On Controllability and Observability of Multivariable Linear Time-invariant Systems

WANG Cheng-Hong SONG Su
(*Department of Information Science, National Natural Science Foundation of China, Beijing* 100085)
(E-mail: songsu@nsfc.gov.cn)

Abstract Some results on linear system theory are reported. Based on these results, necessary and sufficient conditions for the controllability and observability of both continuous-time and its corresponding discrete-time multivariable linear time-invariant systems are presented.

Key words Multivariable linear time-invariant system, controllability, observability

1 Introduction

When a multivariable linear time-invariant system is controllable and observable, does its corresponding discrete system also have the same properties? This problem has drawn researcher′s attention for a long time. In the 1st IFAC congress of 1960, Kalman proved a sufficient and necessary condition for the controllability and observability of SISO system. In 1975, Guan and Chen proved a sufficient condition for the controllability and observability of MIMO system. In 1987, Hu *et al.* proved a necessary condition for the controllability and observability of MIMO system. In 1985, Ackermann proved a sufficient and necessary condition for the controllability and observability of SISO system in another form. Up to the present, the sufficient and necessary condition for the controllability and observability of MIMO system has not been reported yet. This paper reports some new results on linear system theory, and presents necessary and sufficient conditions for the controllability and observability of both continuous-time and its corresponding discrete-time multivariable linear time-invariant systems.

2 System model and preliminaries

The dynamic equation of a multivariable linear time-invariant system can be represented as

$$\begin{cases} \dot{\boldsymbol{X}}(t) = A\boldsymbol{X}(t) + B\boldsymbol{U}(t) \\ \boldsymbol{Y}(t) = C\boldsymbol{X}(t) + D\boldsymbol{U}(t) \end{cases} \tag{1}$$

where $\boldsymbol{X}(t) \in R^n, \boldsymbol{U}(t) \in R^p, \boldsymbol{Y}(t) \in R^q$ $(1 \leqslant p, q \leqslant n)$ are the state, input and output vectors of the system, respectively, and $A \in R^{n\times n}, B \in R^{n\times p}, C \in R^{q\times n}, D \in R^{q\times p}$ are the system matrix, input matrix, output matrix and feed-forward matrix of the system, respectively. The corresponding discrete-time system (with the sampling period T) of the dynamic (1) is

$$\begin{cases} \boldsymbol{X}((k+1)T) = \varPhi(T)\boldsymbol{X}(kT) + G(T)\boldsymbol{U}(kT) \\ \boldsymbol{Y}(kT) = C\boldsymbol{X}(kT) + D\boldsymbol{U}(kT) \end{cases} \tag{2}$$

where $\varPhi(T) = e^{A\mathrm{T}}$, and $G(T) = \int_0^{\mathrm{T}} e^{A\tau}\mathrm{d}\tau B$.

The necessary and sufficient conditions of the controllability and observability of continuous-time system $\{A, B, C, D\}$ are

$$\mathrm{rank}([B, AB, A^2B, \cdots, A^{n-1}B]) = n \tag{3}$$

$$\mathrm{rank}([C^{\mathrm{T}}, A^{\mathrm{T}}C^{\mathrm{T}}, (A^{\mathrm{T}})^2C^{\mathrm{T}}, \cdots, (A^{\mathrm{T}})^{n-1}C^{\mathrm{T}}]) = n \tag{4}$$

The necessary and sufficient conditions of the controllability and observability of discrete-time system $\{\varPhi, G, C, D\}$ are

$$\mathrm{rank}([G, \varPhi G, \varPhi^2G, \cdots, \varPhi^{n-1}G]) = n \tag{5}$$

$$\mathrm{rank}([C^{\mathrm{T}}, \varPhi^{\mathrm{T}}C^{\mathrm{T}}, (\varPhi^{\mathrm{T}})^2C^{\mathrm{T}}, \cdots, (\varPhi^{\mathrm{T}})^{n-1}C^{\mathrm{T}}]) = n \tag{6}$$

Lemma 1[1]**.** Suppose that the system $\{A, B, C, D\}$ is controllable. Then, the necessary condition of the discrete-time system $\{\varPhi, G, C, D\}$ being controllable is that $j2K\pi/T$ is not an eigenvalue of A, where k is a non-zero integer, and j is the imaginary unit.

