New Approach to Designing Constrained Predictive Controllers[1)]

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Abstract Using the framework of predictive control algorithms and the analysis results of unconstrained predictive control systems, the desired pole placement is achieved by the coefficient mapping between the characteristic polynomials of the closed-loop system and the open-loop plant. The designed control law can not only ensure the dynamical performance of the closed-loop system, but also provide plentiful degrees of freedom to satisfy input-output constraints. Based on the theory of invariant set, this paper derives some sufficient conditions for the satisfaction of constraints with these degrees of freedom and presents an approach to design corresponding constrained controller. The constrained controller with performance guarantee can be designed off-line. Furthermore, it has convenient on-line computation and satisfies all constraints. A simulation example is presented to illustrate the proposed approach.

Key words Constraints, invariant set, controller design, stability

1 Introduction

Model predictive control (MPC) has been found wide applications in industry[1]. The key idea of MPC is: at each sampling instant, based on the system equation as the predictive model, solve a finite-horizon optimization problem to obtain a control sequence whose first control is applied to the plant. At the next sampling instant, repeat the above procedure and a receding horizon optimization scheme is then established. Due to its efficiency in handling input and state constraints, MPC has attracted wide attention in recent years[2,3]. For an overview on MPC, please refer to [4]. With a time-varying ellipsoid invariant set, [5] established the closed-loop stability of MPC for linear discrete systems. Considering the real-time requirement of practical implementations, [6] presented a robust model predictive controller based on aggregation of optimization variables, which can reduce the on-line computation load significantly.

Generally speaking, quadratic performance functions are often adopted in MPC algorithms. For the unconstrained case, the analytical solution to the finite-horizon optimization problem can be determined off-line, which makes it possible to have an performance analysis for the closed-loop system. Furthermore, the on-line optimization is reduced into the simple vector-matrix computation which is well acceptable in practical applications. However, there exist various constraints in industrial applications. The finite-horizon optimization at each time is practically a nonlinear optimization problem mixed with quadratic performance function and linear constraints, which must be solved by quadratic programming on-line. In such cases, not only heavy on-line computation demand is inevitable, but also the exact evaluation of the system performance becomes impossible.

For the MPC of single input and single output (SISO) unconstrained systems, many conclusions about performance analysis have been reported in the literature. For example, [7,8] studied dynamic matrix control (DMC) and the generalized predictive control (GPC) in internal model control (IMC) structure, respectively, and derived some conclusions about stability and the quantitative relationship between deadbeat and design parameters. However, these results are not available to the constrained case. The main reason is that the unconstrained controller is resulted from optimization of the quadratic performance function. The uniqueness of the solution leads to no guarantee of constraints. For the constrained case, all constraint conditions have to be imposed in the on-line optimization and the optimal solution should be derived through a nonlinear programming procedure, which prevents the analytic results of unconstrained systems from applying to the constrained case. Based on the above analysis, this paper will design a constrained MPC controller from a new view. The main idea is: firstly, we design an unconstrained control law using the analytic results and quantitative representations.

1) Supported by Shanghai Science & Technology Development Fund (04DZ11008)
Received June 7, 2004; in revised form January 25, 2005

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Different from the previous method, the control law is derived by pole placement rather than by quadratic performance optimization, so it is not unique and has redundant freedoms. Next, we use these degrees of freedom to satisfy constraints based on the theory of invariant set. In this way, the designed controller can satisfy constraints. The quantitative analysis for the closed-loop system is still available while keeping analytical form. Therefore, it is an off-line designed, easily on-line solved controller together with performance guarantee and constraints satisfaction.

2 Design of pole placement for unconstrained MPC

In this paper, we use DMC as an example to illustrate the design procedure of the new type of controller. For simplicity, the details of DMC can be found in [9]. In the unconstrained case, Fig. 1 shows the flow of a DMC algorithm.

