbol{\lambda}(T) = 0 \quad (15)$$

One necessary condition for existence of saddle point is given by

$$\Delta > 0 \quad (16)$$

It follows from (14) that
$$\boldsymbol{u}^* = \Delta^{-1}(B^{\mathrm{T}}\boldsymbol{\lambda} - F^{\mathrm{T}}\boldsymbol{x} + \gamma^2 D^{\mathrm{T}}\boldsymbol{y} - L_u^{\mathrm{T}}\hat{\boldsymbol{z}}) \tag{17}$$

Substituting $\boldsymbol{u}^*$ into (1) and (14), we obtain the following Hamilton accompany system
$$\begin{bmatrix}\dot{\boldsymbol{x}}\\ \dot{\boldsymbol{\lambda}}\end{bmatrix} = \begin{bmatrix} A - B\Delta^{-1}F^{\mathrm{T}} & B\Delta^{-1}B^{\mathrm{T}}\\ \gamma^2 C^{\mathrm{T}}C - L_x^{\mathrm{T}}L_x - F\Delta^{-1}F^{\mathrm{T}} & -A^{\mathrm{T}} + F\Delta^{-1}B^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{\lambda}\end{bmatrix} + \begin{bmatrix} B\Delta^{-1}(\gamma^2 D^{\mathrm{T}}\boldsymbol{y} - L_u^{\mathrm{T}}\hat{\boldsymbol{z}})\\ F\Delta^{-1}(\gamma^2 D^{\mathrm{T}}\boldsymbol{y} - L_u^{\mathrm{T}}\hat{\boldsymbol{z}}) + (L_x\hat{\boldsymbol{z}} - \gamma^2 C^{\mathrm{T}}\boldsymbol{y})\end{bmatrix} \tag{18}$$

The above equation is a linear two-point boundary value problem (TPBVP), whose solution is
$$\boldsymbol{x}^* = \hat{\boldsymbol{x}} + Q\boldsymbol{\lambda} \tag{19}$$

Setting $\hat{\boldsymbol{x}}(0) = 0$, the border condition $Q(0) = 0$ of RDE is then obvious. Taking differentiation in both sides of (19) and noting (18), we have the following equation
$$\begin{cases}\dot{\hat{\boldsymbol{x}}} = A\hat{\boldsymbol{x}} + \gamma^2[(B - QF)\Delta^{-1}D^{\mathrm{T}} + QC^{\mathrm{T}}](\boldsymbol{y} - C\hat{\boldsymbol{x}}) + [(B - QF)\Delta^{-1}L_u^{\mathrm{T}} + QL_x^{\mathrm{T}}](L_x\hat{\boldsymbol{x}} - \hat{\boldsymbol{z}})\\ \dot{Q} = (A - B\Delta^{-1}F^{\mathrm{T}})Q + Q(A - B\Delta^{-1}F^{\mathrm{T}})^{\mathrm{T}} - Q(\gamma^2 C^{\mathrm{T}}C - L_x^{\mathrm{T}}L_x - F\Delta^{-1}F^{\mathrm{T}})Q + B\Delta^{-1}B^{\mathrm{T}}\end{cases} \tag{20}$$

where the lower equation is exactly RDE (8) in Theorem 1.

If (20) is used to solve differential game, the result will be $\hat{\boldsymbol{z}}^* = L_x\hat{\boldsymbol{x}}$, *i.e.*, the optimal solution of the estimator′s output signal $\hat{\boldsymbol{z}}$ is only state observation without any information about unknown input. In order to tackle this difficulty, a key matrix transformation is introduced here.
$$\gamma^2[(B - QF)\Delta^{-1}D^{\mathrm{T}} + QC^{\mathrm{T}}] - [(B - QF)\Delta^{-1}L_u^{\mathrm{T}} + QL_x^{\mathrm{T}}]H = (\gamma^2 QC^{\mathrm{T}} + BD^{\mathrm{T}})(I + DD^{\mathrm{T}})^{-1} \tag{21}$$

Using (21), the upper equation in (20) is equivalent to
$$\dot{\hat{\boldsymbol{z}}} = A\hat{\boldsymbol{x}} + (\gamma^2 QC^{\mathrm{T}} + BD^{\mathrm{T}})(I + DD^{\mathrm{T}})^{-1}(\boldsymbol{y} - C\hat{\boldsymbol{x}}) + [(B - QF)\Delta^{-1}L_u^{\mathrm{T}} + QL_x^{\mathrm{T}}][H(\boldsymbol{y} - C\hat{\boldsymbol{x}}) + L_x\hat{\boldsymbol{x}} - \hat{\boldsymbol{z}}] \tag{22}$$

By comparing the upper equation in (20) with (22), it is obvious that the matrix transformation (21) leads to the plus and minus item $[(B - QF)\Delta^{-1}L_u^{\mathrm{T}} + QL_x^{\mathrm{T}}]H(\boldsymbol{y} - C\hat{\boldsymbol{x}})$ simultaneous in the upper equation in (20). The final effect is to introduce unknown input estimation part in the estimate $\hat{\boldsymbol{z}}$.

