The Maximum Principle for Fully Coupled Forward-backward Stochastic Control System[1)]

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Abstract The maximum principle for fully coupled forward-backward stochastic control system in the global form is proved, under the assumption that the forward diffusion coefficient does not contain the control variable, but the control domain is not necessarily convex.

Key words The maximum principle, fully coupled forward-backward stochastic control system, spike variation

1 Introduction

It is well known that the maximum principle, the necessary condition of the optimal control, which is a milestone-like result in the optimal control theory was established for the deterministic control system by Pontryakin′s group[1] in the 1950′s and 1960′s. Since then, a lot of work has been done on the forward stochastic control system such as Bensoussan A[2], Bismut J M[3], Kushner H J[4], Peng S[5] *etc.* Peng[6] firstly studied one kind of forward-backward stochastic control system which had the economic background and could be used to study the recursive utility problem in the mathematical finance. He obtained the maximum principle for this kind of control system with the control domain being convex. And then Xu[7] studied the non-convex control domain case and obtained the corresponding maximum principle. But he assumed that the diffusion coefficient in the forward control system does not contain the control variable.

However, the forward-backward control system they studied is not fully coupled, that is, their forward system does not contain the backward state variables. As for the fully coupled forward-backward stochastic control systems, it is difficult to ensure the existence and uniqueness of the solution with an arbitrarily fixed long time duration for a given admissible control. To overcome this difficulty, we need the result of the fully coupled forward-backward stochastic differential equations (FBSDE in short). Using partial differential equations method, Ma J, Protter and Yong J[8] obtained the existence and uniqueness of FBSDE. But they required the forward stochastic differential equations to be non-degenerate and the coefficients not to be randomly disturbed. Hu Y and Peng S[9] proved the existence and uniqueness of solution to FBSDE when the forward and backward variables had the same dimensions under some monotonic assumptions. Peng and Wu[10] extended their result to different dimensional FBSDE and weakened the monotonic assumptions so that the results could be used widely. In this paper we use the result in [10] to overcome the above difficulty we mentioned.

Another difficulty to get the maximum principle for the fully coupled forward-backward stochastic control system with non-convex control domain is how to use the spike variational method for the variational equations with enough higher estimate order and use the duality technique for the adjoint equations. In our paper we use the result of FBSDE to ensure the existence and uniqueness of the solution to the adjoint forward-backward stochastic differential equations which are obtained by using duality technique to the variational equation. And also we use the technique of FBSDE to obtain the estimate for the solution of the variational equations and then for the difference between the solution of the perturbed state equations with the sum of the solution of the optimal state equations and the variational equations with enough higher order.

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Under the assumption that the forward diffusion coefficient does not contain the control variable we obtain in our paper the maximum principle for the fully coupled forward-backward stochastic control system with non-convex control domain. We hope our results can have some applications in practice such as in mathematical finance.

This paper is organized as follows. In section 2, we state the problem and our main assumptions. In section 3, we study the variational equations and the variational inequality. In section 4, we obtain the maximum principle in the global form for the fully coupled forward-backward stochastic control system.

2 Statement of the problem

Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $\{\mathcal{F}_t\}$ and $(\boldsymbol{B}_t)_{t\geqslant 0}$ be a R^d-valued standard Brownian motion. We assume $\mathcal{F}_t \doteq \sigma\{(\boldsymbol{B}_s), 0 \leqslant s \leqslant t\}$ and consider the following fully coupled forward-backward stochastic control system:

$$\begin{cases} d\boldsymbol{x}(t) = b(t, \boldsymbol{x}(t), \boldsymbol{y}(t), \boldsymbol{z}(t), \boldsymbol{v}(t))dt + \sigma(t, \boldsymbol{x}(t), \boldsymbol{y}(t), \boldsymbol{z}(t))d\boldsymbol{B}_t \\ -d\boldsymbol{y}(t) = f(t, \boldsymbol{x}(t), \boldsymbol{y}(t), \boldsymbol{z}(t), \boldsymbol{v}(t))dt - \boldsymbol{z}(t)d\boldsymbol{B}_t, \ 0 \leqslant t \leqslant T \\ \boldsymbol{x}(0) = \boldsymbol{x}_0, \quad \boldsymbol{y}(T) = \boldsymbol{h}(\boldsymbol{x}(T)) \end{cases} \tag{1}$$

where $(\boldsymbol{x}(\cdot), \boldsymbol{y}(\cdot), \boldsymbol{z}(\cdot)) \in R^n \times R^m \times R^{m\times d}$, $\boldsymbol{x}_0 \in R^n$, $T > 0$, $b : [0,T] \times R^n \times R^m \times R^{m\times d} \times R^k \longrightarrow R^n$, $\sigma : [0,T] \times R^n \times R^m \times R^{m\times d} \longrightarrow R^{n\times d}$, $f : [0,T] \times R^n \times R^m \times R^{m\times d} \times R^k \longrightarrow R^m$, $\boldsymbol{h} : R^n \longrightarrow R^m$.

Let $\mathcal{U}$ be a nonempty subset of R^k. We define the admissible control set $\mathcal{U}_{ad} \doteq \{v(\cdot) \in M^2(0,T;R^k);$ $\boldsymbol{v}(t) \in \mathcal{U}, 0 \leqslant t \leqslant T, a.e., a.s.\}$.

Our optimal control problem is to minimize the cost function:

$$J(\boldsymbol{v}(\cdot)) \doteq E[\int_0^{\mathrm{T}} l(t, \boldsymbol{x}(t), \boldsymbol{y}(t), \boldsymbol{z}(t), \boldsymbol{v}(t))dt + \Phi(\boldsymbol{x}(T)) + \gamma(\boldsymbol{y}(0))] \tag{2}$$

over $\mathcal{U}_{ad}$, where $l : [0,T] \times R^n \times R^m \times R^{m\times d} \times R^k \longrightarrow R$, $\Phi : R^n \longrightarrow R, \gamma : R^m \longrightarrow R$.

That is to say, we want to find a $u(\cdot)$, such that

$$J(\boldsymbol{u}(\cdot)) = \inf_{v(\cdot)\in\mathcal{U}_{ad}} J(\boldsymbol{v}(\cdot)) \tag{3}$$

An admissible control $\boldsymbol{u}(\cdot)$ is called an optimal control if it attains the minimum. (1) is called the state equation, the solution $(\boldsymbol{x}(\cdot), \boldsymbol{y}(\cdot), \boldsymbol{z}(\cdot))$ corresponding to $\boldsymbol{u}(\cdot)$ is called the optimal trajectory.

We are given an $m \times n$ full-rank matrix G and the notations:

$$\boldsymbol{\lambda} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix}, \quad A(t, \boldsymbol{\lambda}) = \begin{pmatrix} -\boldsymbol{G}^{\tau} f \\ \boldsymbol{G}b \\ \boldsymbol{G}\sigma \end{pmatrix} (t, \boldsymbol{\lambda})$$

where $\boldsymbol{G\lambda} = (\boldsymbol{G\sigma}_1, \boldsymbol{G\sigma}_2, \dots, \boldsymbol{G\sigma}_d)$. We use the usual inner product $\langle \cdot, \cdot \rangle$ and Euclidean norm $|\cdot|$ in R^n, R^m, and $R^{m\times d}$. We assume that

H_1:
$$\begin{cases} \text{i) } A(t,\boldsymbol{\lambda}) \text{ is uniformly Lipschitz with repect to } \boldsymbol{\lambda} \\ \text{ii) for each } \boldsymbol{\lambda} \in R^{n+m+m\times d}, \ A(t,\boldsymbol{\lambda}) \in M^2(0,T;R^{n+m+m\times d}), \ t \in [0,T] \\ \text{iii) } \boldsymbol{h}(\boldsymbol{x}) \text{ is uniformly Lipschitz with repect to } \boldsymbol{x} \in R^n \\ \text{iv) for } \boldsymbol{x} \in R^n, \ \boldsymbol{h}(\boldsymbol{x}) \in L^2(\Omega, \mathcal{F}_T, P; R^n) \end{cases}$$

The following monotonic conditions were firstly introduced in [10], and is the necessary assumption in this paper.