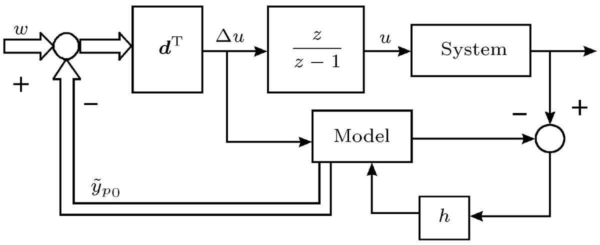


Fig. 1 The structure of DMC algorithm

In Fig. 1 $\tilde{\boldsymbol{y}}_{p0}$ is the initial predictive vector based on the step-response model at time k, $\boldsymbol{w}$ denotes the desired output, p represents the prediction horizon, *i.e.*, the time horizon in which the on-line optimization is considered, $\boldsymbol{h}$ is the error-correction vector, and $\boldsymbol{d}^{\mathrm{T}} = [d_1 \cdots d_p]$ is the control vector. In the unconstrained DMC, $\boldsymbol{d}^{\mathrm{T}}$ is derived by optimizing the quadratic performance function and has the following analytical form:

$$\boldsymbol{d}^{\mathrm{T}} = [1 \quad 0 \quad \cdots \quad 0](A^{\mathrm{T}}QA + R)^{-1}A^{\mathrm{T}}Q \tag{1}$$

where A denotes the dynamic matrix, Q and R are the weighing matrices in performance function[9]. From (1), it is easy to see that $\boldsymbol{d}^{\mathrm{T}}$ has been determined uniquely.

From Fig. 1, an unconstrained control law can be calculated by:

$$\Delta\boldsymbol{u}(k) = \boldsymbol{d}^{\mathrm{T}}[w_p(k) - \tilde{y}_{p0}(k)] \tag{2a}$$

$$\boldsymbol{u}(k) = u(k-1) + \Delta\boldsymbol{u}(k) \tag{2b}$$

Apparently, the on-line computation for the above control law (2) is easy and simple.

For unconstrained DMC, [10] analyzed Fig. 1 in an internal model control structure and derived the following conclusion.

Lemma 1[10]**.** For a system:

$$G_p(z^{-1}) = \frac{m(z)}{P(z)} = \frac{m_1 z^{-1} + \cdots + m_n z^{-n}}{1 + p_1 z^{-1} + \cdots + p_n z^{-n}} \tag{3}$$

with the unconstrained DMC control law (2), the closed-loop transfer function can be described as

$$F(z^{-1}) = \frac{d_s m(z^{-1})}{P^*(z^{-1})} = \frac{d_s(m_1 z^{-1} + \cdots + m_n z^{-n})}{1 + p_1^* z^{-1} + \cdots + p_{n+1}^* z^{-(n+1)}} \tag{4}$$

where

$$d_s = \sum_{i=1}^{p} d_i \tag{5}$$

$$\begin{bmatrix} 1 \\ p_1^* \\ \vdots \\ p_{n+1}^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ b_2 - 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n+2} - b_{n+1} & b_{n+1} - b_n & \cdots & b_2 - 1 \end{bmatrix} \begin{bmatrix} 1 \\ p_1 \\ \vdots \\ p_n \end{bmatrix} \tag{6}$$

$$b_i = \sum_{j=1}^{p} d_j a_{i+j-1}, \quad i = 2, 3 \cdots, n+2 \tag{7}$$

and $a_{i+j-1}(i = 2, 3 \cdots, n+2)$ denote the unit step-response coefficients of the system.

Lemma 1 reveals the coefficient mapping between the characteristic polynomials of the close-loop system designed by the algorithm in Fig. 1 and the open-loop plant. Based on this mapping, we can make quantitative performance analysis for predictive control systems in the frequency domain. Just based on that and the expression of $\boldsymbol{d}^{\mathrm{T}}$ in the original DMC algorithm, [10] presented a number of quantitative relationships about the closed-loop performance and design parameters for predictive control systems.

It should be mentioned that during transforming Fig. 1 into IMC structure and deriving (6), no assumption on $\boldsymbol{d}^{\mathrm{T}}$ having the form (1) is given. That means (6) is independent of the expression of $\boldsymbol{d}^{\mathrm{T}}$. It will be the key in our design. Note that the dimension of $\boldsymbol{d}^{\mathrm{T}} \in R^p$ represents the length of prediction horizon in the original DMC and there are no restrictions on it. We can select the dimension of $\boldsymbol{d}^{\mathrm{T}}$ arbitrarily. In order to ensure enough degrees of freedom of $\boldsymbol{d}^{\mathrm{T}}$, we will not use the unique $\boldsymbol{d}^{\mathrm{T}}$ derived from quadratic performance optimization with the form (1), but design the closed-loop predictive control system by pole placement.