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Lemma 2[3]**.** Suppose that the system $\{A, B, C, D\}$ is controllable (observable). Then the sufficient condition of the discrete-time system $\{\Phi, G, C, D\}$ being controllable (observable) is that there is no integer k except zero that can make $(\lambda_1 - \lambda_2)T = 2K\pi j$ hold for arbitrary two different eigenvalues λ_1 and λ_2 of A.

Before we present the sufficient and necessary conditions for the controllability and observability of both system $\{A, B, C, D\}$ and system $\{\Phi, G, C, D\}$, we present some results on linear system theory.

3 Results on linear system theory

Theorem 1. $\forall A \in R^{n\times n}$, let $\varphi(\lambda)$ be the minimal characteristic polynomial of A, and m be the power of $\varphi(\lambda)$. If $m \leqslant n$, then matrixes $I, A, A^2, \cdots, A^{m-1}$ are linearly independent, where I is the unit matrix.

Proof. Suppose that there exist m real numbers $k_i \in R(i = 0, 1, 2, \cdots, m-1)$ not all equal to 0 such that

$$k_0 I + k_1 A + k_2 A^2 + \cdots + k_{m-1} A^{m-1} = 0 \tag{7}$$

Assume $k_{m-1} \neq 0$ (otherwise, assume $k_{m-2} \neq 0, \cdots$), and let $\alpha_i = k_i/k_{m-1}(i = 0, 1, 2, \cdots, m-2)$. Then (7) can be rewritten in the following form

$$\alpha_0 I + \alpha_1 A + \alpha_2 A^2 + \cdots + \alpha_{m-2} A^{m-2} + A^{m-1} = 0 \tag{8}$$

In (8), the coefficient of the first term is 1 and the highest power is $m-1$. This contradicts the assumption that the power of $\varphi(\lambda)$ is m. This ends the proof of Theorem 1. □

Theorem 2. $\forall T \in R_+, A \in R^{n\times n}$, let $\varphi(\lambda)$ be the minimal characteristic polynomial of A, m be the power of $\varphi(\lambda)$, $p(\alpha)$ be the minimal characteristic polynomial of $e^{A\mathrm{T}}$ and $h(h \leqslant n)$ be the power of $p(\alpha)$. Then, $h \leqslant m$.

Proof. Because $\deg[p(\alpha)] = h$, there exist $h+1$ real numbers β_i not all equal to 0 such that

$$\beta_0 + \beta_1 e^{A\mathrm{T}} + \beta_2 e^{A2\mathrm{T}} + \cdots + \beta_h e^{Ah\mathrm{T}} = 0 \tag{9}$$

According to the Cayley-Hamilton theorem

$$e^{Ai\mathrm{T}} = \sum_{j=0}^{n-1} \gamma_{ji}(T) A^j, \quad i = 1, 2, \cdots, h \tag{10}$$

Since $\varphi(A) = 0, A^m, A^{m+1}, \cdots, A^{n-1}$ can be linearly represented by $I, A, \cdots, A^{m-1}$, (10) can be rewritten as

$$e^{Ai\mathrm{T}} = \sum_{j=0}^{m-1} \alpha_{ji}(T) A^j, \quad i = 1, 2, \cdots, h \tag{11}$$

Combining (11) and (9), we obtain

$$\left(\beta_0 + \sum_{i=1}^{h} \beta_i \alpha_{0i}\right) I + \left(\sum_{i=1}^{h} \beta_i \alpha_{1i}\right) A + \cdots + \left(\sum_{i=1}^{h} \beta_i \alpha_{m-1,i}\right) A^{m-1} = 0 \tag{12}$$

From Theorem 1 and the assumption made, we know that (12) holds when all its coefficients equal 0, we obtain

$$\begin{bmatrix} 1 & \alpha_{01} & \alpha_{02} & \cdots & \alpha_{0,h-2} & \alpha_{0,h-1} & \alpha_{0,h} \\ 0 & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1,h-2} & \alpha_{1,h-1} & \alpha_{1,h} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & \alpha_{m-1,1} & \alpha_{m-1,2} & \cdots & \alpha_{m-1,h-2} & \alpha_{m-1,h-1} & \alpha_{m-1,h} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_h \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{13}$$