H_2:
$$\langle A(t,\boldsymbol{\lambda}) - A(t,\bar{\boldsymbol{\lambda}}), \boldsymbol{\lambda} - \bar{\boldsymbol{\lambda}} \rangle \leqslant -\beta_1 |\, G\hat{\boldsymbol{x}}\,|^2 - \beta_2 (|\, G^{\tau}\hat{\boldsymbol{y}}\,|^2 + |\, G^{\tau}\hat{\boldsymbol{z}}\,|^2)$$
$$\langle \boldsymbol{h}(\boldsymbol{x}) - \boldsymbol{h}(\bar{\boldsymbol{x}}), G(\boldsymbol{x} - \bar{\boldsymbol{x}}) \rangle \geqslant \mu_1 |\, G\hat{\boldsymbol{x}}\,|^2$$

or

H$_2'$:
$$\langle A(t,\boldsymbol{\lambda})-A(t,\bar{\boldsymbol{\lambda}}),\boldsymbol{\lambda}-\bar{\boldsymbol{\lambda}}\rangle \geqslant \beta_1 | G\hat{\boldsymbol{x}}|^2+\beta_2(| G^\tau \hat{\boldsymbol{y}}|^2+| G^\tau \hat{\boldsymbol{z}}|^2)$$
$$\langle \boldsymbol{h}(\boldsymbol{x})-\boldsymbol{h}(\bar{\boldsymbol{x}}), G(\boldsymbol{x}-\bar{\boldsymbol{x}})\rangle \leqslant -\mu_1 | G\hat{\boldsymbol{x}}|^2$$
$$\forall \boldsymbol{\lambda}=(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}), \bar{\boldsymbol{\lambda}}=(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}},\bar{\boldsymbol{z}}), \hat{\boldsymbol{x}}=\boldsymbol{x}-\bar{\boldsymbol{x}}, \hat{\boldsymbol{y}}=\boldsymbol{y}-\bar{\boldsymbol{y}}, \hat{\boldsymbol{z}}=\boldsymbol{z}-\bar{\boldsymbol{z}}$$

where β_1,β_2,μ_1 are nonnegative constants with $\beta_1+\beta_2>0, \beta_2+\mu_1>0$. Moreover, we have $\beta_1>0, \mu_1>0$ (resp. $\beta_2>0$), when $m>n$ (resp. $m<n$).

Lemma 1. For any given admissible control $\boldsymbol{u}(\cdot)$, we assume H$_1$ and H$_2$ (or H$_2'$) hold. Then equation (1) has the unique adapted solution $(\boldsymbol{x}(\cdot),\boldsymbol{y}(\cdot),\boldsymbol{z}(\cdot))\in M^2(0,T;R^{n+m+m\times d})$.

The proof under assumptions H$_1$ and H$_2$ was given in [10]. The proof under assumptions H$_1$ and H$_2'$ is similar. We also assume:

H$_3$:
$$\begin{cases} \text{i) } b,\boldsymbol{\lambda},f,h,l,\varPhi,\gamma \ \text{ are continuously differentiable} \\ \text{ii) the derivatives of } b,\boldsymbol{\lambda},f,h \text{ are bounded} \\ \text{iii) the derivatives of } l \text{ are bounded by } \ C(1+|\boldsymbol{x}|+|\boldsymbol{y}|+|\boldsymbol{z}|+|\boldsymbol{v}|) \\ \text{iv) the derivatives of } \varPhi \text{ and } \gamma \text{ are bounded by } C(1+|\boldsymbol{x}|) \text{ and } C(1+|\boldsymbol{y}|), \text{respectively} \end{cases}$$

3 Variational equations and variational inequality

Suppose $(\boldsymbol{u}(\cdot),\boldsymbol{x}(\cdot),\boldsymbol{y}(\cdot),\boldsymbol{z}(\cdot))$ is the solution to our optimal control problem. We introduce the spike variation with respect to $\boldsymbol{u}(\cdot)$ as follows:

$$\boldsymbol{u}^\varepsilon(t)\doteq\begin{cases} \boldsymbol{v}, \text{ if } \tau\leqslant t\leqslant\tau+\varepsilon \\ \boldsymbol{u}(t), \text{ otherwise}\end{cases} \tag{4}$$

where $\varepsilon>0$ is sufficiently small, $\boldsymbol{v}\in\mathcal{U}$ is an $\mathcal{F}_\tau-$ measurable random variable, and $\sup\limits_{\omega\in\varOmega}|\boldsymbol{v}(\omega)|<+\infty$, $0\leqslant t\leqslant T$.

Suppose $(\boldsymbol{x}^\varepsilon(\cdot),\boldsymbol{y}^\varepsilon(\cdot),\boldsymbol{z}^\varepsilon(\cdot))$ is the trajectory of (1) corresponding to $\boldsymbol{u}^\varepsilon(\cdot)$. We introduce the following variational equations:

$$\begin{cases} d\boldsymbol{x}^1(t)=[\boldsymbol{b}_x\boldsymbol{x}^1(t)+\boldsymbol{b}_y\boldsymbol{y}^1(t)+\boldsymbol{b}_z\boldsymbol{z}^1(t)+b(\boldsymbol{u}^\varepsilon(t))-b(\boldsymbol{u}(t))]dt+[\boldsymbol{\lambda}_x\boldsymbol{x}^1(t)+\boldsymbol{\lambda}_y\boldsymbol{y}^1(t)+\boldsymbol{\lambda}_z\boldsymbol{z}^1(t)]d\boldsymbol{B}_t \\ -d\boldsymbol{y}^1(t)=[f_x\boldsymbol{x}^1(t)+f_y\boldsymbol{y}^1(t)+f_z\boldsymbol{z}^1(t)+f(\boldsymbol{u}^\varepsilon(t))-f(\boldsymbol{u}(t))]dt-\boldsymbol{z}^1(t)d\boldsymbol{B}_t, \ 0\leqslant t\leqslant T \\ \boldsymbol{x}^1(0)=0, \quad \boldsymbol{y}^1(T)=h_x(\boldsymbol{x}(T))\boldsymbol{x}^1(T)\end{cases} \tag{5}$$

For convenience, we use the following notations $g_x=g_x(t,\boldsymbol{x}(t),\boldsymbol{y}(t),\boldsymbol{z}(t),\boldsymbol{u}(t))$, $g(\boldsymbol{u}^\varepsilon(t))=g(t,\boldsymbol{x}(t),\boldsymbol{y}(t),\boldsymbol{z}(t),\boldsymbol{u}^\varepsilon(t))$, $g(\boldsymbol{u}(t))=g(t,\boldsymbol{x}(t),\boldsymbol{y}(t),\boldsymbol{z}(t),\boldsymbol{u}(t))$, $g=b,\sigma,f,l$, respectively.

