Finite-time Control of Linear Singular Systems with Parametric Uncertainties and Disturbances[1)]

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Abstract The concept of finite-time stability for linear singular system is induced in this paper. Finite-time control problem is considered for linear singular systems with time-varying parametric uncertainties and exogenous disturbances. The disturbance satisfies a dynamical system with parametric uncertainties. A sufficient condition is presented for robust finite-time stabilization *via* state feedback. The condition is translated to a feasibility problem involving restricted linear matrix inequalities (LMIs). A detailed solving method is proposed for the restricted linear matrix inequalities. Finally, an example is given to show the validity of the results.

Key words Finite-time stability, linear singular systems, parametric uncertainties, exogenous disturbances, LMI

1 Introduction

In the last decade, big effort has been spent on studying the robust stability problem for linear systems subject to uncertainties. The work of many control scientists and engineers has mainly focused on robust Lyapunov stability. In practice, one is not only interested in system stability(*e.g.* in the sense of Lyapunov), but also in bounds of system trajectories. A system could be stable but completely useless because it possesses undesirable transient performances. To study the transient performances of system, the concept of finite-time stability was proposed by Dorato[1]. Some work has been done on the finite time control of linear systems, such as [2,3].

On the other hand, singular systems have comprehensive practical background and great progress has been made in the theory and its applications since 1974[4~9]. In this paper, based on [2] , we first generalize the concept of finite-time stability to linear singular systems. Then finite-time control problem is considered for linear singular systems with time-varying parametric uncertainties and exogenous disturbances.

2 Problem statement

Consider the following linear singular systems:

$$\begin{cases} E\dot{\boldsymbol{x}} = A(p)\boldsymbol{x}(t) + B(p)\boldsymbol{u}(t) + G(p)\boldsymbol{w}(t) \\ \dot{\boldsymbol{w}} = S(p)\boldsymbol{w}(t) \end{cases} \tag{1}$$

where, $\boldsymbol{x}(t) \in R^n$ is the state, $\boldsymbol{u}(t) \in R^m$ is control input, $\boldsymbol{w}(t) \in R^l$ is the disturbance; E is a singular matrix with $rankE = r < n$, other matrices $A(p), B(p)$, $G(p), S(p)$ are of appropriate dimensions. In this paper, we assume the following:

A1. The parametric vector function $\boldsymbol{p}(\cdot) = (p_1(\cdot), p_2(\cdot), \cdots, p_q(\cdot))^{\mathrm{T}}$ is any Lebesgue measurable function $\boldsymbol{p}(\cdot) : [0, T] \to \Re$, (Lebesgue measurement is more general than continuous property, the former also can gurantee the existence of system solution.) where, $\Re = [\underline{p_1}, \bar{p}_1] \times [\underline{p_2}, \bar{p}_2] \times \cdots \times [\underline{p_q}, \bar{p}_q]$. We denote the vertices of $\Re$ by $p_{(i)}, i = 1, 2, \cdots, 2^q$.

A2. The matrix valued functions $A(\cdot), B(\cdot), G(\cdot), S(\cdot)$ are given by multiaffine matrix valued function; for instance, $A(p) = \sum\limits_{i_1, i_2, \cdots, i_q = 0}^{1} A_{i_1 i_2 \cdots i_q} p_1^{i_1} p_2^{i_2} \cdots p_q^{i_q}$, where, $i_1, i_2, \cdots, i_1 \in \{0, 1\}$, particularly, if $q = 2$, then $\boldsymbol{p}(\cdot) = (p_1(\cdot), p_2(\cdot))^{\mathrm{T}}$, $A(p) = A_{00} + A_{10}p_1 + A_{01}p_2 + A_{11}p_1p_2$.

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A3. The initial value of exogenous disturbances $w(0)$ satisfies the constraint: $\boldsymbol{w}^{\mathrm{T}}(0)\boldsymbol{w}(0) \leqslant d, d \geqslant 0$.