(13) has at least 2 coefficients that can be arbitrarily selected when $h \geqslant m+1$. Suppose there are only 2 coefficients that can be arbitrarily selected, say β_{h-1}, β_h. Let (β_{h-1}, β_h) be $(1, 1)$ and $(0, 1)$, respectively. According to the solution′s structure theory, (13) has two linearly independent solutions $(\beta_0', \beta_1', \cdots, \beta_{n-2}', 1, 1)$ and $(\beta_0'', \beta_1'', \cdots, \beta_{n-2}'', 0, 1)$. Substituting these two solutions into (9) and performing subtraction between them, we have

$$(\beta_0' - \beta_0'')I + (\beta_1' - \beta_1'')e^{A\mathrm{T}} + \cdots + (\beta_{n-2}' - \beta_{n-2}'')e^{A(h-2)\mathrm{T}} + e^{A(h-1)\mathrm{T}} = 0 \tag{14}$$

In (14), the highest power is $h-1$, the coefficient of the first term is 1 and the coefficients of other terms are not all equal to 0. This contradicts the assumption that the power of $p(\alpha)$ is h. Thus, we have $h \leqslant m$. The same conclusion can be reached when there are more than 2 coefficients that can be arbitrarily selected. This ends the proof of Theorem 2. □

Theorem 3. $\forall T \in R_+, A \in R^{n\times n}$, let $\varphi(\lambda)$ be the minimal characteristic polynomial of A, m be the power of $\varphi(\lambda)$, $p(\alpha)$ be the minimal characteristic polynomial of e^{AA}, and h be the power of $p(\alpha)$. If the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_s (s \leqslant n)$ of A are different from each other, and $\varphi(\lambda) = (\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_s)^{m_s}\left(\sum_{i=1}^{s} m_i = m\right)$, then the sufficient and necessary condition for $h = m$ is $(\lambda_p - \lambda_q)T \neq 2K\pi j (1 \leqslant p < q \leqslant s;\ k = \pm1, \pm2, \cdots)$.

Proof. Necessity. If $h = m$, then there exist a group of numbers $\beta_0, \beta_1, \cdots, \beta_m (\beta_m = 1)$ such that

$$\beta_0 I + \beta_1 e^{A\mathrm{T}} + \beta_2 e^{A2\mathrm{T}} + \cdots + \beta_m e^{Am\mathrm{T}} = 0 \tag{15}$$

According to the canonical form[2] of $e^{A\mathrm{T}}$, $e^{A\mathrm{T}}$ can also be written in the following form

$$e^{A\mathrm{T}} = \sum_{i=1}^{s} [p_{i0}(A) + Tp_{i1}(A) + \cdots + T^{m_i-1} p_{i,m_i-1}(A)] e^{\lambda_i \mathrm{T}} \tag{16}$$

where matrixes $p_{ij}(A)(i = 1, 2, \cdots, s;\ j = 0, 1, \cdots, m-1)$ are called the components of A, they are linearly independent and none of them equal to 0. Using (16), we arrive at

$$I = \lim_{T\to 0} e^{A\mathrm{T}} = \sum_{i=1}^{s} p_{i0}(A) \tag{17}$$

Combining, and, we obtain

$$\sum_{k=0}^{m} \beta_k \sum_{i=1}^{s} [p_{i0}(A) + kTp_{i1}(A) + \cdots + (kT)^{m_i-1} p_{i,m_i-1}(A)] e^{\lambda_i k\mathrm{T}} = 0 \tag{18}$$