It is easy to know that there exists a unique adapted solution $(\boldsymbol{x}^1(t),\boldsymbol{y}^1(t),\boldsymbol{z}^1(t))\in R^n\times R^m\times R^{m\times d}$, $0\leqslant t\leqslant T$, satisfying the variational (5).

We want to give the estimate for the solution of the variational (5). For this we give one result.

Lemma 2. For the following stochastic differential equations

$$\begin{cases} d\tilde{\boldsymbol{x}}_t=(\boldsymbol{A}_t\tilde{\boldsymbol{x}}_t+a_t)dt+(\boldsymbol{C}_t\tilde{\boldsymbol{x}}_t+\boldsymbol{b}_t)d\boldsymbol{B}_t, \ 0\leqslant t\leqslant T \\ \tilde{\boldsymbol{x}}_0 \ =0\end{cases}$$

$\boldsymbol{A}.,\boldsymbol{C}.\in L^\infty(0,T;R^{n\times n})$, $\boldsymbol{a}.,\boldsymbol{b}.\in L^2(0,T;R^n)$, there exists a constant K_1, such that the unique solution satisfies

$$E[\sup_{0\leqslant t\leqslant T}|\tilde{\boldsymbol{x}}_t|^2]\leqslant K_1(E\int_0^{\mathrm{T}}|\boldsymbol{a}_s|^2\,ds+E\int_0^{\mathrm{T}}|\boldsymbol{b}_s|^2\,ds) \tag{6}$$

For the following backward stochastic differential equation:

$$\begin{cases} -d\tilde{\boldsymbol{y}}_t=(\boldsymbol{D}_t\tilde{\boldsymbol{y}}_t+\boldsymbol{E}_t\tilde{\boldsymbol{z}}_t+\boldsymbol{F}_t)dt-\tilde{\boldsymbol{z}}_t d\boldsymbol{B}_t, \ 0\leqslant t\leqslant T \\ \tilde{\boldsymbol{y}}_T=\boldsymbol{\xi}\end{cases}$$

where $\boldsymbol{D}.,\boldsymbol{E}. \in L^{\infty}(0,T;R^{m\times m})$, $\boldsymbol{F}. \in L^2(0,T;R^m)$, its adapted solution $(\tilde{\boldsymbol{y}}.,\tilde{\boldsymbol{z}}.)$ exists uniquely. And also there exists a constant K_2, such that

$$E[\sup_{0\leqslant t\leqslant T} \mid \tilde{\boldsymbol{y}}_t \mid^2] \leqslant K_2(E\int_0^{\mathrm{T}} \mid \tilde{\boldsymbol{z}}_s \mid^2 ds + E\int_0^{\mathrm{T}} \mid \boldsymbol{F}_s \mid^2 ds + E \mid \boldsymbol{\xi} \mid^2) \tag{7}$$

Proof. It can be easily proved by B-D-G inequality and the Gronwall′s inequality. □

Then we first have

Lemma 3. We assume H_1, H_2, and H_3 hold. Then

$$\sup_{0\leqslant t\leqslant T} E \mid \boldsymbol{y}^1(t) \mid^2 \leqslant C\varepsilon, \quad E\int_0^{\mathrm{T}} \mid \boldsymbol{z}^1(t) \mid^2 dt \leqslant C\varepsilon \tag{8}$$

$$E[\sup_{0\leqslant t\leqslant T} \mid \boldsymbol{x}^1(t) \mid^2] \leqslant C\varepsilon, \quad E[\sup_{0\leqslant t\leqslant T} \mid \boldsymbol{y}^1(t) \mid^2] \leqslant C\varepsilon \tag{9}$$

Proof. Using Itô formula to $\langle G\boldsymbol{x}^1(t),\boldsymbol{y}^1(t)\rangle$, we get

$$\begin{aligned} E\langle h_x(\boldsymbol{x}(T))\boldsymbol{x}^1(T), G\boldsymbol{x}^1(T)\rangle = {} & E\int_0^{\mathrm{T}} [\langle G(b_x\boldsymbol{x}^1(t) + b_y\boldsymbol{y}^1(t) + b_z\boldsymbol{z}^1(t)), \boldsymbol{y}^1(t)\rangle - \\ & \langle G^{\tau}(f_x\boldsymbol{x}^1(t) + f_y\boldsymbol{y}^1(t) + f_z\boldsymbol{z}^1(t)), \boldsymbol{x}^1(t)\rangle + \langle G^{\tau}(\sigma_x\boldsymbol{x}^1(t) + \sigma_y\boldsymbol{y}^1(t) + \sigma_z\boldsymbol{z}^1(t)), \boldsymbol{z}^1(t)\rangle]dt - \\ & E\int_0^{\mathrm{T}} \langle G^{\tau}(f(\boldsymbol{u}^{\varepsilon}) - f(u)), \boldsymbol{x}^1(t)\rangle dt + E\int_0^{\mathrm{T}} \langle G(b(\boldsymbol{u}^{\varepsilon}) - b(u)), \boldsymbol{y}^1(t)\rangle dt \end{aligned}$$

From the monotonic conditions H_2, we get

$$\begin{aligned} \mu_1 E\,|G\boldsymbol{x}^1(T)\,|^2 + \beta_1 E\int_0^{\mathrm{T}} \mid G\boldsymbol{x}^1(t) \mid^2 dt + \beta_2 E\int_0^{\mathrm{T}} (\mid G\boldsymbol{y}^1(t) \mid^2 + \mid G\boldsymbol{z}^1(t) \mid^2) dt \leqslant \\ E\int_0^{\mathrm{T}} \langle G(b(\boldsymbol{u}^{\varepsilon}) - b(u)), \boldsymbol{y}^1(t)\rangle dt - E\int_0^{\mathrm{T}} \langle G^{\tau}(f(\boldsymbol{u}^{\varepsilon}) - f(u)), \boldsymbol{x}^1(t)\rangle dt \end{aligned} \tag{10}$$

Using similar technique in [7], we can get the conclusion for the cases $m > n$ and $m \leqslant n$, respectively. □

However, the order of the estimate for $(\boldsymbol{x}^1(\cdot),\boldsymbol{y}^1(\cdot),\boldsymbol{z}^1(\cdot))$ is too low to get the variational inequality. We need to give a more elaborate estimate using the technique of FBSDE again.

