H_∞ Optimal Model Reduction for Singular Fast Subsystems[1)]

WANG Jing[1] ZHANG Qing-Ling[1] LIU Wan-Quan[2] ZHOU Yue[3]

[1](*College of Sciences, Northeastern University, Shenyang* 110004)

[2](*School of Computing, Curtin University of Technology, Australia, WA* 6102)

[3](*College of Information Sciences and Engineering, Shenyang University of Technology, Shenyang* 110004)

(E-mail: wj-and-zhang@163.com,)

Abstract In this paper, H_∞ optimal model reduction for singular fast subsystems will be investigated. First, error system is established to measure the error magnitude between the original and reduced systems, and it is demonstrated that the new feature for model reduction of singular systems is to make H_∞ norm of the error system finite and minimal. The necessary and sufficient condition is derived for the existence of the H_∞ suboptimal model reduction problem. Next, we give an exact and practicable algorithm to get the parameters of the reduced subsystems by applying the matrix theory. Meanwhile, the reduced system may be also impulsive. The advantages of the proposed algorithm are that it is more flexible in a straight-forward way without much extra computation, and the order of the reduced systems is as minimal as possible. Finally, one illustrative example is given to illustrate the effectiveness of the proposed model reduction approach.

Key words Singular system, model reduction, Silverman-Ho algorithm, H_∞ norm, transfer function

1 Introduction

In recent years, singular systems have been extensively investigated due to their applications. But in reality, the orders of many singular systems are so large that they add the difficulty in designing and controlling them. Thus model reduction is vital for analysis and design of controller for such systems[1].

The initial investigation of model reduction for singular systems was the chained aggregation method in [2]. Following this, many papers discussing the model reduction of such systems appeared, such as [3~10]. In [3], Perev and Shafai considered model reduction *via* balanced realizations. Unfortunately, their method ignored the impulsive behavior which is of paramount importance to singular systems. In [4], Liu and Sreeram used the Nehari′s approximation algorithm and overcame the problem. Recently, Zhang *et al.*[9] discussed the H_∞ suboptimal model reduction problem for singular systems.

In this paper, we will discuss the model reduction problem again and present an approach for the H_∞ optimal model reduction for singular fast subsystem via solving the minimum rank problem for a matrix set. A necessary and sufficient condition will be obtained for the existence of a stable reduced-order system. Further, an algorithm has been designed for the H_∞ optimal model reduction.

2 Main results

Let the following nth-order system be the singular fast subsystem to be reduced

$$G(s) = P_0 + sP_1 + \cdots + s^{h-1}P_{h-1}, \quad P_{h-1} \neq 0 \tag{1}$$

The H_∞ optimal model reduction problem of singular fast subsystems is to find a lower-order singular fast subsystem $G_r(s)$ such that H_∞ norm of the error system $G_e(s) = G(s) - G_r(s)$ is finite and minimal.

Let the reduced-order singular fast system $G_r(s)$ has the following form

$$G_r(s) = P_{r0} + sP_{r1} + \cdots + s^{v-1}P_{r(v-1)}, \quad P_{r(v-1)} \neq 0 \tag{2}$$

Then H_∞ norm of the error system is finite if and only if $G(s) - G_r(s) = P_0 - P_{r0}$ holds, which can hold if and only if

$$v = h, \text{ and } P_i = P_{ri}, \quad i = 1, 2, \cdots, h-1 \tag{3}$$

Then, H_∞ optimal model reduction problem is converted into obtaining matrix P_{r0} such that the following conditions hold:

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1) Equality (3) holds

2) The order of system (2), n_r, is less than the order of system (1), n

3) H_∞ norm of the error system $G_e(s)$ is minimal

Assume $n_{r\,\min}$ is the minimal order of fast subsystem $G_r(s)$ satisfying equality (3).

While obtaining the matrix P_{r0}, the method in [9] is that for the fixed nilpotent matrix N_r, $\dim N_r < n$, one gets parameters C_r and B_r satisfying (3). In this paper, we will provide a new reduction algorithm, and its advantage is that it can get parameters (C_r, N_r, B_r) simultaneously and it can guarantee that the order of the reduced system is minimal, that is, $n_r = n_{r\,\min}$.

From Silverman-Ho algorithm in [11], the key of reducing system $G(s)$ is to find P_{r0} such that

$$n = \mathrm{rank}[M_0] > \mathrm{rank}[M_{r0}] = n_r \tag{4}$$

where

$$M_0 = \begin{bmatrix} -P_0 & -P_1 & \cdots & -P_{h-2} & -P_{h-1} \\ -P_1 & -P_2 & \cdots & -P_{h-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -P_{h-2} & -P_{h-1} & \cdots & 0 & 0 \\ -P_{h-1} & 0 & \cdots & 0 & 0 \end{bmatrix} \in R^{hm\times hm}$$

$$M_{r0} = \begin{bmatrix} -P_{r0} & -P_1 & \cdots & -P_{h-2} & -P_{h-1} \\ -P_1 & -P_2 & \cdots & -P_{h-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -P_{h-2} & -P_{h-1} & \cdots & 0 & 0 \\ -P_{h-1} & 0 & \cdots & 0 & 0 \end{bmatrix} \in R^{hm\times hm}$$

Then, the existence of the reduced system depends on the existence of matrix M_{r0} satisfying (4). Next, we will give a sufficient and necessary condition such that such matrix M_{r0} exists.

