# Quadratic Stability of Switched Nonlinear Systems in Block-triangular Form[1)]

ZHAO Sheng-Zhi[1,2] ZHAO Jun[1]

[1](*Key Laboratory of Process Industry Automation, Ministry of Education, Northeastern University, Shenyang* 110004)

[2](*Department of Mathematics, Liaoning University, Shenyang* 110036)

(E-mail: zzsszz2002@163.com)

**Abstract** The problem of globally quadratic stability of switched nonlinear systems in block-triangular form under arbitrary switching is addressed. Under the assumption that all block-subsystems are zero input-to-state stable, a sufficient condition for the problem to be solvable is presented. A common Lyapunov function is constructed iteratively by using the Lyapunov functions of block-subsystems.

**Key words** Switched nonlinear systems, common Lyapunov functions, quadratic stability

## 1 Introduction

In recent years, the stability problem of switched systems has attracted great attention[1~3]. It is well known that switching between stable subsystems may lead to instability and switching between unstable subsystems can give rise to stability. Stability of switched systems under arbitrary switching is a desirable property because this property enables one to seek for other system performances by switching without changing stability. Many results concerning this issue have appeared. The stability of switched systems under arbitrary switching can be assured by a common Lyapunov function. In linear case, many approaches, such as Lie algebra condition under which all subsystems can be simultaneously transformed into triangular form[4], gradient algorithms[5] and solving a group of linear inequalities[6], have been presented to construct common Lyapunov functions. For switched nonlinear systems the problem turns to be more complicated and relatively fewer results have been available by now. By using approximation linearization, [4] gave a solution of local stability. Some systems with commutative vector fields and in triangular form were investigated in [7], and a global stability result was obtained.

A switched linear system, whose subsystems are all in stable triangular form, is known to share a common Lyapunov function and is thus globally asymptotically stable under arbitrary switching. For a switched nonlinear system, however, this conclusion is no longer valid (see the counterpart in [7]). In this paper, we study switched nonlinear systems in block-triangular form. Under the assumption that all block-subsystems are zero input-to-state stable, a sufficient condition for a switched system of this class to be globally quadratically stable under arbitrary switching is presented. This result extends the stability results of switched linear systems in triangular form to the nonlinear case. Also, a common Lyapunov function is constructed iteratively by using the Lyapunov functions of block-subsystems. An example is given to demonstrate the proposed result.

## 2 Main results

Consider the switched system

$$\dot{\boldsymbol{x}} = f_i(\boldsymbol{x}), \quad i = 1, \cdots, m \tag{1}$$

We first give the definition of globally quadratic stability.

**Definition 1.** System (1) is said to be globally quadratically stable under arbitrary switching if there exists a positive definite quadratic function $U(x) = \boldsymbol{x}^{\mathrm{T}} P \boldsymbol{x}$ such that $L_{f_i(x)} U(x) \leqslant -\alpha \|\boldsymbol{x}\|^2$, $i = 1, \cdots, m$, for some constant $\alpha > 0$.

Consider a switched nonlinear system described as

$$\dot{\boldsymbol{x}} = \begin{pmatrix} f_{i1}(\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_k) \\ f_{i2}(\boldsymbol{x}_2, \cdots, \boldsymbol{x}_k) \\ \cdots \\ f_{ik}(\boldsymbol{x}_k) \end{pmatrix}, \quad i = 1, \cdots, m \tag{2}$$

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where $\boldsymbol{x} = (\boldsymbol{x}_1^{\mathrm{T}}, \cdots, \boldsymbol{x}_k^{\mathrm{T}})^{\mathrm{T}} \in R^n$, $\boldsymbol{x}_j \in R^{n_j}$, $j = 1, \cdots, k$, $n_1 + \cdots + n_k = n$, $f_{ij}(\boldsymbol{x}_j, \cdots, \boldsymbol{x}_k) \in R^{n_j}$, $f_{ij}(0) = 0$, $i = 1, \cdots, m$, $j = 1, \cdots, k$. We study the globally quadratic stability of system (2) under zero input-to-state stability of each block-subsystem, that is, if $(\boldsymbol{x}_{j+1}, \cdots, \boldsymbol{x}_k)$ is viewed as an input $\boldsymbol{u}$, the switched system $\dot{\boldsymbol{x}}_j = f_{ij}(\boldsymbol{x}_j, \boldsymbol{u})$, $i = 1, \cdots, m$, is globally asymptotically stable under arbitrary switching when $\boldsymbol{u} \equiv 0$. We assume that $f_{ij}$, $i = 1, \cdots, m$, $j = 1, \cdots, k$, are global Lipschitz vector fields.

**Theorem 1.** If there exist positive definite quadratic functions $V_j(\boldsymbol{x}_j) = \boldsymbol{x}_j^{\mathrm{T}} P_j \boldsymbol{x}_j$ and constants $\alpha_j > 0$, $j = 1, \cdots, k$ such that $L_{f_{ik}(\boldsymbol{x}_k)} V_k(\boldsymbol{x}_k) \leqslant -\alpha_k \|\boldsymbol{x}_k\|^2$, $i = 1, \cdots, m$, and $L_{f_{ij}(\boldsymbol{x}_j, \mathbf{0}, \cdots, \mathbf{0})} V_j(\boldsymbol{x}_j) \leqslant -\alpha_j \|\boldsymbol{x}_j\|^2$, $i = 1, \cdots, m$, $j = 1, \cdots, k-1$, then switched system (2) is globally quadratically stable under arbitrary switching.