Lemma 4. We assume H_1, H_2, and H_3 hold. Then we have

$$E\int_0^{\mathrm{T}} \mid \boldsymbol{x}^1(t) \mid^2 dt \leqslant C\varepsilon^{\frac{3}{2}} \tag{11}$$

$$E\int_0^{\mathrm{T}} \mid \boldsymbol{y}^1(t) \mid^2 dt \leqslant C\varepsilon^{\frac{3}{2}} \tag{12}$$

$$E\int_0^{\mathrm{T}} \mid \boldsymbol{z}^1(t) \mid^2 dt \leqslant C\varepsilon^{\frac{3}{2}} \tag{13}$$

Proof. By (10), we have

$$\begin{aligned} & \mu_1 E \mid G\boldsymbol{x}^1(T) \mid^2 + \beta_1 E\int_0^{\mathrm{T}} \mid G\boldsymbol{x}^1(t) \mid^2 dt + \beta_2 E\int_0^{\mathrm{T}} (\mid G^{\tau}\boldsymbol{y}^1(t) \mid^2 + \mid G^{\tau}\boldsymbol{z}^1(t) \mid^2) dt \leqslant \\ & E\int_0^{\mathrm{T}} \langle G(b(\boldsymbol{u}^{\varepsilon}) - b(u)), \boldsymbol{y}^1(t)\rangle dt - E\int_0^{\mathrm{T}} \langle G^{\tau}(f(\boldsymbol{u}^{\varepsilon}) - f(u)), \boldsymbol{x}^1(t)\rangle dt \leqslant \\ & E[\sup_{0\leqslant t\leqslant T} \mid \boldsymbol{x}^1(t) \mid \int_0^{\mathrm{T}} \mid G^{\tau}(f(\boldsymbol{u}^{\varepsilon}) - f(u)) \mid dt] + E[\sup_{0\leqslant t\leqslant T} \mid \boldsymbol{y}^1(t) \mid \int_0^{\mathrm{T}} \mid G(b(\boldsymbol{u}^{\varepsilon}) - b(u)) \mid dt] \leqslant \\ & [E(\sup_{0\leqslant t\leqslant T} \mid \boldsymbol{x}^1(t) \mid^2)]^{\frac{1}{2}} [E(\int_0^{\mathrm{T}} \mid G^{\tau}(f(\boldsymbol{u}^{\varepsilon}) - f(u)) \mid dt)^2)]^{\frac{1}{2}} + \end{aligned}$$

$$[E(\sup_{0\leqslant t\leqslant T} \mid \boldsymbol{y}^1(t)\mid^2)]^{\frac{1}{2}}[E(\int_0^{\mathrm{T}} \mid G(b(\boldsymbol{u}^\varepsilon)-b(u))\mid dt)^2)]^{\frac{1}{2}} \leqslant C\varepsilon^{\frac{3}{2}}$$

In the case of $m \geqslant n$, $\beta_1 > 0, \beta_2 \geqslant 0$ and $\mu_1 > 0$, we have

$$\mu_1 E \mid G\boldsymbol{x}^1(T)\mid^2 + \beta_1 E\int_0^{\mathrm{T}} \mid G\boldsymbol{x}^1(t)\mid^2 dt \leqslant C\varepsilon^{\frac{3}{2}}$$

Thus (11) is obtained. Using the method in Lemma 3, we can prove (12) and (13).

In the case of $m < n$, $\beta_1 \geqslant 0, \beta_2 > 0$ and $\mu_1 \geqslant 0$,

$$\beta_2 E\int_0^{\mathrm{T}} (\mid G^\tau \boldsymbol{y}^1(t)\mid^2 + \mid G^\tau \boldsymbol{z}^1(t)\mid^2) dt \leqslant C\varepsilon^{\frac{3}{2}}$$

so (12) and (13) are obtained. From (5), we have

$$\begin{aligned}
E\mid \boldsymbol{x}^1(t)\mid^2 = & E\{\int_0^t [b_x\boldsymbol{x}^1(s) + b_y\boldsymbol{y}^1(s) + b_z\boldsymbol{z}^1(s) + b(\boldsymbol{u}^\varepsilon(s)) - b(\boldsymbol{u}(s))]ds + \\
& \int_0^t [\sigma_x\boldsymbol{x}^1(s) + \sigma_y\boldsymbol{y}^1(s) + \sigma_z\boldsymbol{z}^1(s)]d\boldsymbol{B}_s\}^2 \leqslant \\
& 7[E(\int_0^{\mathrm{T}} b_x\boldsymbol{x}^1(s)ds)^2 + E(\int_0^{\mathrm{T}} b_y\boldsymbol{y}^1(s)ds)^2 + E(\int_0^{\mathrm{T}} b_z\boldsymbol{z}^1(s)ds)^2 + E(\int_0^{\mathrm{T}} \sigma_x\boldsymbol{x}^1(s)ds)^2 + \\
& E(\int_0^{\mathrm{T}} \sigma_y\boldsymbol{y}^1(s)ds)^2 + E(\int_0^{\mathrm{T}} \sigma_z\boldsymbol{z}^1(s)ds)^2 + E(\int_0^{\mathrm{T}} (b(\boldsymbol{u}^\varepsilon(s)) - b(\boldsymbol{u}(s)))ds)^2] \leqslant \\
& 7C(E\int_0^{\mathrm{T}} \mid \boldsymbol{x}^1(t)\mid^2 dt + E\int_0^{\mathrm{T}} \mid \boldsymbol{y}^1(t)\mid^2 dt + E\int_0^{\mathrm{T}} \mid \boldsymbol{z}^1(t)\mid^2 dt) + C\varepsilon^2 \leqslant \\
& 7CE\int_0^{\mathrm{T}} \mid \boldsymbol{x}^1(t)\mid^2 dt + C\varepsilon^{\frac{3}{2}}
\end{aligned}$$

By Gronwall′s inequality, we have $E\mid \boldsymbol{x}^1(t)\mid^2 \leqslant C\varepsilon^{\frac{3}{2}}$, and thus (11) is obtained. □

Now we can give the estimate of the difference between the perturbed state equation solution with the sum of the optimal state and the variational equation solution.

Lemma 5. We assume H_1, H_2, and H_3 hold. Then we have

$$\sup_{0\leqslant t\leqslant T} E\mid \boldsymbol{x}^\varepsilon(t) - \boldsymbol{x}(t) - \boldsymbol{x}^1(t)\mid^2 \leqslant C\varepsilon^{\frac{3}{2}} \tag{14}$$

$$\sup_{0\leqslant t\leqslant T} E\mid \boldsymbol{y}^\varepsilon(t) - \boldsymbol{y}(t) - \boldsymbol{y}^1(t)\mid^2 \leqslant C\varepsilon^{\frac{3}{2}} \tag{15}$$

$$E\int_0^{\mathrm{T}} \mid \boldsymbol{z}^\varepsilon(t) - \boldsymbol{z}(t) - \boldsymbol{z}^1(t)\mid^2 dt \leqslant C\varepsilon^{\frac{3}{2}} \tag{16}$$

