# Delay-dependent Robust Stability and Stabilization Criteria for Uncertain Neutral Systems1)

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**Abstract** This paper concerns problem of the delay-dependent robust stability and stabilization for uncertain neutral systems. Some new delay-dependent stability criteria are derived by taking the relationship between the terms in the Leibniz-Newton formula into account. Free weighting matrices are given to express the relationship between the terms in the Leibniz-Newton formula and the new criteria are based on linear matrix inequalities such that the free weighting matrices can be easily obtained. Moreover, the stability criteria are also used to design the state-feedback controller. Numerical examples demonstrates that the proposed criteria are effective and are an improvement over the previous papers.

**Key words** Neutral systems, stability, stabilization, time-varying structured uncertainties, delay-dependent criterion, linear matrix inequality

## 1 Introduction

During the last decade, considerable attention has been devoted to the problem of delay-dependent stability analysis and controller design for retarded and neutral systems[1~10]. Before 1999, the main methods on this topic were the standard bounding method and matrix measure and matrix norm, which led to considerable conservatism[1,3,5,6]. Recently Park presented a new inequality to improve the standard bounding method and obtained the delay-dependent criteria for systems with time-invariant delays[2]; and Moon *et al.* extended Park′s results to a more general form[4]. In addition, Fridman and Shaked presented a descriptor model transformation method and some more efficient stability criteria were derived by combining Park and Moon′s inequalities with it[7~10]. In the derivative of Lyapunov functional, they used the Leibniz-Newton formula and replaced the term $\boldsymbol{x}(t-h)$ with $\boldsymbol{x}(t)-\int_{t-h}^{t}\dot{\boldsymbol{x}}(s)\mathrm{d}s$ in some places, but kept it in other places. For example, in [4], $\boldsymbol{x}(t-h)$ in the expression $2\boldsymbol{x}^{\mathrm{T}}(t)PA_1\dot{\boldsymbol{x}}(t)$ is replaced with $\boldsymbol{x}(t)-\int_{t-h}^{t}\dot{\boldsymbol{x}}(s)\mathrm{d}s$; but not in $\tau\dot{\boldsymbol{x}}^T(t)Z\dot{\boldsymbol{x}}(t)$. In fact, both $\boldsymbol{x}(t-h)$ and $\boldsymbol{x}(t)-\int_{t-h}^{t}\dot{\boldsymbol{x}}(s)\mathrm{d}s$ affect the result, and there must be some relationship between them. However, the above papers ignored this problem.

In this paper, some new delay-dependent stability criteria that take the relationship between $\boldsymbol{x}(t-h)$ and $\boldsymbol{x}(t)-\int_{t-h}^{t}\dot{\boldsymbol{x}}(s)\mathrm{d}s$ into account are presented for neutral systems with time-varying delays. Some free weighting matrices that express the influences of the terms in the Leibniz-Newton formula are determined based on linear matrix inequalities (LMIs). Then, the stability criterion can be extended to the systems with time-varying structured uncertainties. In addition, they are also applied to the state-feedback controller design to solve the problem of stabilizing the systems. Finally, some numerical examples demonstrate that the results obtained in this paper are effective and are a significant improvement over the existing criteria.

## 2 Notation and preliminaries

Consider a time-delay system $\Sigma$

$$\Sigma:\begin{cases}\dot{\boldsymbol{x}}(t)-C\dot{\boldsymbol{x}}(t-\tau)=(A+\Delta A(t))\boldsymbol{x}(t)+(A_d+\Delta A_d(t))\boldsymbol{x}(t-d(t))+B\boldsymbol{u}(t),\ t>0\\ \boldsymbol{x}(t)=\boldsymbol{\phi}(t),t\in[-r,0]\end{cases}\tag{1}$$

where $\boldsymbol{x}(t)\in\mathcal{R}^n$ is the state vector, $\boldsymbol{u}(t)\in\mathcal{R}^m$ is the control input. The matrices $A, A_d, B, C$ are constant matrices with appropriate dimensions. The time-varying structured uncertainties are of the

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form

$$[\Delta A(t)\ \Delta A_d(t)] = DF(t)[E_a\ E_{ad}] \tag{2}$$

where $D, E_a, E_{ad}$ are appropriate dimensional constant matrices and $F(t)$ is an unknown real and possibly time-varying matrix with Lsbesgue measurable elements satisfying

$$\|F(t)\| \leqslant 1,\ \forall t \tag{3}$$

The time delay $d(t)$ is a time-varying continues function and satisfies

$$0 \leqslant d(t) \leqslant h \tag{4}$$

and

$$\dot{d}(t) \leqslant \mu < 1 \tag{5}$$

where $h, \tau$ and $\mu$ are constants and $r$=max$(\tau, h)$. The initial condition $\boldsymbol{\phi}(t)$ denotes a continuous vector-valued initial function of $t \in [-r, 0]$. We are interested in designing a memoryless state-feedback controller

$$\boldsymbol{u}(t) = K\boldsymbol{x}(t) \tag{6}$$

where $K \in \mathcal{R}^{m\times n}$ is a constant gain matrix. To obtain results for the system with time-varying structured uncertainties, the following lemma is needed.