According to (6), we can select $n+1$ parameters $b_2, \cdots, b_{n+2}$ to assign the poles of the closed-loop system in order to achieve the desired dynamical response. Equivalently, we can do this by assigning coefficients $p_1^*, \cdots, p_{n+1}^*$ of the closed-loop characteristic polynomial.

From (6), it is easy to have

$$\begin{bmatrix} p_1^* \\ p_2^* \\ \vdots \\ p_{n+1}^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ p_1 - 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_n - p_{n-1} & p_{n-1} - p_{n-2} & \cdots & 1 \end{bmatrix} \begin{bmatrix} b_2 \\ b_3 \\ \vdots \\ b_{n+2} \end{bmatrix} + \begin{bmatrix} p_1 - 1 \\ p_2 - p_1 \\ \vdots \\ -p_n \end{bmatrix} \tag{8}$$

Then parameters b_i satisfying the pole placement condition can be solved by

$$\begin{bmatrix} b_2 \\ b_3 \\ \vdots \\ b_{n+2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ p_1 - 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_n - p_{n-1} & p_{n-1} - p_{n-2} & \cdots & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} p_1^* \\ p_2^* \\ \vdots \\ p_{n+1}^* \end{bmatrix} - \begin{bmatrix} p_1 - 1 \\ p_2 - p_1 \\ \vdots \\ -p_n \end{bmatrix} \right) \tag{9}$$

Since the parameters b_i are defined by (7), *i.e.*,

$$b_i = \sum_{j=1}^{p} d_j a_{i+j-1} = [d_1 \ \cdots \ d_p] \begin{bmatrix} a_i \\ a_{i+1} \\ \vdots \\ a_{i+p-1} \end{bmatrix}, \quad i = 2, 3, \cdots, n+2 \tag{10}$$

We have

$$\begin{bmatrix} a_2 & a_3 & \cdots & a_{p+1} \\ a_3 & a_4 & \cdots & a_{p+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+2} & a_{n+3} & \cdots & a_{n+p=1} \end{bmatrix}_{(n+1)\times p} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_p \end{bmatrix} = \begin{bmatrix} b_2 \\ b_3 \\ \vdots \\ b_{n+2} \end{bmatrix} \tag{11}$$

Since $\{a_i\}$ are coefficients of the unit-step response, the rank of the matrix in (11) is $n+1(p \geqslant n+1)$[11]. By the basic property of linear algebra, the solution of $\boldsymbol{d}^{\mathrm{T}}$ in (11) is unique if $p = n+1$; but if $p > n+1$, there are infinite solutions for $\boldsymbol{d}^{\mathrm{T}}$. The general solution for $\boldsymbol{d}^{\mathrm{T}}$ satisfying (11) can be expressed as

$$\boldsymbol{d}^{\mathrm{T}} = \boldsymbol{d}_0^{\mathrm{T}} + s^{\mathrm{T}} K \tag{12}$$

where $\boldsymbol{d}_0^{\mathrm{T}}$ is a special solution to (11), and $K = [k_1 \quad k_2 \quad \cdots \quad k_{p-n-1}]$ is a basic solution set of the equation below:

$$\begin{bmatrix} a_2 & a_3 & \cdots & a_{p+1} \\ a_3 & a_4 & \cdots & a_{p+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+2} & a_{n+3} & \cdots & a_{n+p+1} \end{bmatrix}_{(n+1)\times p} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{13}$$

$\boldsymbol{s}^{\mathrm{T}} = [s_1 \quad s_2 \quad \cdots \quad s_{p-n-1}]$ is an arbitrary vector. Owing to the existence of $\boldsymbol{s}^{\mathrm{T}}$, there are still redundant freedoms after placing the desired closed-loop poles with $\boldsymbol{d}^{\mathrm{T}}$. In the next section, we will use these redundant freedoms to satisfy all constraints.