Proof. We have

$$\begin{aligned}
& \int_0^{\mathrm{T}} b(\boldsymbol{x}+\boldsymbol{x}^1, \boldsymbol{y}+\boldsymbol{y}^1, \boldsymbol{z}+\boldsymbol{z}^1, \boldsymbol{u}^\varepsilon)ds + \int_0^{\mathrm{T}} \sigma(\boldsymbol{x}+\boldsymbol{x}^1, \boldsymbol{y}+\boldsymbol{y}^1, \boldsymbol{z}+\boldsymbol{z}^1)d\boldsymbol{B}_s = \int_0^{\mathrm{T}} [b(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{u}^\varepsilon) + \\
& \int_0^1 b\boldsymbol{x}(\boldsymbol{x}+\boldsymbol{\lambda}\boldsymbol{x}_1, \boldsymbol{y}+\boldsymbol{\lambda}y_1, \boldsymbol{z}+\boldsymbol{\lambda}z_1, \boldsymbol{u}^\varepsilon)d\boldsymbol{\lambda}\boldsymbol{x}^1 + \int_0^1 b\boldsymbol{y}(\boldsymbol{x}+\boldsymbol{\lambda}\boldsymbol{x}_1, \boldsymbol{y}+\boldsymbol{\lambda}y_1, \boldsymbol{z}+\boldsymbol{\lambda}z_1, \boldsymbol{u}^\varepsilon)d\boldsymbol{\lambda}\boldsymbol{y}^1 + \\
& \int_0^1 b\boldsymbol{z}(\boldsymbol{x}+\boldsymbol{\lambda}\boldsymbol{x}_1, \boldsymbol{y}+\boldsymbol{\lambda}y_1, \boldsymbol{z}+\boldsymbol{\lambda}z_1, \boldsymbol{u}^\varepsilon)d\boldsymbol{\lambda}\boldsymbol{z}^1]ds + \int_0^{\mathrm{T}} [\sigma(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) + \int_0^1 \sigma\boldsymbol{x}(\boldsymbol{x}+\boldsymbol{\lambda}\boldsymbol{x}_1, \boldsymbol{y}+\boldsymbol{\lambda}y_1, \boldsymbol{z}+\boldsymbol{\lambda}z_1)d\boldsymbol{\lambda}\boldsymbol{x}^1 + \\
& \int_0^1 \sigma\boldsymbol{y}(\boldsymbol{x}+\boldsymbol{\lambda}\boldsymbol{x}_1, \boldsymbol{y}+\boldsymbol{\lambda}y_1, \boldsymbol{z}+\boldsymbol{\lambda}z_1)d\boldsymbol{\lambda}\boldsymbol{y}^1 + \int_0^1 \sigma\boldsymbol{z}(\boldsymbol{x}+\boldsymbol{\lambda}\boldsymbol{x}_1, \boldsymbol{y}+\boldsymbol{\lambda}y_1, \boldsymbol{z}+\boldsymbol{\lambda}z_1)d\boldsymbol{\lambda}\boldsymbol{z}^1]d\boldsymbol{B}_s = \\
& \int_0^{\mathrm{T}} b(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{u})ds + \int_0^{\mathrm{T}} \sigma(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})d\boldsymbol{B}_s + \int_0^{\mathrm{T}} [b_x\boldsymbol{x}^1(s) + b_y\boldsymbol{y}^1(s) + b_z\boldsymbol{z}^1(s) + b(\boldsymbol{u}^\varepsilon) - b(u)]ds +
\end{aligned}$$

$$\int_0^{\mathrm{T}}[\sigma_x\boldsymbol{x}^1(s)+\sigma_y\boldsymbol{y}^1(s)+\sigma_z\boldsymbol{z}^1(s)]d\boldsymbol{B}_s+\int_0^{\mathrm{T}}A^{\varepsilon}(s)ds+\int_0^{\mathrm{T}}B^{\varepsilon}(s)d\boldsymbol{B}_s=$$
$$\boldsymbol{x}(t)-\boldsymbol{x}_0+\boldsymbol{x}^1(t)+\int_0^{\mathrm{T}}A^{\varepsilon}(s)ds+\int_0^{\mathrm{T}}B^{\varepsilon}(s)d\boldsymbol{B}_s$$

where

$$\begin{aligned}A^{\varepsilon}(s)=&\int_0^1[b\boldsymbol{x}(\boldsymbol{x}+\boldsymbol{\lambda x}^1,\boldsymbol{y}+\boldsymbol{\lambda y}^1,\boldsymbol{z}+\boldsymbol{\lambda z}^1,\boldsymbol{u}^{\varepsilon})-b_x]d\boldsymbol{\lambda x}^1+\int_0^1[b\boldsymbol{y}(\boldsymbol{x}+\boldsymbol{\lambda x}^1,\boldsymbol{y}+\boldsymbol{\lambda y}^1,\boldsymbol{z}+\\&\boldsymbol{\lambda z}^1,\boldsymbol{u}^{\varepsilon})-b_y]d\boldsymbol{\lambda y}^1+\int_0^1[b\boldsymbol{z}(\boldsymbol{x}+\boldsymbol{\lambda x}^1,\boldsymbol{y}+\boldsymbol{\lambda y}^1,\boldsymbol{z}+\boldsymbol{\lambda z}^1,\boldsymbol{u}^{\varepsilon})-b_z]d\boldsymbol{\lambda z}^1\\B^{\varepsilon}(s)=&\int_0^1[\sigma\boldsymbol{x}(\boldsymbol{x}+\boldsymbol{\lambda x}^1,\boldsymbol{y}+\boldsymbol{\lambda y}^1,\boldsymbol{z}+\boldsymbol{\lambda z}^1)-\sigma_x]d\boldsymbol{\lambda x}^1+\int_0^1[\sigma\boldsymbol{y}(\boldsymbol{x}+\boldsymbol{\lambda x}^1,\boldsymbol{y}+\boldsymbol{\lambda y}^1,\boldsymbol{z}+\boldsymbol{\lambda z}^1)-\\&\sigma_y]d\boldsymbol{\lambda y}^1+\int_0^1[\sigma\boldsymbol{z}(\boldsymbol{x}+\boldsymbol{\lambda x}^1,\boldsymbol{y}+\boldsymbol{\lambda y}^1,\boldsymbol{z}+\boldsymbol{\lambda z}^1)-\sigma_z]d\boldsymbol{\lambda z}^1\end{aligned}$$

By Lemma 4 we have

$$\sup_{0\leqslant t\leqslant T}E\{(\int_0^{\mathrm{T}}A^{\varepsilon}(s)ds)^2+(\int_0^{\mathrm{T}}B^{\varepsilon}(s)d\boldsymbol{B}_s)^2\}\leqslant C\varepsilon^{\frac{3}{2}}\tag{17}$$

and

$$\boldsymbol{x}^{\varepsilon}(t)-\boldsymbol{x}(t)-\boldsymbol{x}^1(t)=\int_0^{\mathrm{T}}C^{\varepsilon}(s)(\boldsymbol{x}^{\varepsilon}-\boldsymbol{x}-\boldsymbol{x}^1)ds+\int_0^{\mathrm{T}}D^{\varepsilon}(s)(\boldsymbol{x}^{\varepsilon}-\boldsymbol{x}-\boldsymbol{x}^1)ds+\int_0^{\mathrm{T}}A^{\varepsilon}(s)ds+\int_0^{\mathrm{T}}B^{\varepsilon}(s)d\boldsymbol{B}_s\tag{18}$$