**Lemma 1**[11]. Given matrices $Q = Q^{\mathrm{T}}, H, E$ and $R = R^{\mathrm{T}} > 0$ of appropriate dimensions,

$$Q + HFE + E^{\mathrm{T}}F^{\mathrm{T}}H^{\mathrm{T}} < 0$$

for all $F$ satisfying $F^{\mathrm{T}}F \leqslant R$, if and only if there exists some $\varepsilon > 0$ such that

$$Q + \varepsilon^{-1}HH^{\mathrm{T}} + \varepsilon E^{\mathrm{T}}RE < 0$$

First, we consider the nominal system $\Sigma_0$ of $\Sigma$, which is described as

$$\Sigma_0 : \begin{cases} \dot{\boldsymbol{x}}(t) - C\dot{\boldsymbol{x}}(t-\tau) = A\boldsymbol{x}(t) + A_d\boldsymbol{x}(t-d(t)) + B\boldsymbol{u}(t),\ t > 0 \\ \boldsymbol{x}(t) = \boldsymbol{\phi}(t),\ t \in [-r, 0] \end{cases} \tag{7}$$

## 3 Stability

In this section, for nominal system $\Sigma_0$ with $\boldsymbol{u}(t) = \mathbf{0}$, the relationship between the terms in the Leibniz-Newton formula is taken into account. Specifically, the term $2[\boldsymbol{x}^{\mathrm{T}}(t)N_1 + \boldsymbol{x}^{\mathrm{T}}(t-d(t))N_2 + \dot{\boldsymbol{x}}^{\mathrm{T}}(t)N_3 + \dot{\boldsymbol{x}}^{\mathrm{T}}(t-\tau)N_4][\boldsymbol{x}(t) - \int_{t-d(t)}^{t} \dot{\boldsymbol{x}}(s)\mathrm{d}s - \boldsymbol{x}(t-d(t))]$, which is equal to zero, is added into the derivative of Lyapunov functional. The free weighting matrices $N_j$ $(j = 1, \cdots, 4)$ are used to indicate the relationship between the terms in the Leibniz-Newton formula. In addition, they can easily be determined by solving linear matrix inequalities. Now, the following conclusion for nominal system $\Sigma_0$ with time-varying delay satisfying (4) and (5) can be derived.

**Theorem 1.** Given scalars $h > 0$ and $\mu < 1$, the nominal system $\Sigma_0$ of $\Sigma$ with $\boldsymbol{u}(t) = 0$ and time-varying delay satisfying (4) and (5) is asymptotically stable for any $\tau > 0$ if there exist $P = P^{\mathrm{T}} > 0$, $Q = Q^{\mathrm{T}} \geqslant 0,\ R = R^{\mathrm{T}} \geqslant 0,\ Z = Z^{\mathrm{T}} \geqslant 0,\ X = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{12}^{\mathrm{T}} & X_{22} & X_{23} & X_{24} \\ X_{13}^{\mathrm{T}} & X_{23}^{\mathrm{T}} & X_{33} & X_{34} \\ X_{14}^{\mathrm{T}} & X_{24}^{\mathrm{T}} & X_{34}^{\mathrm{T}} & X_{44} \end{bmatrix} \geqslant 0$, and any appropriate dimensional matrices $N_j$ $(j = 1, \cdots, 4)$ and $T_j$ $(j = 1, \cdots, 4)$ such that the following LMIs (8) and (9) hold

$$\varXi = \begin{bmatrix} \varXi_{11} & \varXi_{12} & \varXi_{13} & \varXi_{14} \\ \varXi_{12}^{\mathrm{T}} & \varXi_{22} & \varXi_{23} & \varXi_{24} \\ \varXi_{13}^{\mathrm{T}} & \varXi_{23}^{\mathrm{T}} & \varXi_{33} & \varXi_{34} \\ \varXi_{14}^{\mathrm{T}} & \varXi_{24}^{\mathrm{T}} & \varXi_{34}^{\mathrm{T}} & \varXi_{44} \end{bmatrix} < 0 \tag{8}$$