3 The constrained control law based on invariant set

In practical applications, the control input and its increment are always limited by the physical properties of the actuator, *i.e.*,

$$|u| \leqslant u_{\max} \tag{14}$$

$$|\Delta u| \leqslant \Delta u_{\max} \tag{15}$$

At the same time, the system output is often asked to be constrained in a limited range around the setpoint, *i.e.*,

$$|w - y| \leqslant y_{\max} \tag{16}$$

At time k, we assume the initial predictive output vector is given by $\tilde{\boldsymbol{y}}_{p0}(k) = [y_0(k+1|k) \ \cdots \ y_0(k+p|k)]^{\mathrm{T}}$, and the control law is given by (2), where $\boldsymbol{d}^{\mathrm{T}}$ is the control gain given by (12), and $\tilde{\boldsymbol{w}}_p(k)$ is the desired output vector. For the simplicity of discussion, we suppose $\tilde{\boldsymbol{w}}_p(k)$ is a constant vector with all elements being w.

According to the DMC algorithm[10], the system output at time k is

$$\tilde{\boldsymbol{y}}_{p1}(k) = \tilde{\boldsymbol{y}}_{p0}(k) + a\Delta u(k) \tag{17}$$

where $\tilde{\boldsymbol{y}}_{p1}(k) = [y_1(k+1|k) \ \cdots \ y_1(k+p|k)]^{\mathrm{T}}$ is the output prediction after the implementation of $\Delta u(k)$, and $a = [a_1 \quad \cdots \quad a_p]^{\mathrm{T}}$ is the step-response vector. Then, at time $k+1$, the initial predictive output vector $\tilde{\boldsymbol{y}}_{p0}(k+1)$ can be obtained by the shift of $\tilde{\boldsymbol{y}}_{p1}(k)$:

$$\tilde{\boldsymbol{y}}_{p0}(k+1) = S\tilde{\boldsymbol{y}}_{p1}(k) \tag{18}$$

where $S = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \vdots \\ \vdots & \vdots & \vdots & 1 \\ 0 & 0 & \cdots & 1 \end{bmatrix}$ is the shift matrix.

To establish the connections between vectors at time k and those at time $k+1$, we introduce

$$\boldsymbol{x}(k) = \tilde{w}_p(k) - \tilde{y}_{p0}(k) \tag{19}$$

By (17) and (18), we can get

$$\begin{aligned} \boldsymbol{x}(k+1) = \tilde{w}_p(k+1) - \tilde{y}_{p0}(k+1) = \tilde{w}_p(k+1) - S(\tilde{y}_{p0}(k) + a\Delta u(k)) = \\ \tilde{w}_p(k+1) - S(\tilde{w}_p(k) - x(k) + a\Delta u(k)) = Sx(k) - Sa\Delta u(k) \end{aligned} \tag{20}$$

$$\Delta u(k+1) = d^{\mathrm{T}}(\tilde{w}_p(k+1) - \tilde{y}_{p0}(k+1)) = d^{\mathrm{T}}Sx(k) - d^{\mathrm{T}}Sa\Delta u(k) \tag{21}$$

Combining (2b) with (20)(21), we have

$$\tilde{\boldsymbol{x}}(k+1) = A\tilde{\boldsymbol{x}}(k) \tag{22}$$

where

$$\tilde{\boldsymbol{x}}(k) = \begin{bmatrix} x(k) \\ \Delta u(k) \\ u(k) \end{bmatrix}, \quad A = \begin{bmatrix} S & -Sa & 0 \\ d^{\mathrm{T}}S & -d^{\mathrm{T}}Sa & 0 \\ d^{\mathrm{T}}S & -d^{\mathrm{T}}Sa & 1 \end{bmatrix} \tag{23}$$

Next, we will discuss the problem that if all the elements of $\tilde{\boldsymbol{x}}(k)$ meet the following constraints at time k, *i.e.*,

$$|x(k)|_i \leqslant y_{\max}, \qquad |\Delta u(k)| \leqslant \Delta u_{\max}, \qquad |u(k)| \leqslant u_{\max} \tag{24}$$

then how to choose $\boldsymbol{s}^{\mathrm{T}}$ in $\boldsymbol{d}^{\mathrm{T}}$ such that all the element of $\tilde{\boldsymbol{x}}(k+1)$ meet the above conditions. We have the following theorem.