Since all $p_{ij}(A)$ in (18) are linearly independent, the coefficients of the corresponding terms should be equal to 0 to make the above equation hold. Because $\beta_m = 1$, we obtain

$$\begin{bmatrix}
1 & e^{\lambda_1\mathrm{T}} & e^{2\lambda_1\mathrm{T}} & \cdots & e^{(m-1)\lambda_1\mathrm{T}} \\
0 & Te^{\lambda_1\mathrm{T}} & 2Te^{2\lambda_1\mathrm{T}} & \cdots & (m-1)Te^{(m-1)\lambda_1\mathrm{T}} \\
\vdots & \vdots & \vdots & & \vdots \\
0 & T^{m_1-1}e^{\lambda_1\mathrm{T}} & (2T)^{m_1-1}e^{2\lambda_1\mathrm{T}} & \cdots & [(m-1)T]^{m_1-1}e^{(m-1)\lambda_1\mathrm{T}} \\
1 & e^{\lambda_2\mathrm{T}} & e^{2\lambda_2\mathrm{T}} & \cdots & e^{(m-1)\lambda_2\mathrm{T}} \\
0 & Te^{\lambda_2\mathrm{T}} & 2Te^{2\lambda_2\mathrm{T}} & \cdots & (m-1)Te^{(m-1)\lambda_2\mathrm{T}} \\
\vdots & \vdots & \vdots & & \vdots \\
0 & T^{m_2-1}e^{\lambda_2\mathrm{T}} & (2T)^{m_2-1}e^{2\lambda_2\mathrm{T}} & \cdots & [(m-1)T]^{m_2-1}e^{(m-1)\lambda_2\mathrm{T}} \\
\vdots & \vdots & \vdots & & \vdots \\
1 & e^{\lambda_s\mathrm{T}} & e^{2\lambda_s\mathrm{T}} & \cdots & e^{(m-1)\lambda_s\mathrm{T}} \\
0 & Te^{\lambda_s\mathrm{T}} & 2Te^{2\lambda_s\mathrm{T}} & \cdots & (m-1)Te^{(m-1)\lambda_s\mathrm{T}} \\
\vdots & \vdots & \vdots & & \vdots \\
0 & T^{m_s-1}e^{\lambda_s\mathrm{T}} & (2T)^{m_s-1}e^{2\lambda_s\mathrm{T}} & \cdots & [(m-1)T]^{m_s-1}e^{(m-1)\lambda_s\mathrm{T}}
\end{bmatrix}
\begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{m_1-1} \\ \bullet \\ \bullet \\ \vdots \\ \bullet \\ \vdots \\ \bullet \\ \bullet \\ \vdots \\ \beta_{m-1} \end{bmatrix}
=
\begin{bmatrix} -e^{m\lambda_1\mathrm{T}} \\ -mTe^{m\lambda_1\mathrm{T}} \\ \vdots \\ -(mT)^{m_1-1}e^{m\lambda_1\mathrm{T}} \\ -e^{m\lambda_2\mathrm{T}} \\ -mTe^{m\lambda_2\mathrm{T}} \\ \vdots \\ -(mT)^{m_2-1}e^{m\lambda_2\mathrm{T}} \\ \vdots \\ -e^{m\lambda_s\mathrm{T}} \\ -mTe^{m\lambda_s\mathrm{T}} \\ \vdots \\ -(mT)^{m_s-1}e^{m\lambda_s\mathrm{T}} \end{bmatrix} \tag{19}$$

It is easy to notice that the elements of the $2^{\mathrm{nd}}, 3^{\mathrm{rd}}, \cdots, m_1$-th rows in the coefficient matrix are the $1^{\mathrm{st}}, 2^{\mathrm{nd}}, (m_1-1)$-th derivative of the corresponding elements in the 1^{st} row with the independent variable of λ_1; the elements from the (m_1+2)-th row to the (m_1+m_2)-th row in the coefficient matrix are the $1^{\mathrm{st}}, \cdots, (m_2-1)$-th derivatives of the corresponding elements in the (m_1+1)-th row with the independent variable of λ_2, the elements from the $(m-m_s+2)$-th row to the m-th row in the coefficient matrix are the $1^{\mathrm{st}}, \cdots, (m_s-1)$-th derivatives of the corresponding elements in the $(m-m_s+1)$-th row with the independent variable of λ_s, and so on. Obviously, (19) has an exclusive solution only if its′ coefficient matrix is of full rank, *i.e.*, $e^{\lambda p\mathrm{T}} \neq e^{\lambda q\mathrm{T}}$, this is equivalent to $(\lambda p - \lambda q)T \neq 2K\pi j (1 \leqslant p < q \leqslant s;\ k = \pm1, \pm2, \cdots)$.