and

$$\varPsi = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} & N_1 \\ X_{12}^{\mathrm{T}} & X_{22} & X_{23} & X_{24} & N_2 \\ X_{13}^{\mathrm{T}} & X_{23}^{\mathrm{T}} & X_{33} & X_{34} & N_3 \\ X_{14}^{\mathrm{T}} & X_{24}^{\mathrm{T}} & X_{34}^{4} & X_{44} & N_4 \\ N_1^{\mathrm{T}} & N_2^{\mathrm{T}} & N_3^{\mathrm{T}} & N_4^{\mathrm{T}} & Z \end{bmatrix} \geqslant 0 \tag{9}$$

where

$$\varXi_{11} = Q + N_1 + N_1^{\rm T} - A^{\rm T}T_1^{\rm T} - T_1A + hX_{11}, \quad \varXi_{12} = N_2^{\rm T} - N_1 - A^{\rm T}T_2^{\rm T} - T_1A_d + hX_{12}$$
$$\varXi_{13} = P + N_3^{\rm T} + T_1 - A^{\rm T}T_3^{\rm T} + hX_{13}, \quad \varXi_{14} = N_4^{\rm T} - A^{\rm T}T_4^{\rm T} - T_1C + hX_{14}$$
$$\varXi_{22} = -(1-\mu)Q - N_2 - N_2^{\rm T} - T_2A_d - A_d^{\rm T}T_2^{\rm T} + hX_{22}, \quad \varXi_{23} = -N_3^{\rm T} + T_2 - A_d^{\rm T}T_3^{\rm T} + hX_{23}$$
$$\varXi_{24} = -N_4^{\rm T} - A_d^{\rm T}T_4^{\rm T} - T_2C + hX_{24}, \quad \varXi_{33} = R + hZ + T_3 + T_3^{\rm T} + hX_{33}$$
$$\varXi_{34} = T_4^{\rm T} - T_3C + hX_{34}, \quad \varXi_{44} = -R - C^{\rm T}T_4^{\rm T} - T_4C + hX_{44}$$

**Proof.** Choose the Lyapunov functional candidate as

$$V(\boldsymbol{x}_t) := \boldsymbol{x}^{\rm T}(t)P\boldsymbol{x}(t) + \int_{t-d(t)}^{t} \boldsymbol{x}^{\rm T}(s)Q\boldsymbol{x}(s){\rm d}s + \int_{t-\tau}^{t} \dot{\boldsymbol{x}}^{\rm T}(s)R\dot{\boldsymbol{x}}(s){\rm d}s + \int_{-h}^{0}\int_{t+\theta}^{t} \dot{\boldsymbol{x}}^{\rm T}(s)Z\dot{\boldsymbol{x}}(s){\rm d}s{\rm d}\theta \quad (10)$$

where $P = P^{\rm T} > 0$, $Q = Q^{\rm T} \geqslant 0$, $R = R^{\rm T} \geqslant 0$ and $Z = Z^{\rm T} \geqslant 0$ are to be determined. For any appropriate dimensional matrices $N_j (j = 1, \cdots, 4)$, by using the Leibniz-Newton formula one has

$$2[x^{\rm T}(t)N_1 + x^{\rm T}(t-d(t))N_2 + \dot{x}^{\rm T}(t)N_3 + \dot{x}^{\rm T}(t-\tau)N_4]\left[\boldsymbol{x}(t) - \int_{t-d(t)}^{t} \dot{x}(s){\rm d}s - \boldsymbol{x}(t-d(t))\right] = 0 \quad (11)$$

In addition, according to (7), for any appropriate dimensional matrices $T_j$ $(j = 1, \cdots, 4)$, one has

$$2\left[\boldsymbol{x}^{\rm T}(t)T_1 + \boldsymbol{x}^{\rm T}(t-d(t))T_2 + \dot{\boldsymbol{x}}^{\rm T}(t)T_3 + \dot{\boldsymbol{x}}^{\rm T}(t-\tau)T_4\right][\dot{\boldsymbol{x}}(t) - A\boldsymbol{x}(t) - A_d\boldsymbol{x}(t-d(t)) - C\dot{\boldsymbol{x}}(t-\tau)] = 0 \quad (12)$$