Step 2. To seek optimal solution $\boldsymbol{y}^*$, $\hat{\boldsymbol{z}}^*$ of y and $\hat{\boldsymbol{z}}$, respectively.

Define $\tilde{\boldsymbol{y}} = \boldsymbol{y} - C\hat{\boldsymbol{x}}$, $\tilde{\boldsymbol{z}} = H(\boldsymbol{y} - C\hat{\boldsymbol{x}}) + L_x\hat{\boldsymbol{x}} - \hat{\boldsymbol{z}}$, and denote $L(\boldsymbol{u}^*, y, \hat{\boldsymbol{z}})$ and $\dot{\boldsymbol{\lambda}}$ in (18) in the form of $\boldsymbol{\lambda}$, $\tilde{\boldsymbol{y}}$, $\tilde{\boldsymbol{z}}$. Through a rather involved algebraic operation, we can obtain
$$\int_0^{\mathrm{T}} L(\boldsymbol{u}^*, y, \hat{\boldsymbol{z}})\mathrm{d}t + \int_0^{\mathrm{T}} \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\lambda}^{\mathrm{T}}Q\boldsymbol{\lambda})\mathrm{d}t = -\gamma^2\|\tilde{\boldsymbol{y}}\|_M^2 + \|\tilde{\boldsymbol{z}}\|_N^2 \tag{23}$$

where $M = (I + DD^{\mathrm{T}})^{-1}$ and $N = (I + L_u\Delta^{-1}L_u^{\mathrm{T}})$. Noting that $\int_0^{\mathrm{T}} \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\lambda}^{\mathrm{T}}Q\boldsymbol{\lambda})\mathrm{d}t$ is zero, we have
$$J(\boldsymbol{u}^*, y, \hat{\boldsymbol{z}}) = -\gamma^2\|\boldsymbol{y} - C\hat{\boldsymbol{x}}\|_M^2 + \|H(\boldsymbol{y} - C\hat{\boldsymbol{x}}) + L_x\hat{\boldsymbol{x}} - \hat{\boldsymbol{z}}\|_N^2 \tag{24}$$

After $\boldsymbol{u}^*$ is determined, the optimal game solution to the next game object $\min\limits_{\hat{\boldsymbol{z}}}\max\limits_{\boldsymbol{y}} J(\boldsymbol{u}^*, y, \hat{\boldsymbol{z}})$ is
$$\boldsymbol{y}^* = c\hat{\boldsymbol{x}}, \quad \hat{\boldsymbol{z}}^* = H(\boldsymbol{y} - C\hat{\boldsymbol{x}}) + L_x\hat{\boldsymbol{x}} \tag{25,26}$$

Step 3. To verify $(\boldsymbol{u}^*, \boldsymbol{y}^*, \hat{\boldsymbol{z}}^*)$ satisfy saddle point condition (6)

It follows from (24)∼(26) and definition of $\boldsymbol{u}^*$ that
$$J(\boldsymbol{u}, \boldsymbol{y}, \hat{\boldsymbol{z}}^*) \leqslant J(\boldsymbol{u}^*, \hat{y}^*, \hat{\boldsymbol{z}}^*) \leqslant J(\boldsymbol{u}^*, \boldsymbol{y}^*, \hat{\boldsymbol{z}}) \tag{27}$$

Finally, substituting (25) and (26) into (20) will lead to estimator (9). By synthesizing RDE in the necessary condtions (16) and (20), the proof of Theorem 1 is evident. □

5 Numerical example

Let us consider a fault system Σ_1 described by (1) and (2), where the parameter matrices of the system are as follows
$$A = \begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix},\ \boldsymbol{B} = \begin{bmatrix}0\\ 1\end{bmatrix},\ \boldsymbol{C} = [3 \quad 3],\ D = 7,\ \boldsymbol{L}_x = [1 \quad 1],\ L_u = 0.7$$

where $\boldsymbol{u}$ denotes the fault signal, v is band-limited white noise with the value of power spectrum being 0.1. The appropriate choice of γ is 1.7. By Theorem 1, we need to compute RDE (9) and its stead$\boldsymbol{y}$-state solution is $Q = \mathrm{diag}[0.0017 \quad 0.0025]$. Next, the gain matrix and output mapping matrix of H_∞ hybrid estimator are constructed from matrix function Q as $\boldsymbol{K} = [0.0003 \quad 0.1404]^{\mathrm{T}}$, $H = 0.0980$. The hybrid estimator′s output signal $\hat{\boldsymbol{z}}$ is composed of two parts, *i.e.*, state estimation $\hat{\boldsymbol{z}}_x = L_x\hat{\boldsymbol{x}}$ and unknown input estimated part $\hat{\boldsymbol{z}}_u = H(\boldsymbol{y} - C\hat{\boldsymbol{x}})$.