Theorem 1. If there exist a symmetric matrix $Q \geqslant 0$ and a vector $\boldsymbol{s}^{\mathrm{T}}$ such that the following conditions are true

$$\begin{bmatrix} -y_{\max}^2 Q & S_i^{\mathrm{T}} \\ S_i & -1 \end{bmatrix} \leqslant 0, \quad i = 1, 2 \cdots p \tag{25}$$

$$\begin{bmatrix} -\Delta u_{\max}^2 Q & S_{p+1}^{\mathrm{T}} + S_0^{\mathrm{T}} s \\ S_{p+1} + s^{\mathrm{T}} S_0 & -1 \end{bmatrix} \leqslant 0 \tag{26}$$

$$\begin{bmatrix} -u_{\max}^2 Q & S_{p+2}^{\mathrm{T}} + S_0^{\mathrm{T}} s \\ S_{p+2} + s^{\mathrm{T}} S_0 & -1 \end{bmatrix} \leqslant 0 \tag{27}$$

$$A^{\mathrm{T}} Q A - Q \leqslant 0 \tag{28}$$

$$\tilde{x}^{\mathrm{T}}(0) Q \tilde{x}(0) < 1 \tag{29}$$

where $S_0 = [KS \quad -KSa \quad 0]$, $S_i = [S \quad -Sa \quad 0]_i$, $S_{p+1} = [d_0^{\mathrm{T}} S \quad -d_0^{\mathrm{T}} Sa \quad 0]$, and $S_{p+2} = [d_0^{\mathrm{T}} S \quad -d_0^{\mathrm{T}} Sa \quad 1]$, then if $\tilde{\boldsymbol{x}}(k)$ satisfies constraint (24) at time k, all future $\tilde{\boldsymbol{x}}(k+i)$ will never violate (24).

Proof. For any k, if there exists $Q \geqslant 0$ satisfying (28), then according to (22) and (29), we can get

$$\tilde{\boldsymbol{x}}^{\mathrm{T}}(k+1) Q \tilde{\boldsymbol{x}}(k+1) \leqslant \tilde{\boldsymbol{x}}^{\mathrm{T}}(k) Q \tilde{\boldsymbol{x}}(k) \leqslant \cdots \leqslant \tilde{\boldsymbol{x}}^{\mathrm{T}}(0) Q \tilde{\boldsymbol{x}}(0) < 1 \tag{30}$$

and the following formula holds for all $k \geqslant 0$:

$$R(k) = \{x(k) | \tilde{\boldsymbol{x}}^{\mathrm{T}}(k) Q \tilde{\boldsymbol{x}}(k) < 1\} \subseteq R(0) \tag{31}$$

Namely, $R(k)$ is an invariant set.

Suppose $\tilde{\boldsymbol{x}}(k)$ in set $R(k)$ at time k meets the following constraints

$$|\tilde{x}(k)|_i \leqslant (\tilde{x}_{\max})_i \tag{32}$$

where $\tilde{x}_{\max}$ denotes the upper bound of $\tilde{\boldsymbol{x}}(k)$, $|\cdot|_i$ represents the absolute value of the ith element, and $(\cdot)_i$ is the ith element of a vector. Then at time $k+1$, the sufficient and necessary condition for $\tilde{x}(k+1)$ to meet constraint (32) is

$$|A\tilde{x}(k)|_i \leqslant (\tilde{x}_{\max})_i \tag{33}$$

Since

$$|A\tilde{x}(k)|_i^2 = |A_i Q^{-1/2} Q^{1/2} \tilde{x}(k)|^2 \leqslant \tilde{x}^{\mathrm{T}}(k) Q \tilde{x}(k) A_i Q^{-1} A_i^{\mathrm{T}} \leqslant \tilde{x}^{\mathrm{T}}(0) Q \tilde{x}(0) A_i Q^{-1} A_i^{\mathrm{T}} \leqslant A_i Q^{-1} A_i^{\mathrm{T}} \tag{34}$$

where $\boldsymbol{A}_i$ denotes the ith row vector of matrix A, (33) is implied by

$$\boldsymbol{A}_i Q^{-1} \boldsymbol{A}_i^{\mathrm{T}} \leqslant (\tilde{x}_{\max})_i^2 \tag{35}$$

Dividing both sides of (35) by positive scalars $(\tilde{x}_{\max})_i^2$ and using Schur complement, we can get the equivalent form of (35) as

$$\begin{bmatrix} -(\tilde{x}_{\max})_i^2 Q & \boldsymbol{A}_i^{\mathrm{T}} \\ \boldsymbol{A}_i & -1 \end{bmatrix} \leqslant 0 \tag{36}$$