On the other hand, for any appropriate dimensional semi-positive definite matrix $X \geqslant 0$, the following holds

$$h\boldsymbol{\xi}^{\rm T}(t)X\boldsymbol{\xi}(t) - \int_{t-d(t)}^{t} \boldsymbol{\xi}^{\rm T}(t)X\boldsymbol{\xi}(t){\rm d}s \geqslant 0 \quad (13)$$

where $\boldsymbol{\xi}(t) = [\boldsymbol{x}^{\rm T}(t) \;\; \boldsymbol{x}^{\rm T}(t-d(t)) \;\; \dot{\boldsymbol{x}}^{\rm T}(t) \;\; \dot{\boldsymbol{x}}^{\rm T}(t-\tau)]^{\rm T}$. Then, for $X = X^{\rm T} \geqslant 0$, and any matrices $N_j(j = 1, \cdots, 4)$ and $T_j(j = 1, \cdots, 4)$, using (11), (12) and (13) and calculating the derivative of $V(x_t)$ along the solutions of system $\Sigma_0$ yield

$$\begin{aligned}
\dot{V}(x_t) = {} & 2\boldsymbol{x}^{\rm T}(t)P\dot{\boldsymbol{x}}(t) + \boldsymbol{x}^{\rm T}(t)Q\boldsymbol{x}(t) - (1-\dot{d}(t))\boldsymbol{x}^{\rm T}(t-d(t))Q\boldsymbol{x}(t-d(t)) + \\
& \dot{\boldsymbol{x}}^{\rm T}(t)R\dot{\boldsymbol{x}}(t) - \dot{\boldsymbol{x}}^{\rm T}(t-\tau)R\dot{\boldsymbol{x}}(t-\tau) + h\dot{\boldsymbol{x}}^{\rm T}(t)Z\dot{\boldsymbol{x}}(t) - \int_{t-h}^{t} \dot{\boldsymbol{x}}^{\rm T}(s)Z\dot{\boldsymbol{x}}(s){\rm d}s \leqslant \\
& 2\boldsymbol{x}^{\rm T}(t)P\dot{\boldsymbol{x}}(t) + \boldsymbol{x}^{\rm T}(t)Q\boldsymbol{x}(t) - (1-\mu)\boldsymbol{x}^{\rm T}(t-d(t))Q\boldsymbol{x}(t-d(t)) + \\
& \dot{\boldsymbol{x}}^{\rm T}(t)R\dot{\boldsymbol{x}}(t) - \dot{\boldsymbol{x}}^{\rm T}(t-\tau)R\dot{\boldsymbol{x}}(t-\tau) + h\dot{\boldsymbol{x}}^{\rm T}(t)Z\dot{\boldsymbol{x}}(t) - \int_{t-d(t)}^{t} \dot{\boldsymbol{x}}^{\rm T}(s)Z\dot{\boldsymbol{x}}(s){\rm d}s + \\
& 2\left[\boldsymbol{x}^{\rm T}(t)N_1 + \boldsymbol{x}^{\rm T}(t-d(t))N_2 + \dot{\boldsymbol{x}}^{\rm T}(t)N_3 + \dot{\boldsymbol{x}}^{\rm T}(t-\tau)N_4\right] \cdot \\
& \left[\boldsymbol{x}(t) - \int_{t-d(t)}^{t} \dot{\boldsymbol{x}}(s){\rm d}s - \boldsymbol{x}(t-d(t))\right] + \\
& 2\left[\boldsymbol{x}^{\rm T}(t)T_1 + \boldsymbol{x}^{\rm T}(t-d(t))T_2 + \dot{\boldsymbol{x}}^{\rm T}(t)T_3 + \dot{\boldsymbol{x}}^{\rm T}(t-\tau)T_4\right] \cdot \\
& [\dot{\boldsymbol{x}}(t) - A\boldsymbol{x}(t) - A_d\boldsymbol{x}(t-d(t)) - C\dot{\boldsymbol{x}}(t-\tau)] + \\
& h\boldsymbol{\xi}^{\rm T}(t)X\boldsymbol{\xi}(t) - \int_{t-d(t)}^{t} \boldsymbol{\xi}^{\rm T}(t)X\boldsymbol{\xi}(t){\rm d}s := \\
& \boldsymbol{\xi}^{\rm T}(t)\varXi\boldsymbol{\xi}(t) - \int_{t-d(t)}^{t} \boldsymbol{\zeta}^{\rm T}(t,s)\varPsi\boldsymbol{\zeta}(t,s){\rm d}s
\end{aligned} \quad (14)$$