Theorem 1. The matrix P_{r0} exists, satisfying (4), if and only if

$$\mathrm{rank}[M_{21} \quad M_{22}] + \mathrm{rank}\begin{bmatrix} M_{12} \\ M_{22} \end{bmatrix} < \mathrm{rank}[M_{22}] + n \tag{5}$$

where

$$M_{12} = [-P_1 \quad -P_2 \quad \cdots \quad -P_{h-1}],\ M_{21} = [-P_1^{\mathrm{T}} \quad -P_2^{\mathrm{T}} \quad \cdots \quad -P_{h-1}^{\mathrm{T}}]^{\mathrm{T}}$$

$$M_{22} = \begin{bmatrix} -P_2 & -P_3 & \cdots & -P_{h-1} & 0 \\ -P_3 & -P_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -P_{h-1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Further, if the inequality (5) holds, there exists matrix $\bar{K}$ satisfying the following:

$$\mathrm{rank}\begin{bmatrix} \bar{K} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \min_{K} \mathrm{rank}\begin{bmatrix} K & M_{12} \\ M_{21} & M_{22} \end{bmatrix} < n \tag{6}$$

In this case, let $P_{r0} = -\bar{K}$. Then P_{r0} satisfies (5).

Theorem 2. System $G(s)$ can be reduced if and only if the inequality (5) holds, and the minimal order of the reduced system $G_r(s)$ is

$$n_{r\,\min} = \mathrm{rank}[M_{21} \quad M_{22}] + \mathrm{rank}\begin{bmatrix} M_{12} \\ M_{22} \end{bmatrix} - \mathrm{rank}[M_{22}]$$

where the notations are the same as the ones in Theorem 1.

Next, we will give the parametric form of matrix $\bar{K}$, such that

$$\mathrm{rank}\begin{bmatrix} \bar{K} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \min_{K} \mathrm{rank}\begin{bmatrix} K & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{7}$$

Before getting it, we will have the following important theorem.

Let $B \in R^{m\times p}$, $\mathrm{rank}[B] = r_b$, $C \in R^{q\times n}$, $\mathrm{rank}[C] = r_c$, and without loss of generality, assume that $B = [B_1 \quad B_2]$, $C = [C_1 \quad C_2]$, where $B_1 \in R^{m\times r_b}$, $C_1 \in R^{q\times r_c}$ are both of full-column rank. Then there exists matrix D_1, such that $B_1 D_1 = B_2$. Accordingly, F can be partitioned into $F = [F_1 \quad F_2]$. Then we have the following theorem.

Theorem 3. Let the set $\vec{F} = \{F | F \in R^{q\times p}, \mathrm{rank}\begin{bmatrix} 0 & B \\ C & F \end{bmatrix} = r_b + r_c\}$; then it has the following parametric form

$$\vec{F}(F_1, D_2) = \{[F_1 \quad F_1 D_1 + C D_2] | F_1 \in R^{q\times r_b}, D_2 \in R^{n\times(p-r_b)}\}$$

Proof. From the above analysis, it follows that

$$\begin{bmatrix} 0 & B \\ C & F \end{bmatrix} = \begin{bmatrix} 0 & 0 & B_1 & B_2 \\ C_1 & C_2 & F_1 & F_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & B_1 & B_1 D_1 \\ C_1 & C_2 & F_1 & F_2 \end{bmatrix}$$

then, it is enough that F_2 satisfies the following

$$\mathrm{rank}\begin{bmatrix} 0 & B_1 & B_1 D_1 \\ C_1 & F_1 & F_2 \end{bmatrix} = \mathrm{rank}\begin{bmatrix} B_1 \\ F_1 \end{bmatrix}$$

Thus this theorem is obvious. □

Next, we will obtain $\bar{K}$ satisfying (7).

Assume $\mathrm{rank}[M_{22}] = r_a$; then there exist nonsingular matrices T_1 and T_2, such that

$$\begin{bmatrix} I_q & -M_{12}^1 & 0 \\ 0 & I_{r_a} & 0 \\ 0 & 0 & I_{hq-q-r_a} \end{bmatrix} \begin{bmatrix} I_q & 0 \\ 0 & T_1 \end{bmatrix} \begin{bmatrix} K & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} I_m & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} I_m & 0 & 0 \\ -M_{21}^1 & I_{r_a} & 0 \\ 0 & 0 & I_{hm-m-r_a} \end{bmatrix} = \begin{bmatrix} K - M_{12}^1 M_{21}^1 & 0 & M_{12}^2 \\ 0 & I_{r_a} & 0 \\ M_{21}^2 & 0 & 0 \end{bmatrix} \tag{8}$$

where $T_1[M_{22}]T_2 = \mathrm{diag}[I_r, 0]$, $T_1[M_{21}] = \begin{bmatrix} M_{21}^1 \\ M_{21}^2 \end{bmatrix}$, $[M_{12}]T_2 = [M_{12}^1 \quad M_{12}^2]$.