By replacing $\boldsymbol{A}_i$ with the row vector of A in (23) and substituting $y_{\max}, u_{\max}$ and $\Delta u_{\max}$ into $\tilde{x}_{\max}$, (36) holds if (25)$\sim$(27) are satisfied. □

Remark 1. Theorem 1 proposes an efficient approach to design constrained predictive controllers. Once the free vector $\boldsymbol{s}^{\mathrm{T}}$ is designed off-line, the corresponding $\boldsymbol{d}^{\mathrm{T}}$ can be determined. Then, only simple computation is needed on-line. Therefore, the resulted closed-loop system has desired dynamical performance. Moreover, its input and output always satisfy constraints.

Although Theorem 1 presents an approach to design $\boldsymbol{s}^{\mathrm{T}}$, both matrix Q and vector $\boldsymbol{s}^{\mathrm{T}}$ are unknown in (25)$\sim$(29). It implies that these conditions construct a set of nonlinear matrix inequalities which are difficult to solve. Next, we give an efficient algorithm for designing $\boldsymbol{s}^{\mathrm{T}}$ by Theorem 1.

Algorithm 1.

Step 1. Calculate the unit step-response coefficients $\{a_i\}$ of the system based on its transfer function and give a set of desired closed-loop poles, then solve parameters $\{b_i\}$ according to (9).

Step 2. Select a prediction horizon $p > n+1$, where n is the dimension of the system, determine $\boldsymbol{d}_0^{\mathrm{T}}$ using (11). Simultaneously, solve the set of the basic solution $K = [k_1 \quad k_2 \quad \cdots \quad k_{p-n-1}]$ with (13).

Step 3. Select a symmetric matrix $Q_1 \geqslant 0$ and substitute it into (25)~(27), then solve the LMI-based optimization problem, *i.e.*,

$$\min\ a \quad \text{s.t.} \begin{cases} \text{Inequaties } (25 \sim 27) \\ Q_1 \leqslant aI_{p+2} \end{cases} \tag{37}$$

Step 4. If there exist feasible solutions in Step 3, then substitute the determined $\boldsymbol{s}^{\mathrm{T}}$ into (12) to obtain $\boldsymbol{d}^{\mathrm{T}}$, and into (23) to obtain A. Solve the following LMIs:

$$\begin{cases} A^{\mathrm{T}}Q_2A - Q_2 \leqslant 0 \\ Q_1 \leqslant Q_2 \\ \tilde{\boldsymbol{x}}^{\mathrm{T}}(0)Q_2\tilde{\boldsymbol{x}}(0) < 1 \end{cases} \tag{38}$$

Step 5. If the problems in Step 3 and 4 are feasible, $\boldsymbol{d}^{\mathrm{T}}$ is the final result for the on-line control law (2). Otherwise, increase the prediction horizon p and go to Step 2.

Remark 2. In the proof of Theorem 1, we concluded that (25)~(27) are equivalent to condition (35). Therefore, if there exists a feasible solution in Step 3, we can get $A_iQ_1^{-1}A_i^{\mathrm{T}} \leqslant (\tilde{x}_{\max})_i^2$. Next, if the LMIs in Step 4 are feasible, then $Q_1 \leqslant Q_2$ ensures that $A_iQ_2^{-1}A_i^{\mathrm{T}} \leqslant A_iQ_1^{-1}A_i^{\mathrm{T}} \leqslant (\tilde{x}_{\max})_i^2$ is true, *i.e.*, (25)~(27) are still satisfied when Q_1 is replaced by Q_2.

4 Simulation example

In this section, an illustrative example is presented to verify the feasibility of Algorithm 1. The software used in this example is LMI Toolbox of Matlab.