where $\boldsymbol{\zeta}(t,s) = [\boldsymbol{x}^{\rm T}(t) \;\; \boldsymbol{x}^{\rm T}(t-d(t)) \;\; \dot{\boldsymbol{x}}^{\rm T}(t) \;\; \dot{\boldsymbol{x}}^{\rm T}(t-\tau) \;\; \dot{\boldsymbol{x}}^{\rm T}(s)]^{\rm T}$, and $\varXi$, $\varPsi$ are defined in (8) and (9), and $\boldsymbol{\xi}(t)$ is defined in (13). If $\varXi < 0$ and $\varPsi \geqslant 0$, $\dot{V}(\boldsymbol{x}_t) < 0$ for any $\boldsymbol{\xi}(t) \neq 0$. So $\Sigma_0$ is asymptotically stable if LMIs (8) and (9) are true. This completes the proof. □

**Remark 1.** Some simple transforms show that the condition in Theorem 1 includes the condition in Lemma 1 in [8] for a single time-delay. In addition, Theorem 1 gives a delay-dependent stability criterion through quite simple and nature form. In [8], the approach that Park and Moon′s inequalities were combined with descriptor model transformation was more complicated than ours.

The above result can be extended to the system with time-varying structured uncertainties. Now, based on Theorem 1, the following theorem provides robust stability analysis of the unforced system $\Sigma$ with $\boldsymbol{u}(t)=0$.

**Theorem.** Given scalars $h>0$ and $\mu<1$, the uncertain system $\Sigma$ with $\boldsymbol{u}(t)=\mathbf{0}$ and time-varying delay satisfying (4) and (5) is robustly stable for any $\tau>0$ if there exist $P=P^{\mathrm{T}}>0$, $Q=Q^{\mathrm{T}}\geqslant 0$, $R=R^{\mathrm{T}}\geqslant 0$, $Z=Z^{\mathrm{T}}\geqslant 0$, $X=X^{\mathrm{T}}\geqslant 0$, and any appropriate dimensional matrices $N_j(j=1,\cdots,4)$ and $T_j(j=1,\cdots,4)$ and a scalar $\lambda>0$ such that the following LMIs (15) and (9) hold,

$$\begin{bmatrix} \Xi_{11}+\lambda E_a E_a^{\mathrm{T}} & \Xi_{12}+\lambda E_a E_{ad}^{\mathrm{T}} & \Xi_{13} & \Xi_{14} & -T_1 D \\ \Xi_{12}^{\mathrm{T}}+\lambda E_{ad} E_{ad}^{\mathrm{T}} & \Xi_{22}+\lambda E_{ad} E_{ad}^{\mathrm{T}} & \Xi_{23} & \Xi_{24} & -T_2 D \\ \Xi_{13}^{\mathrm{T}} & \Xi_{23}^{\mathrm{T}} & \Xi_{33} & \Xi_{34} & -T_3 D \\ \Xi_{14}^{\mathrm{T}} & \Xi_{24}^{\mathrm{T}} & \Xi_{34}^{\mathrm{T}} & \Xi_{44} & -T_4 D \\ -D^{\mathrm{T}}T_1^{\mathrm{T}} & -D^{\mathrm{T}}T_2^{\mathrm{T}} & -D^{\mathrm{T}}T_3^{\mathrm{T}} & -D^{\mathrm{T}}T_4^{\mathrm{T}} & -\lambda I \end{bmatrix}<0 \tag{15}$$

where $\Xi_{ij}(i=1,\cdots,4;i\leqslant j\leqslant 4)$ are defined in (8).

**Proof.** Replacing $A$ and $A_d$ in (8) are replaced with $A+DF(t)E_a$ and $A_d+DF(t)E_{ad}$, respectively, then (8) for the uncertain system $\Sigma$ with $\boldsymbol{u}(t)=0$ is equivalent to the following condition

$$\Xi+\Gamma_d^{\mathrm{T}}F(t)\Gamma_e+\Gamma_e^{\mathrm{T}}F^{\mathrm{T}}(t)\Gamma_d<0 \tag{16}$$

where

$$\Gamma_d=[-D^{\mathrm{T}}T_1 \quad -D^{\mathrm{T}}T_2 \quad -D^{\mathrm{T}}T_3 \quad -D^{\mathrm{T}}T_4],\quad \Gamma_e=[E_a \quad E_{ad} \quad 0 \quad 0]$$

By Lemma 1, a necessary and sufficient condition for (16) for the uncertain system $\Sigma$ is that there exists a $\lambda>0$ such that

$$\Xi+\lambda^{-1}\Gamma_d^{\mathrm{T}}\Gamma_d+\lambda\Gamma_e^{\mathrm{T}}\Gamma_e<0 \tag{17}$$

Applying Schur complements, (17) is equivalent to (15). □

## 4 Stabilization

The results of Theorem 1 can also be used to verify the stability of the closed loop obtained by applying (6) to system $\Sigma_0$(with $\boldsymbol{u}(t)\neq 0$).