where

$$\begin{aligned}C^{\varepsilon}(s)=&\int_0^1[b_x(\boldsymbol{x}+\boldsymbol{x}^1+\boldsymbol{\lambda}(\boldsymbol{x}^{\varepsilon}-\boldsymbol{x}-\boldsymbol{x}^1),\boldsymbol{y}+\boldsymbol{y}^1+\boldsymbol{\lambda}(\boldsymbol{y}^{\varepsilon}-\boldsymbol{y}-\boldsymbol{y}^1),\boldsymbol{z}+\boldsymbol{z}^1+\boldsymbol{\lambda}(\boldsymbol{z}^{\varepsilon}-\boldsymbol{z}-\boldsymbol{z}^1),\boldsymbol{u}^{\varepsilon})+\\&b_y(\boldsymbol{x}+\boldsymbol{x}^1+\boldsymbol{\lambda}(\boldsymbol{x}^{\varepsilon}-\boldsymbol{x}-\boldsymbol{x}^1),\boldsymbol{y}+\boldsymbol{y}^1+\boldsymbol{\lambda}(\boldsymbol{y}^{\varepsilon}-\boldsymbol{y}-\boldsymbol{y}^1),\boldsymbol{z}+\boldsymbol{z}^1+\boldsymbol{\lambda}(\boldsymbol{z}^{\varepsilon}-\boldsymbol{z}-\boldsymbol{z}^1),\boldsymbol{u}^{\varepsilon})+\\&b_z(\boldsymbol{x}+\boldsymbol{x}^1+\boldsymbol{\lambda}(\boldsymbol{x}^{\varepsilon}-\boldsymbol{x}-\boldsymbol{x}^1),\boldsymbol{y}+\boldsymbol{y}^1+\boldsymbol{\lambda}(\boldsymbol{y}^{\varepsilon}-\boldsymbol{y}-\boldsymbol{y}^1),\boldsymbol{z}+\boldsymbol{z}^1+\boldsymbol{\lambda}(\boldsymbol{z}^{\varepsilon}-\boldsymbol{z}-\boldsymbol{z}^1),\boldsymbol{u}^{\varepsilon})]d\boldsymbol{\lambda}\\D^{\varepsilon}(s)=&\int_0^1[\sigma_x(\boldsymbol{x}+\boldsymbol{x}^1+\boldsymbol{\lambda}(\boldsymbol{x}^{\varepsilon}-\boldsymbol{x}-\boldsymbol{x}^1),\boldsymbol{y}+\boldsymbol{y}^1+\boldsymbol{\lambda}(\boldsymbol{y}^{\varepsilon}-\boldsymbol{y}-\boldsymbol{y}^1),\boldsymbol{z}+\boldsymbol{z}^1+\boldsymbol{\lambda}(\boldsymbol{z}^{\varepsilon}-\boldsymbol{z}-\boldsymbol{z}^1))+\\&\sigma_y(\boldsymbol{x}+\boldsymbol{x}^1+\boldsymbol{\lambda}(\boldsymbol{x}^{\varepsilon}-\boldsymbol{x}-\boldsymbol{x}^1),\boldsymbol{y}+\boldsymbol{y}^1+\boldsymbol{\lambda}(\boldsymbol{y}^{\varepsilon}-\boldsymbol{y}-\boldsymbol{y}^1),\boldsymbol{z}+\boldsymbol{z}^1+\boldsymbol{\lambda}(\boldsymbol{z}^{\varepsilon}-\boldsymbol{z}-\boldsymbol{z}^1))+\\&\sigma_z(\boldsymbol{x}+\boldsymbol{x}^1+\boldsymbol{\lambda}(\boldsymbol{x}^{\varepsilon}-\boldsymbol{x}-\boldsymbol{x}^1),\boldsymbol{y}+\boldsymbol{y}^1+\boldsymbol{\lambda}(\boldsymbol{y}^{\varepsilon}-\boldsymbol{y}-\boldsymbol{y}^1),\boldsymbol{z}+\boldsymbol{z}^1+\boldsymbol{\lambda}(\boldsymbol{z}^{\varepsilon}-\boldsymbol{z}-\boldsymbol{z}^1))]d\boldsymbol{\lambda}\end{aligned}$$

Using Gronwall$'$s inequality, we attain the estimate (14). Noticing that

$$\begin{aligned}-\int_t^{\mathrm{T}}f(\boldsymbol{x}+\boldsymbol{x}^1,\boldsymbol{y}+\boldsymbol{y}^1,\boldsymbol{z}+\boldsymbol{z}^1,\boldsymbol{u}^{\varepsilon})ds+\int_t^{\mathrm{T}}(\boldsymbol{z}(s)+\boldsymbol{z}^1(s))d\boldsymbol{B}_s=\\h(\boldsymbol{x}(T))+h_x(\boldsymbol{x}(T))\boldsymbol{x}^1(T)-\boldsymbol{y}(t)-\boldsymbol{y}^1(t)-\int_t^{\mathrm{T}}G^{\varepsilon}(s)ds\end{aligned}$$

where

$$\begin{aligned}G^{\varepsilon}(s)=&\int_0^1[f_x(\boldsymbol{x}+\boldsymbol{\lambda x}^1,\boldsymbol{y}+\boldsymbol{\lambda y}^1,\boldsymbol{z}+\boldsymbol{\lambda z}^1,\boldsymbol{u}^{\varepsilon})-f_x]d\boldsymbol{\lambda x}^1+\int_0^1[f_y(\boldsymbol{x}+\boldsymbol{\lambda x}^1,\boldsymbol{y}+\boldsymbol{\lambda y}^1,\boldsymbol{z}+\boldsymbol{\lambda z}^1,\boldsymbol{u}^{\varepsilon})-\\&f_y]d\boldsymbol{\lambda y}^1+\int_0^1[f_z(\boldsymbol{x}+\boldsymbol{\lambda x}^1,\boldsymbol{y}+\boldsymbol{\lambda y}^1,\boldsymbol{z}+\boldsymbol{\lambda z}^1,\boldsymbol{u}^{\varepsilon})-f_z]d\boldsymbol{\lambda z}^1\end{aligned}$$

we have

$$\begin{aligned}[\boldsymbol{y}^{\varepsilon}(t)-\boldsymbol{y}(t)-\boldsymbol{y}^1(t)]+\int_t^{\mathrm{T}}[\boldsymbol{z}^{\varepsilon}(s)-\boldsymbol{z}(s)-\boldsymbol{z}^1(s)]d\boldsymbol{B}_s=h(\boldsymbol{x}^{\varepsilon}(T)-h(\boldsymbol{x}(T))-h_x(\boldsymbol{x}(T))\boldsymbol{x}^1(T)+\\\int_t^{\mathrm{T}}[f(s,\boldsymbol{x}^{\varepsilon},\boldsymbol{y}^{\varepsilon},\boldsymbol{z}^{\varepsilon},\boldsymbol{u}^{\varepsilon})-f(\boldsymbol{x}+\boldsymbol{x}^1,\boldsymbol{y}+\boldsymbol{y}^1,\boldsymbol{z}+\boldsymbol{z}^1,\boldsymbol{u}^{\varepsilon})]ds+\int_t^{\mathrm{T}}G^{\varepsilon}(s)ds\end{aligned}\tag{19}$$