Similarly, assume that $\mathrm{rank}[M_{21}^2] = r_{21}$, $\mathrm{rank}[M_{12}^2] = r_{12}$, and without loss of generality, assume $M_{21}^2 = [M_{21}^{21} \quad M_{21}^{22}]$, $M_{12}^2 = [M_{12}^{21} \quad M_{12}^{22}]$, where $M_{21}^{21} \in R^{(hq-q-r_a)\times r_{21}}, M_{12}^{21} \in R^{q\times r_{12}}$ are of full-column rank. Then there exists matrix M such that $M_{21}^{21} M = M_{21}^{22}$. Accordingly, K can be partitioned into $K = [K_1 \quad K_2]$. After applying Theorem 3, we can get the following theorem.

Theorem 4. The set $\vec{K}$ consisting of matrix $\bar{K}$ satisfying (7) has the following parametric form:

$$\vec{K} = \vec{K}(K_1, D) = \{M_{12}^1 M_{21}^1 + [K_1 \quad K_1 M + M_{12}^2 D] | K_1 \in R^{q\times r_{21}},\ D \in R^{(hm-m-r_a)\times(m-r_{21})}\}$$

From the above theorem, it is enough to let $\vec{P}_{r0} = -\bar{K}$, $P_{r0} \in \vec{P}_{r0}$, and in this case, the reduced order system is $G_r(s) = P_{r0} + P_1 s + \cdots + P_{h-1}s^{h-1}$. Further, applying Silverman-Ho algorithm, we can get its state space realization (N_r, I_r, B_r, C_r) and $\dim N_r = n_{r\min} = \mathrm{rank}[M_{r0}]$.

From the above, we can get the following reduction algorithm for system (1).

3 Reduction algorithm

1) Appy Theorem 1, and check the inequality (7); if it holds, then apply Theorem 4, and get $P_{r0} \in \vec{P}_{r0}$, which leads to free parameters K_1 and D, such that $\mathrm{rank}[M_{r0}] = n_r < n$; if it does not hold, end

2) Use the function fminunc() in Matlab application software to optimize $\|G_e(s)\|_\infty$, where $G_e(s) = G(s) - G_r(s) = P_0 - P_{r0}$, and get the parameters K_1 and D in matrix P_{r0}, and $G_r(s)$

3) Using Silverman-Ho algorithm, one gets the state space realization (N_r, I_r, B_r, C_r) of system $G_r(s)$.

4 Numerical example

Consider the following singular system:

$$G(s) = -\begin{bmatrix} 7 & 12 \\ 5 & 2 \\ 8 & 9 \end{bmatrix} - s\begin{bmatrix} 3 & 1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} - s^2\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 3 & 0 \end{bmatrix} \tag{9}$$

From Theorem 1, system (9) can be reduced, and the minimal order of the reduced system is 3.

After using the above algorithm, we can get

$$\min_{K,D} \|G_e(s)\|_\infty = 6.3960$$

and in this case,

$$P_{r0} = -\begin{bmatrix} 7 & 5.9092 \\ 5 & 2.9092 \\ 8 & 10.7276 \end{bmatrix}$$

After using the method in [9], we can get the following parameters:

$$\hat{N}_r = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ \hat{C}_{r2} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 0 \\ 3 & -1 & 1 \end{bmatrix},\ \hat{B}_{r2} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

and $\|\hat{G}_e(s)\|_\infty = 15.5753$. Thus $6.3960 = \|G_e(s)\|_\infty < \|\hat{G}_e(s)\|_\infty = 15.5753$, which indicates that the existence of free parameters results in that our method is better.

5 Conclusions

The advantages of the proposed method in this paper are as follows.

1) A sufficient and necessary condition is given to check whether system $G(s)$ can be reduced

2) The H_∞ norm of the error system $G_e(s)$ is minimal

3) The order of the reduced system is minimal

4) The proposed method is valid in the discrete case.

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WANG Jing Ph. D. candidate of control theory and control engineering at Northeastern University. Her research interests include the model reduction, controller reduction, and decentralized control of singular systems.

ZHANG Qing-Ling Ph. D., professor, dean of College of Sciences, Northeastern University. His research interests include decentralized control, fault tolerable control, and robust control of singular systems.

LIU Wan-Quan Senior lecturer in the Department of Computing at Curtin University of Technology. His research interests include dynamics systems and control, network reliability, wavelets and signal processing, and activity recognition.

ZHOU Yue Master, lecture in College of Information Sciences and Engineering at Shenyang University. His research interests include computer control, decentralized control, fault tolerable control, and robust control of singular systems.