Sufficiency: Assuming $(\lambda p - \lambda q)T \neq 2K\pi j$, we can reach a result similar to equations (15)∼(19), in which m is replaced by h. Denote X as the coefficient matrix of (19) (X has m rows and h columns),

Y as the constant vector at the right of equal mark, and $Z = [XY]$ as the augmented matrix. Suppose that $h < m$, then we can reach $\text{rank}(X) = h$ and $\text{rank}(Z) = h + 1$ from $(\lambda p - \lambda q)T \neq 2K\pi j$, and therefore the solution of (19) does not exist. This contradicts the assumption that the power of e^{AT}'s minimal polynomial is h. Combining this result with Theorem 2, we obtain $h = m$. This proves the theorem. □

Theorem 4. $\forall T \in R_+, A \in R^{n\times n}$, if $\deg[\varphi(\lambda)] = \deg[p(\alpha)] = m (m \leqslant n)$, then the matrix group $I, A, A^2, \cdots, A^{m-1}$ and the matrix group $I, e^{A\text{T}}, e^{A2\text{T}}, \cdots, e^{A(m-1)\text{T}}$ can linearly represent each other.

Proof. If $\deg[\varphi(\lambda)] = \deg[p(\alpha)] = m$, according to the Cayley-Hamilton theorem and the derivation from (10) to (11), we can reach the result that $e^{Ai\text{T}} (i = 0, 1, \cdots, m-1)$ can be linearly represented by $I, A, \cdots, A^{m-1}$.

In the following we prove that the converse proposition also holds.

Given $k_j \in R, j = 0, 1, 2, \cdots, m-1$ assume

$$A^i = k_0 I + k_1 e^{A\text{T}} + k_2 e^{A2\text{T}} + \cdots + k_{m-1} e^{A(m-1)\text{T}}, \quad 0 \leqslant i \leqslant m-1 \tag{20}$$

Similar to the derivation from (9) to (13), we obtain

$$\begin{bmatrix} 1 & \alpha_{01} & \alpha_{02} & \cdots & \alpha_{0,m-1} \\ 0 & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1,m-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \alpha_{i1} & \alpha_{i2} & \cdots & \alpha_{i,m-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \alpha_{m-1,1} & \alpha_{m-1,2} & \cdots & \alpha_{m-1,m-1} \end{bmatrix} \begin{bmatrix} k_0 \\ k_1 \\ \vdots \\ k_i \\ \vdots \\ k_{m-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \tag{21}$$

In the coefficient matrix of (21), the elements in the columns form 2 to m are the coefficients of the Cayley-Hamilton expansion (see (11)) of $e^{Ai\text{T}} (1 \leqslant i \leqslant m-1)$ respectively. When $\deg[\varphi(\lambda)] = \deg[p(\alpha)] = m$, we prove (by contradiction) that the m column vectors in the coefficient matrix of (21) are linearly independent. Without loss of generality, suppose that the last column can be linearly represented by other columns, *i.e.*, there are $m-1$ numbers $\beta_i \in R$ not all equal to 0 such that

$$\begin{bmatrix} \alpha_{0,m-1} \\ \vdots \\ \alpha_{i,m-1} \\ \vdots \\ \alpha_{m-1,m-1} \end{bmatrix} = \beta_0 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} + \beta_1 \begin{bmatrix} \alpha_{01} \\ \vdots \\ \alpha_{i1} \\ \vdots \\ \alpha_{m-1,1} \end{bmatrix} + \cdots + \beta_{m-2} \begin{bmatrix} \alpha_{0,m-2} \\ \vdots \\ \alpha_{i,m-2} \\ \vdots \\ \alpha_{m-1,m-2} \end{bmatrix} \tag{22}$$