**Proof.** It is easy to see that there exists $l_j > 0$, such that $\left\| \dfrac{\partial V_j}{\partial \boldsymbol{x}_j} \right\| \leqslant l_j \|\boldsymbol{x}_j\|, j = 1, \cdots, k-1$. For the accordance of symbols, let $W_1 = V_k, \boldsymbol{\xi}_1 = \boldsymbol{x}_k, \tilde{K}_1 = \alpha_k, g_{i1} = f_{ik}$.

Consider system

$$\left\{ \begin{array}{l} \dot{\boldsymbol{x}}_{k-1} = f_{ik-1}(\boldsymbol{x}_{k-1}, \boldsymbol{\xi}_1) \\ \dot{\boldsymbol{\xi}}_1 = g_{i1}(\boldsymbol{\xi}_1) \end{array} \right., \ i = 1, \cdots, m \tag{3}$$

Let $W_2(\boldsymbol{x}_{k-1}, \boldsymbol{\xi}_1) = K_1 W_1 + V_{k-1}$ and choose $K_1 = l^2 l_0^2 / (2\alpha_0 \tilde{K}_1)$, $\theta_1 = \sqrt{\alpha_0/(ll_0) + ll_0/(4K_1\tilde{K}_1)}$, where $\alpha_0 = \min\{\alpha_1, \cdots, \alpha_{k-1}\}$, $l_0 = \max\{l_1, \cdots, l_{k-1}\}$, $l$ is the maximum of all Lipschitz coefficients. Then along the trajectory of (3) for $\forall i = 1, \cdots, m$, using inequality $\|\boldsymbol{x}_k\| \|\boldsymbol{x}_{k-1}\| \leqslant \theta_1^2 \|\boldsymbol{x}_{k-1}\|^2/2 + \|\boldsymbol{x}_k\|^2/(2\theta_1^2)$, we have

$$\begin{aligned} \dot{W}_2 \leqslant & -K_1 \alpha_k \|\boldsymbol{x}_k\|^2 + L_{f_{ik-1}(\boldsymbol{x}_{k-1}, 0)} V_{k-1} + L_{f_{ik-1}(\boldsymbol{x}_{k-1}, \boldsymbol{x}_k) - f_{ik-1}(\boldsymbol{x}_{k-1}, 0)} V_{k-1} \leqslant \\ & -K_1 \alpha_k \|\boldsymbol{x}_k\|^2 - \alpha_0 \|\boldsymbol{x}_{k-1}\|^2 + ll_0 \|\boldsymbol{x}_k\| \|\boldsymbol{x}_{k-1}\| \leqslant -\tilde{K}_2 (\|\boldsymbol{x}_{k-1}\|^2 + \|\boldsymbol{\xi}_1\|^2) \end{aligned} \tag{4}$$

where $\tilde{K}_2 = \min\{K_1\tilde{K}_1 - ll_0/(2\theta_1^2), \alpha_0 - ll_0\theta_1^2/2\} > 0$.

For $j = 2, \cdots, k-1$, let $\boldsymbol{\xi}_j = (\boldsymbol{x}_{k-j+1}^{\mathrm{T}}, \boldsymbol{\xi}_{j-1}^{\mathrm{T}})^{\mathrm{T}}, g_{ij} = (f_{ik-j+1}^{\mathrm{T}}, g_{ij-1}^{\mathrm{T}})^{\mathrm{T}}$. We consider the system

$$\left\{ \begin{array}{l} \dot{\boldsymbol{x}}_{k-j} = f_{ik-j}(\boldsymbol{x}_{k-j}, \boldsymbol{\xi}_j) \\ \dot{\boldsymbol{\xi}}_j = g_{ij}(\boldsymbol{\xi}_j) \end{array} \right. \tag{5}$$

Let $W_{j+1}(\boldsymbol{x}_{k-j}, \boldsymbol{\xi}_j) = K_j W_j + V_{k-j}$, and choose $K_j = l^2 l_0^2/(2\alpha_0 \tilde{K}_j)$, $\theta_j = \sqrt{\alpha_0/(ll_0) + ll_0/(4K_j\tilde{K}_j)}$. Then for $\forall i = 1, \cdots, m$, along the trajectory of (5), we have

$$\dot{W}_{j+1} \leqslant -\tilde{K}_{j+1} (\|\boldsymbol{x}_{k-j}\|^2 + \|\boldsymbol{\xi}_j\|^2) \tag{6}$$

where $\tilde{K}_{j+1} = \min\{(K_j\tilde{K}_j - ll_0/(2\theta_j^2)), (\alpha_0 - ll_0\theta_j^2/2)\} > 0$.