Suppose the transfer function of the plant is

$$G_P(z^{-1}) = \frac{z^{-1}}{1 + 0.85z^{-1}}$$

We select the prediction horizon $p = 5$, the desired output $w = 0.3$. The desired closed-loop characteristic polynomial is $P^*(z) = 1 - 0.3z^{-1} + 0.02z^{-2}$. We consider following constraints:

$$|u| \leqslant 0.8, \qquad |\Delta u| \leqslant 0.4, \qquad |w - y| \leqslant 0.9$$

Other parameters are given as follows

$$S_0 = \begin{bmatrix} 0 & -0.5269 & 0.0730 & 0.8087 & -0.3548 & 0.0833 & 0 \\ 0 & -0.1948 & -0.6544 & 0.0899 & 0.7592 & -0.0155 & 0 \\ 0 & -0.4770 & -0.0362 & -0.1491 & 0.6623 & -0.3752 & \end{bmatrix}$$

$$\boldsymbol{S}_1 = [0 \quad 1.0000 \quad 0 \quad 0 \quad 0 \quad -0.1500 \quad 0]$$

$$\boldsymbol{S}_2 = [0 \quad 0 \quad 1.0000 \quad 0 \quad 0 \quad -0.8725 \quad 0]$$

$$\boldsymbol{S}_3 = [0 \quad 0 \quad 0 \quad 1.0000 \quad 0 \quad -0.2584 \quad 0]$$

$$\boldsymbol{S}_4 = [0 \quad 0 \quad 0 \quad 0 \quad 1.0000 \quad -0.7804 \quad 0]$$

$$\boldsymbol{S}_6 = [0 \quad 1.0771 \quad -0.3571 \quad 0 \quad 0 \quad -0.1500 \quad 0]$$

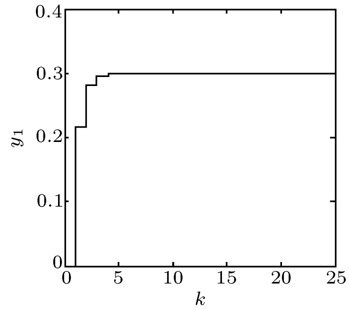
$$\boldsymbol{S}_7 = [0 \quad 1.0771 \quad -0.3571 \quad 0 \quad 0 \quad -0.1500 \quad 1.0000]$$

By solving the optimization problem in Algorithm 1 with the above parameters and initial state $\tilde{\boldsymbol{x}}(0) = [0; 0; 0; 0; 0; 0; 0]$, we can get the predictive control law as

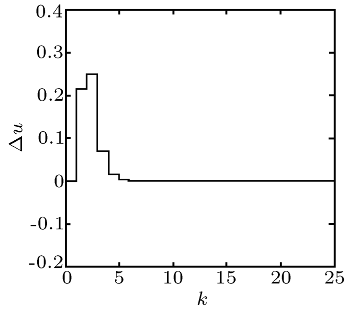
$$\Delta u(k) = [0.2708 \quad 0.0930 \quad 0.2430 \quad -0.8401 \quad 0.9534]\lfloor \tilde{w}_p(k) - \tilde{y}_{P0}(k) \rfloor$$

The trajectories of output, input and input increment are shown in Fig. 2.

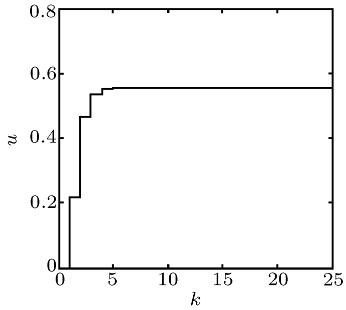
From the simulation above, we can see that the output tracks the desired reference perfectly. Moreover, input, output and input increment do not exceed their constraints during the entire process of implementation.



(a) The trajectory of output



(b) The trajectory of input increment



(c) The trajectory of input

Fig. 2 Simulation results

4 Conclusion

This paper investigates the design of predictive controllers with input and output constraints. The core idea is to assign the closed-loop poles using the coefficient mapping between the characteristic polynomials of the closed-loop system and the open-loop plant. The stability and dynamical performance of the closed-loop system are firstly guaranteed. Next, extend the prediction horizon to increase the degrees of freedom of the control, and use these redundant freedoms to meet all kinds of constraints. In fact, either input, input increment or output has close connection with these freedoms. If appropriate conditions are satisfied with these freedoms, the resulted controller can not only ensure the dynamical performance of the closed-loop system, but also guarantee the satisfaction of constraints. Based on the invariant set method, the sufficient conditions for the existence of the controller are presented, and the feasible algorithm is also given. Using the LMI Toolbox of Matlab, it is easy to design the constrained predictive controllers.

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