And then

$$E\mid \boldsymbol{y}^{\varepsilon}(t)-\boldsymbol{y}(t)-\boldsymbol{y}^{1}(t)\mid^{2}+E\int_{t}^{\mathrm{T}}\mid \boldsymbol{z}^{\varepsilon}(s)-\boldsymbol{z}(s)-\boldsymbol{z}^{1}(s)\mid^{2}ds=E\{h(\boldsymbol{x}^{\varepsilon}(T))-h(\boldsymbol{x}(t))-$$
$$h_x(\boldsymbol{x}(t))\boldsymbol{x}^1(T)+\int_t^{\mathrm{T}}[f(s,\boldsymbol{x}^{\varepsilon},\boldsymbol{y}^{\varepsilon},\boldsymbol{z}^{\varepsilon},\boldsymbol{u}^{\varepsilon})-f(\boldsymbol{x}+\boldsymbol{x}^1,\boldsymbol{y}+\boldsymbol{y}^1,\boldsymbol{z}+\boldsymbol{z}^1,\boldsymbol{u}^{\varepsilon})]ds+\int_t^{\mathrm{T}}G^{\varepsilon}(s)ds\}^2=$$
$$E\{h(\boldsymbol{x}^{\varepsilon}(T))-h(\boldsymbol{x}(T)+\boldsymbol{x}^1(T))+\int_0^1[h_x(\boldsymbol{x}(T)+\boldsymbol{x}^1(T))-h_x(\boldsymbol{x}(T))]d\boldsymbol{\lambda}\boldsymbol{x}^1(T)+$$
$$\int_t^{\mathrm{T}}[f(s,\boldsymbol{x}^{\varepsilon},\boldsymbol{y}^{\varepsilon},\boldsymbol{z}^{\varepsilon},\boldsymbol{u}^{\varepsilon})-f(\boldsymbol{x}+\boldsymbol{x}^1,\boldsymbol{y}+\boldsymbol{y}^1,\boldsymbol{z}+\boldsymbol{z}^1,\boldsymbol{u}^{\varepsilon})]ds+\int_t^{\mathrm{T}}G^{\varepsilon}(s)ds\}^2$$

From Lemma 4 and (14), we obtain

$$\sup_{0\leqslant t\leqslant T}E(\int_t^{\mathrm{T}}G^{\varepsilon}(s)ds)^2\leqslant C\varepsilon^{\frac{3}{2}},\quad E[h(\boldsymbol{x}^{\varepsilon}(T)-h(\boldsymbol{x}(T)+\boldsymbol{x}^1(T))]^2\leqslant C\varepsilon^{\frac{3}{2}}$$

Using the same method in Lemma 3 and Lemma 4, we can get (15) and (16). □

Lemma 6. (variational inequality) We assume H_1, H_2, and H_3 hold; then

$$E\int_0^{\mathrm{T}}[l_x\boldsymbol{x}^1(t)+l_y\boldsymbol{y}^1(t)+l_z\boldsymbol{z}^1(t)+l(\boldsymbol{u}^{\varepsilon}(t))-l(\boldsymbol{u}(t))]dt+E[\Phi_x(\boldsymbol{x}(T))\boldsymbol{x}^1(T)]+E[\gamma_y(\boldsymbol{y}(0))\boldsymbol{y}^1(0)]\geqslant o(\varepsilon) \tag{20}$$

Proof. From $J(\boldsymbol{u}^{\varepsilon}(\cdot))\geqslant J(\boldsymbol{u}(\cdot))$, we have

$$E\int_0^{\mathrm{T}}[l(t,\boldsymbol{x}^{\varepsilon}(t),\boldsymbol{y}^{\varepsilon}(t),\boldsymbol{z}^{\varepsilon}(t),\boldsymbol{u}^{\varepsilon}(t))-l(\boldsymbol{u}(t))]dt+E[\Phi(\boldsymbol{x}^{\varepsilon}(T))-\Phi(\boldsymbol{x}(T))]+E[\gamma(\boldsymbol{y}^{\varepsilon}(0)-\gamma(\boldsymbol{y}(0))]\geqslant 0$$

and

$$0\leqslant E\int_0^{\mathrm{T}}[l(t,\boldsymbol{x}^{\varepsilon}(t),\boldsymbol{y}^{\varepsilon}(t),\boldsymbol{z}^{\varepsilon}(t),\boldsymbol{u}^{\varepsilon}(t))-l(t,\boldsymbol{x}+\boldsymbol{x}^1,\boldsymbol{y}+\boldsymbol{y}^1,\boldsymbol{z}+\boldsymbol{z}^1,\boldsymbol{u}^{\varepsilon})]dt+$$
$$E\int_0^{\mathrm{T}}[l(t,\boldsymbol{x}+\boldsymbol{x}^1,\boldsymbol{y}+\boldsymbol{y}^1,\boldsymbol{z}+\boldsymbol{z}^1,\boldsymbol{u}^{\varepsilon})-l(\boldsymbol{u}(t))]dt+E[\Phi(\boldsymbol{x}^{\varepsilon}(T))-\Phi(\boldsymbol{x}(T)+\boldsymbol{x}^1(T))]+$$
$$E[\Phi(\boldsymbol{x}(T)+\boldsymbol{x}^1(T))-\Phi(\boldsymbol{x}(T))]+E[\gamma(\boldsymbol{y}^{\varepsilon}(0))-\gamma(\boldsymbol{y}(0)+\boldsymbol{y}^1(0))]+E[\gamma(\boldsymbol{y}(0)+\boldsymbol{y}^1(0))-\gamma(\boldsymbol{y}(0))]$$

By H_3 and Lemma 5, we have

$$E\int_0^{\mathrm{T}}[l(t,\boldsymbol{x}^{\varepsilon}(t),\boldsymbol{y}^{\varepsilon}(t),\boldsymbol{z}^{\varepsilon}(t),\boldsymbol{u}^{\varepsilon}(t))-l(t,\boldsymbol{x}+\boldsymbol{x}^1,\boldsymbol{y}+\boldsymbol{y}^1,\boldsymbol{z}+\boldsymbol{z}^1,\boldsymbol{u}^{\varepsilon})]dt+$$
$$E[\Phi(\boldsymbol{x}^{\varepsilon}(T))-\Phi(\boldsymbol{x}(T)+\boldsymbol{x}^1(T))]+E[\gamma(\boldsymbol{y}^{\varepsilon}(0))-\gamma(\boldsymbol{y}(0)+\boldsymbol{y}^1(0))]\leqslant C\varepsilon^{\frac{3}{2}}$$

and

$$0\leqslant E\int_0^{\mathrm{T}}[l(t,\boldsymbol{x}+\boldsymbol{x}^1,\boldsymbol{y}+\boldsymbol{y}^1,\boldsymbol{z}+\boldsymbol{z}^1,\boldsymbol{u}^{\varepsilon})-l(\boldsymbol{u}(t))]dt+E[\Phi(\boldsymbol{x}(t)+\boldsymbol{x}^1(T))-\Phi(\boldsymbol{x}(t))]+$$
$$E[\gamma(\boldsymbol{y}^{\varepsilon}(0))-\gamma(\boldsymbol{y}(0)+\boldsymbol{y}^1(0))]+C\varepsilon^{\frac{3}{2}}=E\int_0^{\mathrm{T}}[l(t,\boldsymbol{x}+\boldsymbol{x}^1,\boldsymbol{y}+\boldsymbol{y}^1,\boldsymbol{z}+\boldsymbol{z}^1,u)-l(\boldsymbol{u}(t))]dt+$$
$$E\int_0^{\mathrm{T}}[l(t,\boldsymbol{x}+\boldsymbol{x}^1,\boldsymbol{y}+\boldsymbol{y}^1,\boldsymbol{z}+\boldsymbol{z}^1,\boldsymbol{u}^{\varepsilon})-l(t,\boldsymbol{x}+\boldsymbol{x}^1,\boldsymbol{y}+\boldsymbol{y}^1,\boldsymbol{z}+\boldsymbol{z}^1,u)]dt+$$
$$E[\Phi(\boldsymbol{x}(T)+\boldsymbol{x}^1(T))-\Phi(\boldsymbol{x}(T))]+E[\gamma(\boldsymbol{y}^{\varepsilon}(0))-\gamma(\boldsymbol{y}(0)+\boldsymbol{y}^1(0))]+$$
$$C\varepsilon^{\frac{3}{2}}=E\int_0^{\mathrm{T}}[l_x\boldsymbol{x}^1(t)+l_y\boldsymbol{y}^1(t)+l_z\boldsymbol{z}^1(t)]dt+E\int_0^{\mathrm{T}}[l(\boldsymbol{u}^{\varepsilon}(t))-l(\boldsymbol{u}(t))]dt+$$
$$E\int_0^{\mathrm{T}}\{[l_x(\boldsymbol{u}^{\varepsilon})-l_x]\boldsymbol{x}^1+[l_y(\boldsymbol{u}^{\varepsilon})-l_y]\boldsymbol{y}^1+[l_z(\boldsymbol{u}^{\varepsilon})-l_z]\boldsymbol{z}^1\}dt+E[\Phi_x(\boldsymbol{x}(T))\boldsymbol{x}^1(T)]+$$