**Theorem 3.** Given scalars $h>0$ and $\mu<1$, the nominal system $\Sigma_0$ of $\Sigma$ with time-varying delay satisfying (4) and (5) is stabilizable by the control law (6) for any $\tau>0$ if there exist symmetric positive definite matrix $L=L^{\mathrm{T}}>0$, symmetric semi-positive definite matrices $M=M^{\mathrm{T}}\geqslant 0$, $J=J^{\mathrm{T}}\geqslant 0$, $W=W^{\mathrm{T}}\geqslant 0$, $Y=\begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{12}^{\mathrm{T}} & Y_{22} & Y_{23} & Y_{24} \\ Y_{13}^{\mathrm{T}} & Y_{23}^{\mathrm{T}} & Y_{33} & Y_{34} \\ Y_{14}^{\mathrm{T}} & Y_{24}^{\mathrm{T}} & Y_{34}^{\mathrm{T}} & Y_{44} \end{bmatrix}\geqslant 0$, and any appropriate dimensional matrices $U_j(j=1,\cdots,4)$ and $S$, $V$ and any scalars $s_2,s_3,s_4$ such that the following matrix inequalities (18) and (19) hold

$$\Phi=\begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} \\ \Phi_{12}^{\mathrm{T}} & \Phi_{22} & \Phi_{23} & \Phi_{24} \\ \Phi_{13}^{\mathrm{T}} & \Phi_{23}^{\mathrm{T}} & \Phi_{33} & \Phi_{34} \\ \Phi_{14}^{\mathrm{T}} & \Phi_{24}^{\mathrm{T}} & \Phi_{34}^{\mathrm{T}} & \Phi_{44} \end{bmatrix}<0 \tag{18}$$

and

$$\Omega=\begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} & U_1 \\ Y_{12}^{\mathrm{T}} & Y_{22} & Y_{23} & Y_{24} & U_2 \\ Y_{13}^{\mathrm{T}} & Y_{23}^{\mathrm{T}} & Y_{33} & Y_{34} & U_3 \\ Y_{13}^{\mathrm{T}} & Y_{23}^{\mathrm{T}} & Y_{34}^{\mathrm{T}} & Y_{44} & U_4 \\ U_1^{\mathrm{T}} & U_2^{\mathrm{T}} & U_3^{\mathrm{T}} & U_4^{\mathrm{T}} & W \end{bmatrix}\geqslant 0 \tag{19}$$

where

$$\begin{aligned}
&\Phi_{11} = M + U_1 + U_1^{\mathrm{T}} - AS^{\mathrm{T}} - BV - SA^{\mathrm{T}} - V^{\mathrm{T}}B^{\mathrm{T}} + hY_{11}\\
&\Phi_{12} = U_2^{\mathrm{T}} - U_1 - s_2(SA^{\mathrm{T}} + V^{\mathrm{T}}B^{\mathrm{T}}) - A_dS^{\mathrm{T}} + hY_{12}\\
&\Phi_{13} = L + U_3^{\mathrm{T}} + S^{\mathrm{T}} - s_3(SA^{\mathrm{T}} + V^{\mathrm{T}}B^{\mathrm{T}}) + hY_{13}, \quad \Phi_{14} = U_4^{\mathrm{T}} - s_4(SA^{\mathrm{T}} + V^{\mathrm{T}}B^{\mathrm{T}}) - CS^{\mathrm{T}} + hY_{14}\\
&\Phi_{22} = -(1-\mu)M - U_2 - U_2^{\mathrm{T}} - s_2(SA_d^{\mathrm{T}} + A_dS^{\mathrm{T}}) + hY_{22}, \quad \Phi_{23} = -U_3^{\mathrm{T}} + s_2S^{\mathrm{T}} - s_3SA_d^{\mathrm{T}} + hY_{23}\\
&\Phi_{24} = -U_4^{\mathrm{T}} - s_4SA_d^{\mathrm{T}} - s_2CS^{\mathrm{T}} + hY_{24}, \quad \Phi_{33} = J + hW + s_3(S + S^{\mathrm{T}}) + hY_{33}\\
&\Phi_{34} = s_4S - s_3CS^{\mathrm{T}} + hY_{34}, \quad \Phi_{44} = -J - s_4(CS^{\mathrm{T}} + SC^{\mathrm{T}}) + hY_{44}
\end{aligned}$$

Moreover, a stabilizing control law is given by $\boldsymbol{u}(t) = V[S^{-1}]^{\mathrm{T}}\boldsymbol{x}(t)$.