From (22) we obtain

$$\begin{cases} \alpha_{0,m-1} = \beta_0 + \sum\limits_{j=1}^{m-2} \beta_j \alpha_{0j} \\ \alpha_{i,m-1} = \sum\limits_{j=1}^{m-2} \beta_j \alpha_{i,j}, \quad i = 1, 2, \cdots, m-1 \end{cases} \tag{23}$$

Combining (23) and the $Cayley-Hamilton$ expression of $e^{A(m-1)\text{T}}$ (see (11)), we see that

$$e^{A(m-1)\text{T}} = \sum_{i=0}^{m-1} \alpha_{i,m-1}(T) A^i = \beta_0 I + \left(\sum_{j=1}^{m-2} \beta_j \alpha_{0j}\right) I + \sum_{i=1}^{m-1} \left(\sum_{j=1}^{m-2} \beta_j \alpha_{ij}\right) A^i =$$
$$\beta_0 I + \sum_{j=1}^{m-2} \left(\sum_{i=0}^{m-1} \beta_j \alpha_{ij} A^i\right) = \beta_0 I + \sum_{j=1}^{m-2} \beta_j \left(\sum_{i=0}^{m-1} \alpha_{ij} A^i\right) =$$
$$\beta_0 I + \sum_{j=1}^{m-2} \beta_j e^{Aj\text{T}} = \beta_0 I + \beta_1 e^{A\text{T}} + \beta_2 e^{A2\text{T}} + \cdots + \beta_{m-2} e^{A(m-2)\text{T}}$$

Leting $\alpha_j = -\beta_j (j = 0, 1, \cdots, m-2)$, we obtain

$$\alpha_0 I + \alpha_1 e^{A\text{T}} + \cdots + \alpha_{m-2} e^{A(m-2)\text{T}} + e^{A(m-1)\text{T}} = 0 \tag{24}$$

(24) means $\deg[p(\alpha)] < m$, and this contradicts the assumption. Thus, the coefficient matrix of (21) is of full rank. *i.e.*, (20) has an exclusive solution, and A^i can be linearly represented by $I, e^{A\mathrm{T}}, \cdots, e^{A(m-1)\mathrm{T}}$. Notice that when i is ergodic on $(0, 1, 2, \cdots, m-1)$, the left side of (21) keeps invariant and the row of the only non-zero element is "1" accordingly ergodic on $(0, 1, 2, \cdots, m-1)$. Therefore, the above statement holds for all $0 \leqslant i \leqslant m-1$. This ends the proof of Theorem 4. □

Corollary 1. $\forall T \in R_+, A \in R^{n\times n}$, if $\deg[\varphi(\lambda)] = \deg[p(\alpha)] = m (m \leqslant n)$, then the matrix group $I, A, A^2, \cdots, A^{n-1}$ and the matrix group $I, e^{A\mathrm{T}}, e^{A2\mathrm{T}}, \cdots, e^{A(n-1)\mathrm{T}}$ can linearly represent each other.

Proof. If $m = n$, then it is just Theorem 4. If $m < n$, then $I, A, \cdots, A^{m-1}$ and $I, A, \cdots, A^{n-1}$ can linearly represent each other; also, $I, e^{A\mathrm{T}}, \cdots, e^{A(m-1)\mathrm{T}}$ and $I, e^{A\mathrm{T}}, \cdots, e^{A(n-1)\mathrm{T}}$ can linearly represent each other. Thus $I, A, \cdots, A^{n-1}$ and $I, e^{A\mathrm{T}}, \cdots, e^{A(n-1)\mathrm{T}}$ can linearly represent each other. This ends the proof Corollary 1. □

Corollary 2. $\forall T \in R_+, A \in R^{n\times n}$, if arbitrary two different latent roots of A, namely λ_p and λ_q, satisfy $(\lambda_p - \lambda_q)T \neq 2K\pi j (1 \leqslant p < q \leqslant s;\ k = \pm 1, \pm 2, \cdots)$, then the matrix group $I, A, \cdots, A_{n-1}$ and the matrix group $I, e^{A\mathrm{T}}, \cdots, e^{A(n-1)\mathrm{T}}$ can linearly represent each other.