The paper′s aim is to find the state feedback controller $\boldsymbol{u}(t) = K\boldsymbol{x}(t)$ such that the closed-loop system is impulse-free and the state is bounded over a finite-time interval, *i.e.*, the closed-loop is finite time stability. Firstly, we need the following definitions:

Definition 1[10]. The time-varying linear singular system $E\dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t) + G(t)\boldsymbol{w}(t)$ is said to be impulse-free in time interval $[0, T]$, if $degreedet(sE - A(t)) = \mathrm{rank}E$.

Definition 2. (FTS) The time-varying linear singular system

$$\left\{\begin{array}{l} E\dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t) + G(t)\boldsymbol{w}(t) \\ \dot{\boldsymbol{w}}(t) = S(t)\boldsymbol{w}(t), \quad \boldsymbol{w}^{\mathrm{T}}(0)\boldsymbol{w}(0) \leqslant d \end{array}\right. \tag{2}$$

is said to be finite time stable with respect to (c_1, c_2, T, R, d) with $c_2 > c_1, R > 0$. If $\boldsymbol{x}^{\mathrm{T}}(0)E^{\mathrm{T}}RE\boldsymbol{x}(0) \leqslant c_1$, then

$$\boldsymbol{x}^{\mathrm{T}}(t)E^{\mathrm{T}}RE\boldsymbol{x}(t) \leqslant c_2, \quad \forall t \in [0, T], \quad \forall \boldsymbol{w}(0), \quad \boldsymbol{w}^{\mathrm{T}}(0)\boldsymbol{w}(0) \leqslant d$$

Remark 1. Definition 2 is generalization of finite time stability[2] to linear singular system.

Remark 2. If impulse-free system is FTS, then the state is no more than one certain bound.

3 Main results

First, a sufficient condition is given, which guarantees that the system (2) is impulse-free and FTS.

Theorem 1. The time-varying linear singular system (2) is impulse-free and FTS with respect to (c_1, c_2, T, R, d), if there exist positive matrices $Q_1 > 0, Q_2 > 0$, nonsingular matrix P, and a scalar $\alpha > 0$ satisfying:

$$PE = E^{\mathrm{T}}P^{\mathrm{T}} \geqslant 0, \quad \begin{bmatrix} PA + A^{\mathrm{T}}P^{\mathrm{T}} - \alpha PE & PG \\ G^{\mathrm{T}}P^{\mathrm{T}} & Q_2S + S^{\mathrm{T}}Q_2 - \alpha Q_2 \end{bmatrix} < 0 \tag{3a,3b}$$

$$PE = E^{\mathrm{T}}R^{\frac{1}{2}}Q_1R^{\frac{1}{2}}E, \quad \mathrm{e}^{\alpha \mathrm{T}}(\lambda_{\max}(Q_1)c_1 + \lambda_{\max}(Q_2)d) < \lambda_{\min}(Q_1)c_2 \tag{3c,3d}$$

Proof is omitted. For the paper′s aim, we have equivalent form for Theorem 1.

Theorem 2. Time-varying linear singular system (2) is impulse-free and FTS with respect to (c_1, c_2, T, R, d), if there exist positive matrices $Q_1 > 0, Q_2 > 0$, nonsingular matrix $\bar{P}$ and scalar $\alpha > 0$ satisfying:

$$E\bar{P}^{\mathrm{T}} = \bar{P}E^{\mathrm{T}} \geqslant 0, \quad \begin{bmatrix} A\bar{P}^{\mathrm{T}} + \bar{P}A^{\mathrm{T}} - \alpha E\bar{P}^{\mathrm{T}} & G \\ G^{\mathrm{T}} & SQ_2 + Q_2S^{\mathrm{T}} - \alpha Q_2 \end{bmatrix} < 0 \tag{4a,4b}$$

$$\bar{P}^{-1}E = E^{\mathrm{T}}R^{\frac{1}{2}}Q_1R^{\frac{1}{2}}E, \quad \mathrm{e}^{\alpha \mathrm{T}}[\lambda_{\max}(Q_1)c_1 + \lambda_{\max}(Q_2)d] < \lambda_{\min}(Q_1)c_2 \tag{4c,4d}$$