**Proof.** With the memoryless state-feedback control law $\boldsymbol{u}(t) = K\boldsymbol{x}(t)$, where the matrix $K \in \mathcal{R}^{m\times n}$ is to be found, the system $\Sigma_0$ becomes

$$\dot{\boldsymbol{x}}(t) - C\dot{\boldsymbol{x}}(t-\tau) = (A+BK)\boldsymbol{x}(t) + A_d\boldsymbol{x}(t-d(t)) \tag{20}$$

Now, we replace $A$ in (8) with $A+BK$ and set $T_1 = T$ and $T_2 = s_2T$, $T_3 = s_3T$, $T_4 = s_4T$. From the fact that in (8) $T_3+T_3^{\mathrm{T}}$ must be negative definite it is obvious that $T_3$ is nonsingular and $T$ is also nonsingular. Then, pre- and postmultiply (8) by $\mathrm{diag}(T^{-1}, T^{-1}, T^{-1}, T^{-1})$ and $\mathrm{diag}([T^{-1}]^{\mathrm{T}}, [T^{-1}]^{\mathrm{T}}, [T^{-1}]^{\mathrm{T}}, [T^{-1}]^{\mathrm{T}})$ and pre- and postmultiply (9) by $\mathrm{diag}(T^{-1}, T^{-1}, T^{-1}, T^{-1}, T^{-1})$ and $\mathrm{diag}([T^{-1}]^{\mathrm{T}}, [T^{-1}]^{\mathrm{T}}, [T^{-1}]^{\mathrm{T}}, [T^{-1}]^{\mathrm{T}}, [T^{-1}]^{\mathrm{T}})$, respectively and change variables such that $S = T^{-1}, L = SPS^{\mathrm{T}}, M = SQS^{\mathrm{T}}, U_i = SN_iS^{\mathrm{T}}(i = 1,\cdots,4), Y_{ij} = SX_{ij}S^{\mathrm{T}}(i = 1,\cdots,4; i \leqslant j \leqslant 4), W = SZS^{\mathrm{T}}, J = SRS^{\mathrm{T}}$ and $K = V[S^{-1}]^{\mathrm{T}}$. Then we obtain (18) and (19). This completes the proof. □

Similar to Theorem 3, the stabilizing memoryless controller (6) for uncertain system $\Sigma$ can also be designed in the following from Theorem 2.

**Theorem 4.** Given scalars $h > 0$ and $\mu < 1$, the uncertain system $\Sigma$ with time-varying delay satisfying (4) and (5) is robustly stabilizable by the control law (6) for any $\tau > 0$ if there exist symmetric positive definite matrix $L = L^{\mathrm{T}} > 0$, symmetric semi-positive definite matrices $M = M^{\mathrm{T}} \geqslant 0$, $J = J^{\mathrm{T}} \geqslant 0$, $W = W^{\mathrm{T}} \geqslant 0$, $Y = Y^{\mathrm{T}} \geqslant 0$, and any appropriate dimensional matrices $U_j(j = 1,\cdots,4)$ and $S$, $V$ and any scalars $s_2, s_3, s_4$ and a scalar $\lambda > 0$ such that the following matrix inequalities (21) and (19) hold

$$\begin{bmatrix}
\Phi_{11} + \lambda DD^{\mathrm{T}} & \Phi_{12} + s_2\lambda DD^{\mathrm{T}} & \Phi_{13} + s_3\lambda DD^{\mathrm{T}} & \Phi_{14} + s_4\lambda DD^{\mathrm{T}} & -SE_a^{\mathrm{T}}\\
\Phi_{12}^{\mathrm{T}} + s_2\lambda DD^{\mathrm{T}} & \Phi_{22} + s_2^2\lambda DD^{\mathrm{T}} & \Phi_{23} + s_2s_3\lambda DD^{\mathrm{T}} & \Phi_{24} + s_2s_4\lambda DD^{\mathrm{T}} & -SE_{ad}^{\mathrm{T}}\\
\Phi_{13}^{\mathrm{T}} + s_3\lambda DD^{\mathrm{T}} & \Phi_{23}^{\mathrm{T}} + s_2s_3\lambda DD^{\mathrm{T}} & \Phi_{33} + s_3^2\lambda DD^{\mathrm{T}} & \Phi_{34} + s_3s_4\lambda DD^{\mathrm{T}} & 0\\
\Phi_{14}^{\mathrm{T}} + s_4\lambda DD^{\mathrm{T}} & \Phi_{24}^{\mathrm{T}} + s_2s_4\lambda DD^{\mathrm{T}} & \Phi_{34}^{\mathrm{T}} + s_3s_4\lambda DD^{\mathrm{T}} & \Phi_{44} + s_4^2\lambda DD^{\mathrm{T}} & 0\\
-E_aS^{\mathrm{T}} & -E_{ad}S^{\mathrm{T}} & 0 & 0 & -\lambda I
\end{bmatrix} < 0 \tag{21}$$