$$E[\gamma_y(\boldsymbol{y}(0))\boldsymbol{y}^1(0)] + C\varepsilon^{\frac{3}{2}} = E\int_0^{\mathrm{T}} [l_x\boldsymbol{x}^1(t) + l_y\boldsymbol{y}^1(t) + l_z\boldsymbol{z}^1(t) + l(\boldsymbol{u}^\varepsilon(t)) - l(\boldsymbol{u}(t))]dt +$$
$$E[\Phi_x(\boldsymbol{x}(t))\boldsymbol{x}^1(T)] + E[\gamma_y(\boldsymbol{y}(0))\boldsymbol{y}^1(0)] + o(\varepsilon)$$

The desired variational inequality (20) can be obtained. □

4 The maximum principle in global form

We first introduce the following adjoint equation with respect to the variational (5) using the dual technique:

$$\begin{cases} d\boldsymbol{p}(t) = [f_y^\tau \boldsymbol{p}(t) - b_y^\tau \boldsymbol{q}(t) - \sigma_y^\tau \boldsymbol{k}(t) - l_y]dt + [f_z^\tau \boldsymbol{p}(t) - b_z^\tau \boldsymbol{q}(t) - \sigma_z^\tau \boldsymbol{k}(t) - l_z]d\boldsymbol{B}_t \\ -d\boldsymbol{q}(t) = [-f_x^\tau \boldsymbol{p}(t) + b_x^\tau \boldsymbol{q}(t) + \sigma_x^\tau \boldsymbol{k}(t) + l_x]dt - \boldsymbol{k}(t)d\boldsymbol{B}_t, \quad 0 \leqslant t \leqslant T \\ p(0) = -\gamma_y(\boldsymbol{y}(0)), \quad \boldsymbol{q}(T) = -h_x(\boldsymbol{x}(t))\boldsymbol{p}(T) + \Phi_x(\boldsymbol{x}(T)) \end{cases} \tag{21}$$

where $(\boldsymbol{p}(\cdot), \boldsymbol{q}(\cdot), \boldsymbol{k}(\cdot)) \in R^m \times R^n \times R^{n\times d}$. From H_3 and the fact that (5) satisfies H_1 and H_2, we can easily verify that the adjoint (21) satisfies H_1 and H_2'. From Lemma 1, we know that (21) has a unique solution $(\boldsymbol{p}(t), \boldsymbol{q}(t), \boldsymbol{k}(t)), 0 \leqslant t \leqslant T$. We define the Hamilton function as

$$H(t,x,y,z,v,p,q,k) \doteq \langle q, b(t,x,y,z,v)\rangle - \langle p, f(t,x,y,z,v)\rangle + \langle k, \boldsymbol{\lambda}(t,x,y,z)\rangle + l(t,x,y,z,v) \tag{22}$$

where $H : [0,T] \times R^n \times R^m \times R^{m\times k} \times R^k \times R^m \times R^n \times R^{n\times d} \longrightarrow R$. Then we can get the following.

Theorem. (Stochastic Maximum Principle)

We assume H_1, H_2, and H_3 hold. If $(\boldsymbol{u}(\cdot), \boldsymbol{x}(\cdot), \boldsymbol{y}(\cdot), \boldsymbol{z}(\cdot))$ is the solution to our optimal control problem and $(\boldsymbol{p}(\cdot), \boldsymbol{q}(\cdot), \boldsymbol{k}(\cdot)$ is the solution to the corresponding adjoint equation (21), then we have

$$H(t,\boldsymbol{x}(t),\boldsymbol{y}(t),\boldsymbol{z}(t),v,\boldsymbol{p}(t),\boldsymbol{q}(t),\boldsymbol{k}(t)) \geqslant H(t,\boldsymbol{x}(t),\boldsymbol{y}(t),\boldsymbol{z}(t),\boldsymbol{u}(t),\boldsymbol{p}(t),\boldsymbol{q}(t),\boldsymbol{k}(t)), \forall \boldsymbol{v} \in \mathcal{U}_{ad},\ a.e.,\ a.s. \tag{23}$$

and (21) can be written as the following stochastic Hamilton system:

$$\begin{cases} d\boldsymbol{p}(t) = -H_y dt - H_z d\boldsymbol{B}_t \\ -d\boldsymbol{q}(t) = H_x dt - \boldsymbol{k}(t)d\boldsymbol{B}_t, \quad 0 \leqslant t \leqslant T \\ p(0) = -\gamma_y(\boldsymbol{y}(0)) \\ \boldsymbol{q}(T) = -h_x(\boldsymbol{x}(T))\boldsymbol{p}(T) + \Phi_x(\boldsymbol{x}(T)) \end{cases} \tag{24}$$

Proof. Using Ito′s formula to $\langle \boldsymbol{p}(t), \boldsymbol{y}^1(t)\rangle + \langle \boldsymbol{q}(t), \boldsymbol{x}^1(t)\rangle$ and noticing the variational equation (5), the adjoint equation (21) and the variational inequality (20), we obtain

$$o(\varepsilon) \leqslant E\int_0^{\mathrm{T}} [l_x\boldsymbol{x}^1(t) + l_y\boldsymbol{y}^1(t) + l_z\boldsymbol{z}^1(t) + l(\boldsymbol{u}^\varepsilon(t)) - l(\boldsymbol{u}(t))]dt + E[\Phi_x(\boldsymbol{x}(T))\boldsymbol{x}^1(T)] + E[\gamma_y(\boldsymbol{y}(0))\boldsymbol{y}^1(0)] =$$
$$E\int_0^{\mathrm{T}} [H(t,\boldsymbol{x}(t),\boldsymbol{y}(t),\boldsymbol{z}(t),\boldsymbol{u}^\varepsilon(t),\boldsymbol{p}(t),\boldsymbol{q}(t),\boldsymbol{k}(t)) - H(t,\boldsymbol{x}(t),\boldsymbol{y}(t),\boldsymbol{z}(t),\boldsymbol{u}(t),\boldsymbol{p}(t),\boldsymbol{q}(t),\boldsymbol{k}(t))]dt$$

The maximum principle (23) follows immediately. The Hamilton system (24) is obvious. □

Remark. When there are initial state constraint for the backward state variable and the final state constraint for the forward state variable, we can also obtain a global maximum principle by using Ekeland′s variational principle. However, if the diffusion coefficients of the forward system contains control variable and the control domain is not necessarily convex, we cannot get the maximum principle for fully coupled forward-backward stochastic control system in the global form. It is still an open problem.

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