Figs. 1∼6 show the simulation results. Figs. 1 and 2 are waveforms of fault signal $\boldsymbol{u}$ and measurement noise v, respectively. Here, the fault signal is a step function. Waveforms of signals $\boldsymbol{z}$, $\hat{\boldsymbol{z}}$ and $(\boldsymbol{z} - \hat{\boldsymbol{z}})$ are shown in Figs. 5 and 6, respectively. It is seen that the proposed H_∞ hybrid estimator, designed by Theorem 1, reconstructed the estimated signal $\boldsymbol{z}$ in a high precision. In essential, the estimation of signal $\boldsymbol{z}$ is realized through estimating state and input respectively. Such a conclusion is clear from the results of Figs. 3 and 4, which demonstrate that both state observation and input estimation are preconditions of the hybrid estimation. At last, the proposed estimator can provide exact fault information in fault diagnosis since it can reconstruct unknown input.

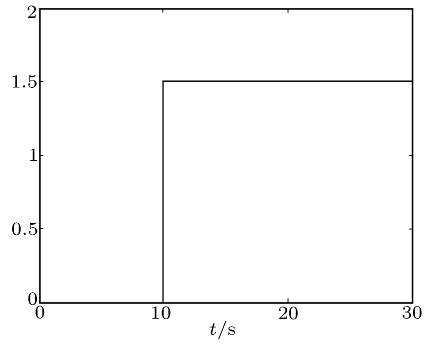

Fig. 1 Fault signal u

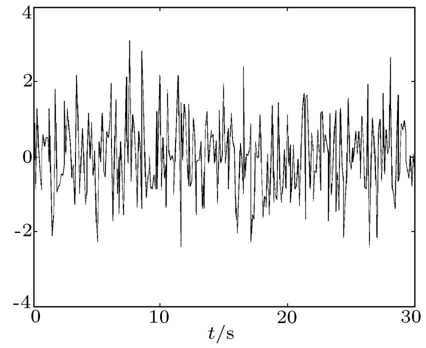

Fig. 2 Measurement noise v

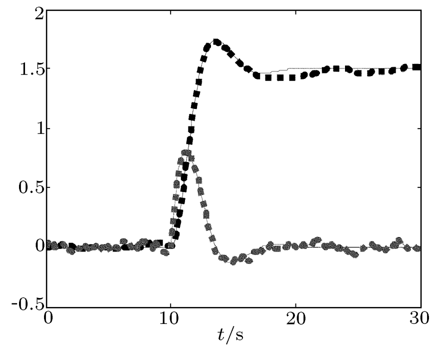

Fig. 3 State x (solid) and estimation $\hat{x}$ (dot)

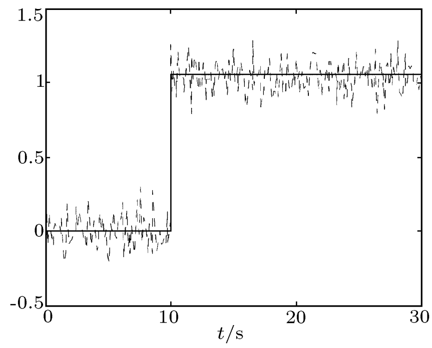

Fig. 4 Signal z_u (solid) and $\hat{z}_u$ (dot)

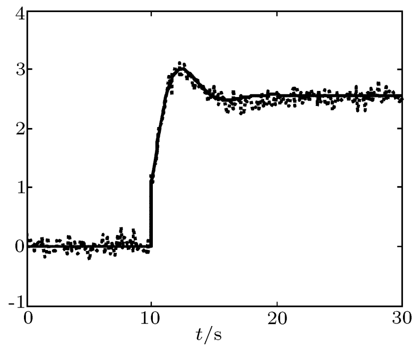

Fig. 5 Signal z (solid) and $\hat{z}$ (dot)

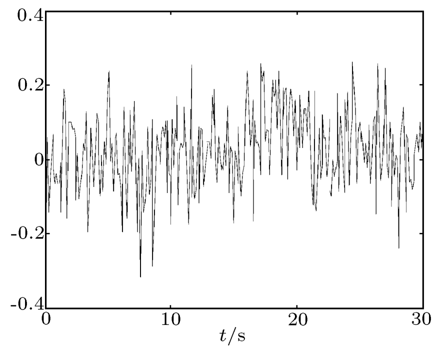

Fig. 6 Hybrid estimation error $z - \hat{z}$

6 Conclusion

A class of state and input simultaneous estimation problem – hybrid estimation is investigated in this paper. Based on differential game theory approach, necessary and sufficient solvable conditions for continuous time-varying H_∞ hybrid estimation are proposed in terms of solution to a Riccati differential equation. One possible suboptimal H_∞ hybrid estimator is provided when the solvable conditions are satisfied. The estimator is characterized by the gain matrix and output mapping matrix, where the latter reflects the linear mapping relation from the output estimation error to the unknown input. Simulation results of the numerical example demonstrate the performance of the proposed H_∞ hybrid estimator is superior. With the input estimation ability, one immediate application area of the proposed estimator is fault diagnosis.

Additionally, same conclusion can be drawn *via* the bounded real lemma, whose deduction procedure is simple but only sufficient solvable conditions can be obtained since there exists freedom to choose the gain matrix. By contrast, these solvable conditions are not only necessary but also sufficient as shown in Theorem 1.

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