where $\Phi_{ij}(i = 1,\cdots,4; i \leqslant j \leqslant 4)$ are defined in (18). Moreover, a stabilizing control law is given by $\boldsymbol{u}(t) = V[S^{-1}]^{\mathrm{T}}\boldsymbol{x}(t)$.

**Remark 2.** If matrices $Q$ in Theorems 1 and 2, and $M$ in Theorems 3 and 4 are set to 0, then the Theorems do not include $\mu$, which is the bound of the derivative of the delay $d(t)$. Thus, the rate-independent and delay-dependent criteria for system $\Sigma$ with time-varying delay satisfying (4) can be derived by following Theorems 1,2,3,4.

## 5 Examples

In this section, two numerical examples are presented to compare with the proposed stabilization methods with previous results.

**Example 1.** Consider the uncertain system $\Sigma$ with

$$A = \begin{bmatrix} 0 & 0\\ 0 & 1\end{bmatrix},\ A_d = \begin{bmatrix} -2 & -0.5\\ 0 & -1\end{bmatrix},\ \boldsymbol{B} = \begin{bmatrix} 0\\ 1\end{bmatrix},\ D = I,\ C = 0,\ E_a = 0.2I,\ E_{ad} = \alpha I$$

In [4] and [8], $\alpha$ is 0.2 and 0, respectively. For the case of $\mu = 0$, the maximum bound of $h$ for which the system is stabilized by a state-feedback was found to be 0.45 and 0.5865 in [4] and [8], respectively ($\alpha = 0$ in [8]). When $\alpha = 0.2$, applying the result of Theorem 4, a maximum bound of $h = 0.7226$ is obtained using $s_2 = 0.35$, $s_3 = 0.95$ and $s_4 = 0$, the corresponding feedback gain matrix $K = -[3.3854\ 2.2804] \times 10^4$. When $\alpha = 0$, the corresponding results are $h = 0.9169$ and $K = -[2.0679\ 0.9166] \times 10^4$ using $s_2 = 0.3$, $s_3 = 1$ and $s_4 = 0$.

Moreover, when $\alpha = 0.2$, applying the result of Theorem 4, the maximum bound of $h$ for varying $\mu$ is listed in Table 1 using $s_2 = 0.35$, $s_3 = 0.95$ and $s_4 = 0$.

Table 1 Upper bounds $h$ of time-delay for varying $\mu$ ($\alpha = 0.2$)

| $\mu$ | 0 | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 | 0.95 | 0.98 | 0.99 | 0.999 |
|---|---|---|---|---|---|---|---|---|---|---|
| Theorem 4 | 0.722 | 0.713 | 0.695 | 0.683 | 0.681 | 0.647 | 0.553 | 0.491 | 0.471 | 0.453 |

**Example 2.** Consider the problem of finding a state-feedback stabilizing controller for nominal system $\Sigma_0$ with

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ A_d = \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix},\ \boldsymbol{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C = 0$$

When $\mu = 0$, the upper bound of $h$ obtained in [9], [8] and [10] were 1.408, 1.51 and 3.2, respectively. Applying Theorem 3, the upper bound of $h$ is 50.03 for $s_2 = 0$, $s_3 = 1.0$ and $s_4 = 0$ and $K = -[7.7843\ 7.7898] \times 10^5$.

## 6 Conclusion

In this paper, new techniques for delay-dependent stability and stabilization of an uncertain neutral system have been developed, in which the relationships between the terms in the Leibniz-Newton formula are taken into account. Then some free-weighting matrices that express the influence of these terms are determined based on linear matrix inequalities. Finally, some numerical examples show that the results obtained in this paper are very effective and are a significant improvement over the existing results.

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