Corollary 1. In system (2) when $E = I, S = 0$, system is FTS with respect (c_1, c_2, T, R, d), if there exist positive matrices $\bar{Q}_1 > 0, \bar{Q}_2 > 0$, scalar $\alpha > 0$ satisfying the following inequalities:

$$\begin{bmatrix} A\bar{P}^{\mathrm{T}} + \bar{P}A^{\mathrm{T}} - \alpha E\bar{P} & G\bar{Q}_2 \\ \bar{Q}_2G^{\mathrm{T}} & -\alpha\bar{Q}_2 \end{bmatrix} < 0, \quad \mathrm{e}^{\alpha \mathrm{T}}\left(\frac{c_1}{\lambda_{\min}\bar{Q}_1} + \frac{d}{\lambda_{\min}\bar{Q}_2}\right) < \frac{c_2}{\lambda_{\max}\bar{Q}_1}$$

where $\bar{P} = R^{-\frac{1}{2}}\bar{Q}_1R^{-\frac{1}{2}}$,

Remark 3. Corollary1 is Lemma 6 in [2].

Using Theorem 2 and [11], we get our main theorem.

Theorem 3. There exists a controller for system (1) such that the closed-loop is impulse-free and FTS with respect to (c_1, c_2, T, R, d), if there are positive matrices $Q_1 > 0, Q_2 > 0$, nonsingular matrix $\bar{P}$, matrix $\bar{K}$, and scalar $\alpha > 0$ satisfying (4a), (4c), (4d), and the following linear matrix inequality:

$$\begin{bmatrix} A(p_{(i)})\bar{P}^{\mathrm{T}} + \bar{P}A^{\mathrm{T}}(p_{(i)}) + B(p_{(i)})\bar{K} + \bar{K}^{\mathrm{T}}B^{\mathrm{T}}(p_{(i)}) - \alpha E\bar{P}^{\mathrm{T}} & G(p_{(i)}) \\ G^{\mathrm{T}}(p_{(i)}) & S(p_{(i)})Q_2 + Q_2S^{\mathrm{T}}(p_{(i)}) - \alpha Q_2 \end{bmatrix} < 0$$

And the feedback controller is $\boldsymbol{u}(t) = \bar{K}\bar{P}^{-\mathrm{T}}\boldsymbol{x}(t)$.

To solve (4a), (4c), (4d), and LMI in Theorem 3, we need translate them to restrict linear matrix inequalities. There exist nonsingular matrices C, D, such that $\bar{E} = CED = \mathrm{diag}\{I_r, 0\}$. Letting $\tilde{P} = C\bar{P}D^{-\mathrm{T}}$, from condition (4a), $\tilde{P}$ is of the form $\tilde{P} = \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix}$, and $P_{11} \geqslant 0, P_{12} \in R^{r\times(n-r)}, P_{22} \in R^{(n-r)\times(n-r)}$. Define matrix $\Phi = D\,[0 \quad I_{n-r}]^{\mathrm{T}}$. It is clear that $\mathrm{rank}\Phi = n-r$, $E\Phi = 0$, and

$$\begin{aligned}
&\bar{P} = C^{-1}\begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix} D^{\mathrm{T}} = \\
&(C^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} D^{-1})(D\begin{bmatrix} P_{11} & 0 \\ 0 & *1 \end{bmatrix} D^{\mathrm{T}}) + (C^{-1}\begin{bmatrix} P_{12} \\ P_{22} \end{bmatrix})([0 \quad I_{n-r}]D^{\mathrm{T}}) = \\
&EDXD^{\mathrm{T}} + C^{-1}Y\Phi^{\mathrm{T}}
\end{aligned}$$