When $j = k-1$, system (5) is exactly system (2). Using (6) gives $\dot{W}_k \leqslant -\tilde{K}_k(\|\boldsymbol{x}_1\|^2 + \|\boldsymbol{\xi}_{k-1}\|^2)$. From the expressions of $W_1, W_2, \cdots, W_{k-1}$, we have $W_k = K_{k-1}K_{k-2}\cdots K_1 V_k + K_{k-1}K_{k-2}\cdots K_2 K_{k-1} + \cdots + K_{k-1}V_2 + V_1$. Therefore, the globally quadratic stability of switched system (2) under arbitrary switching follows. □

**Remark 1.** When all subsystems of system (2) are in linear triangular form, Theorem 1 degenerates into the result in [2,7].

**Remark 2.** Since input-to-state stability implies zero input-to-state stability, Theorem 1 extends the result in [7] where input-to-state stability is assumed.

When system (2) is in the linear block-triangular form:

$$\dot{\boldsymbol{x}} = A_i \boldsymbol{x}, \quad i = 1, \cdots, m \tag{7}$$

where $A_i = \begin{pmatrix} (A_i)_{11} & (A_i)_{12} & \cdots & (A_i)_{1k} \\ 0 & (A_i)_{22} & \cdots & (A_i)_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (A_i)_{kk} \end{pmatrix}$, $i = 1, \cdots, m$, $\boldsymbol{x} = (\boldsymbol{x}_1^{\mathrm{T}}, \cdots, \boldsymbol{x}_k^{\mathrm{T}})^{\mathrm{T}} \in R^n$, $\boldsymbol{x}_j \in R^{n_j}$, $j = 1, \cdots, k$, $n_1 + \cdots + n_k = n$, $(A_i)_{pq} \in R^{n_p \times n_q}$, $p = 1, \cdots, k$, $q = p, \cdots, k$, we have the following corollary which coincides with the result in [8].

**Corollary 1.** For $\forall j = 1, \cdots, k$, if there exists a common quadratic Lyapunov function for system $\dot{\boldsymbol{x}}_j = (A_i)_{jj} \boldsymbol{x}_j$, $i = 1, \cdots, m$, then switched system (7) has a common quadratic Lyapunov function.

## 3 Example

Consider the following switched nonlinear system

$$\begin{cases} \dot{\boldsymbol{x}}_1 = f_{i1}(\boldsymbol{x}_1, \boldsymbol{x}_2) \\ \dot{\boldsymbol{x}}_2 = f_{i2}(\boldsymbol{x}_2) \end{cases}, \quad i = 1, 2 \tag{8}$$

where $\boldsymbol{x}_1 = (z_1, z_2)^{\mathrm{T}}$, $\boldsymbol{x}_2 = (z_3, z_4)^{\mathrm{T}}$, $f_{11} = (-5z_1 - 2z_2 + \sin z_3, 21z_1 - 8z_2)^{\mathrm{T}}$, $f_{12} = (z_4, -3z_3 - 4z_4)^{\mathrm{T}}$, $f_{21} = (-16z_1 + 3z_2 + \sin z_4, -52z_1 + 9z_2)^{\mathrm{T}}$, $f_{22} = (z_4, -2z_3 - 3z_4)^{\mathrm{T}}$.

Choose $V_2 = \boldsymbol{x}_2^{\mathrm{T}} \begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix} \boldsymbol{x}_2$, $V_1 = \boldsymbol{x}_1^{\mathrm{T}} \begin{pmatrix} 7300/21 & -100 \\ -100 & 500/17 \end{pmatrix} x_1$. It is easy to verify that conditions in Theorem 1 are satisfied. Therefore, switched nonlinear system (8) is globally quadratically stable under arbitrary switching. Fig. 1 shows the state response curve of system (8) under a switching law randomly chosen.

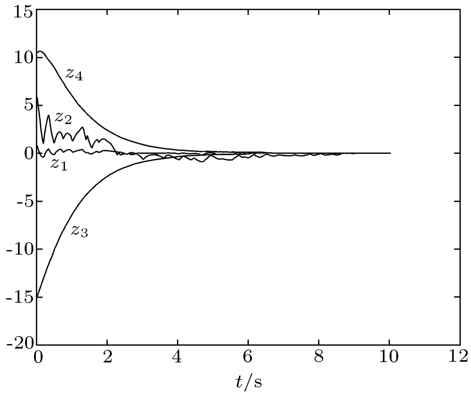


Fig. 1 The state response of system (8)

## 4 Conclusion

A sufficient condition for globally quadratic stability of switched nonlinear systems in block-triangular form has been presented. The result relaxes the requirement of the input-to-state stability which has been commonly used in the literature. A common Lyapunov function is constructed by using the Lyapunov functions of block-subsystems, which transforms the stability problem to that of lower dimensional systems. The result can be easily extended to the stabilization problem for switched nonlinear control systems in block-triangular form.

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**ZHAO Sheng-Zhi** Ph. D. candidate at Northeastern University. Her research interests include switched nonlinear systems and robust control.

**ZHAO Jun** Professor at Northeastern University. His research interests include hybrid systems, geometric control theory, switching control, and robust control.