where $X = \mathrm{diag}\{P_{11}, *1\}$, $Y = [P_{12}^{\mathrm{T}} \quad P_{22}^{\mathrm{T}}]^{\mathrm{T}}$. It is obvious that $\bar{P} = EDXD^{\mathrm{T}} + C^{-1}Y\Phi^{\mathrm{T}}$ satisfies (4a), and (4c) holds when $Q_1 = R^{-\frac{1}{2}}C^{\mathrm{T}}X^{-1}CR^{-\frac{1}{2}}$. For condition (4d), since $\lambda_{\max}(Q_1) = \dfrac{1}{\lambda_{\min}(R^{\frac{1}{2}}C^{-1}XC^{-\mathrm{T}}R^{\frac{1}{2}})}$, $\lambda_{\min}(Q_1) = \dfrac{1}{\lambda_{\max}(R^{\frac{1}{2}}C^{-1}XC^{-\mathrm{T}}R^{\frac{1}{2}})}$, it is easy to check that condition (4d) is guaranteed by imposing the condition

$$\lambda_1 I < R^{\frac{1}{2}}C^{-1}XC^{-\mathrm{T}}R^{\frac{1}{2}} < I, \quad Q_2 < \lambda_2 I, \quad \begin{bmatrix} e^{-\alpha T}c_2 - d\lambda_2 & \sqrt{c_1} \\ \sqrt{c_1} & \lambda_1 \end{bmatrix} > 0 \tag{5a,5b,5c}$$

for some positive numbers λ_1, λ_2. Therefor we have:

Corollary 2. For system (1) there exists a controller such that closed-loop system is impulse-free and FTS with respect to (c_1, c_2, T, R, d), if there exist positive matrices $X = \mathrm{diag}\{P_{11}, *1\} > 0, Q_2 > 0$, matrices $Y \in R^{n\times(n-r)}$, $\bar{K}$, and scalars $\alpha > 0$, $\lambda_i, i = 1, 2$, satisfying LMIs (7) and the following LMI:

$$\begin{bmatrix} \Omega & G(p_{(i)}) \\ G^{\mathrm{T}}(p_{(i)}) & S(p_{(i)})Q_2 + Q_2S^{\mathrm{T}}(p_{(i)}) - \alpha Q_2 \end{bmatrix} < 0 \tag{6}$$

where $\Omega = A(p_{(i)})\beta^{\mathrm{T}}(X,Y) + \beta(X,Y)A^{\mathrm{T}}(p_{(i)}) + B(p_{(i)})\bar{K} + \bar{K}^{\mathrm{T}}B^{\mathrm{T}}(p_{(i)}) - \alpha E\beta^{\mathrm{T}}(X,Y)$, $\beta(X,Y) = EDXD^{\mathrm{T}} + C^{-1}Y\Phi^{\mathrm{T}}$is nonsingular, matrices C, D satisfy $CED = \mathrm{diag}\{I_r, 0\}$, $\Phi = D\begin{bmatrix} 0 & I_{n-r} \end{bmatrix}^{\mathrm{T}}$. Then the feedback controller is $\boldsymbol{u}(t) = \bar{K}\beta^{-\mathrm{T}}(X,Y)\boldsymbol{x}(t)$.

Remark 4. If $\beta(X,Y)$ obtained from LMIs (5,6) is singular, there is a small scalar $\theta(\|\theta\| << 1)$, such that $EDXD^{\mathrm{T}} + C^{-1}Y\Phi^{\mathrm{T}} + C^{-1}[0 \quad \theta I_{n-r}]^{\mathrm{T}}$ is nonsingular. For simplicity denote it by $\beta(X,Y)$ and it still satisfies (5,6).

Remark 5. In this paper the finite time control problem is considered only for impulse-control singular system. For impulse system, from Defintion 2 finite time stability only can gurantee that the state of dynamic part is no more than a certain bound. As for impluse-uncontrol system, finite time control problem needs further study.

Example. Let us consider system (1) with:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A(p) = \begin{bmatrix} p & 1 & p \\ p-1 & p & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{B}(p) = \begin{bmatrix} p \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{G}(p) = \begin{bmatrix} 0.25p \\ 0.1p \\ 0.1 \end{bmatrix}$$

$S(p) = 0.5p - 0.3$, where $p \in [-1, 1]$. When $c_1 = 1,\ c_2 = 10,\ T = 1.3,\ R = I,\ d = 0.5$, we get from LMIs (5,6):

$$X = \begin{bmatrix} 0.7461 & -0.0870 & 0 \\ -0.0870 & 0.8510 & 0 \\ 0 & 0 & 0.7546 \end{bmatrix}, \quad \boldsymbol{Y} = \begin{bmatrix} -0.4235 \\ 0.0275 \\ -0.6639 \end{bmatrix}$$

$$\bar{\boldsymbol{K}} = [-0.7610 \quad -1.0713 \quad 0.2349],\ Q_2 = 1,\ \alpha = 1,\ \lambda_1 = 0.5,\ \lambda_2 = 1.1$$

Then the finite time state feedback controller for system (1) is $\boldsymbol{u}(t) = [-1.1807 \quad -1.3795 \quad 0.3421]\boldsymbol{x}(t)$.

4 Conclusion

The concept of finite time stability for linear singular system is given. Finite-time control problem is studied for linear singular system with time-varying parametric uncertainties and exogenous disturbance. Compared with [2], the exogenous disturbance is not constant, it satisfies one dynamical system with parametric uncertainties. The sufficient condition is attained for robust finite-time stabilization *via* state feedback. This condition is reduced to a feasibility problem involving restricted linear matrix inequalities. A detailed solving method is proposed for such restricted LMIs. Finally, an illustrative example is shown.

References

1 Dorato P. Short time stability in linear time-varying systems. IRE International Convention Record, Part 4. H.E.Schrank (Editor), New York: The Institute of Radio Engineers, Incoporated. 1961. 83~87

2 Amato F, Arolia M,Dorato P. Finite-time control of linear systems subject to parametric uncertainties and disturbance. *Automatica*, 2001, **37**(9): 1459~1463

3 Weiss L, Infate E F. Finite time stability under perturbing forces and on product space. *IEEE Transactions on Automatic Control*, 1967, **12**(1): 54~59

4 Rosenbrock H H. Structural properties of linear dynamical systems. *International Journal of Control*, 1974, **20**(2): 191~202

5 Ho D W C, Gao Z W. Bezout identity related to reduced-order observer-based controllers for singular systems. *Automatica*, 2001, **37**(10): 1655~1662

6 Gao Z W, Wang X L, Li G Q. Proper stabilization of generalized decentralized systems with direct control feedthrough. *Acta Automatica Sinica*, 1998, **24**(6): 754~760

7 Zhu J D, Chen Z L. Output Regulation of Singular Nonlinear Systems *via* Output Dynamic Feedback. *Acta Automatica Sinica*, 2003, **29**(1): 80~85

8 Feng J E, Cheng Z L, Ma S P. Singular Linear-Quadratic Optimal Control Problem for a Class of Discrete Singular Systems with Multiple Time-Delays, *International Journal of Systems and Sciences*, 2003, **34**(4): 293~301

9 Liu G X, Yang W H. A Wavelet-Decomposition-Based Image Fusion Scheme and Its Performance Evaluation. *Acta Automatica Sinica*, 2002, **28**(6): 927~934

10 Takaba K. Robust H_2 control of descriptor system with time-varying uncertainty. *International Journal of Control*, 1998, **71**(4): 559~579

11 Garofalo F, Celentano G, Glielmo L. Stability robustness of interval matrices via quadratic Lyapunov forms. *IEEE Transactions on Automatic Control*, 1993, **38**(2): 281~284

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