Proof. According to Theorem 3, Theorem 4 and Corollary 1, we see that Corollary 2 holds. □

4 Main results

Theorem 5. $\forall A \in R^{n\times n}$, let $\lambda_1, \lambda_2, \cdots \lambda_s$ be the latent roots of A and they are different from one another. Suppose that the continuous system $\{A, B, C, D\}$ is controllable. Then, the necessary and sufficient condition for the corresponding discrete system $\{\Phi, G, C, D\}$ (with sampling period T) being also controllable is: for an arbitrary non-zero integer k, the following two inequalities hold:

$$1) \qquad \lambda_p T \neq 2k\pi j, \quad p = 1, 2, \cdots, s \tag{25}$$

$$2) \qquad (\lambda_p - \lambda_q)T \neq 2k\pi j, \quad 1 \leqslant p < q \leqslant s \tag{26}$$

Proof. Necessity. Combining $\Phi(T) = e^{A\mathrm{T}}, G(T) = \int_0^{\mathrm{T}} e^{A\tau}\mathrm{d}\tau B$ and $V = [G, \Phi G, \cdots, \Phi^{n-1}G]$, and considering $e^{A\mathrm{T}}\int_0^{\mathrm{T}} e^{A\tau}\mathrm{d}\tau = \int_0^{\mathrm{T}} e^{A\tau}\mathrm{d}\tau e^{A\mathrm{T}}$, we obtain

$$V = \int_0^{\mathrm{T}} e^{A\tau}\mathrm{d}\tau [B, e^{A\mathrm{T}}B, \cdots, e^{A(n-1)\mathrm{T}}B] \tag{27}$$

Obviously, if the discrete system is controllable, then we must have $\mathrm{rank}(V) = n$. This is equivalent to

$$\mathrm{rank}\left(\int_0^{\mathrm{T}} e^{A\tau}\mathrm{d}\tau\right) = n \tag{28}$$

$$\mathrm{rank}([B, e^{A\mathrm{T}}B, \cdots, e^{A(n-1)\mathrm{T}}B]) = n \tag{29}$$

Given λ_i as the latent roots of A, (28) is equivalent to $\lambda_i T \neq 2k\pi j (k = \pm 1, \pm 2, \cdots)$[1].

Since the continuous system $\{A, B, C, D\}$ is controllable, we have that $\mathrm{rank}([B, AB, \cdots, A^{n-1}B]) = n$. According to the *Cayley − Hamilton* theorem, $I, e^{A\mathrm{T}}, \cdots, e^{A(n-1)\mathrm{T}}$ can be linearly represented by $I, A, \cdots, A^{n-1}$, thus we obtain $\mathrm{rank}([B, e^{A\mathrm{T}}B, \cdots, e^{A(n-1)\mathrm{T}}B]) \leqslant \mathrm{rank}([B, AB, \cdots, A^{n-1}B]) = n$.

If $\mathrm{rank}([B, e^{A\mathrm{T}}B, \cdots, e^{A(n-1)\mathrm{T}}B]) \geqslant \mathrm{rank}([B, AB, \cdots, A^{n-1}B]) = n$ holds, *i.e.*, (29) holds, then $I, A, \cdots, A^{n-1}$ can be linearly represented by $I, e^{A\mathrm{T}}, \cdots, e^{A(n-1)\mathrm{T}}$. According to Corollary 2, $(\lambda_p - \lambda_q)T \neq 2k\pi j (1 \leqslant p < p \leqslant s)$ must hold. This proves the necessity.

The proof for Sufficiency is seen in [3]. □

Theorem 6. $\forall A \in R^{n\times n}$, let $\lambda_1, \lambda_2, \cdots, \lambda_s$ be all the latent roots of A which are different from each other. Suppose that the continuous system $\{A, B, C, D\}$ is observable. Then, the necessary and sufficient condition for the corresponding discrete system $\{\Phi, G, C, D\}$ (with sampling period T) being also observable is: for an arbitrary non-zero integer k, the inequality $(\lambda_p - \lambda_q)T \neq 2k\pi j (1 \leqslant p < q \leqslant s)$ holds.

Proof. Notice that all the different latent roots of A^{T} are $\lambda_1, \lambda_2, \cdots, \lambda_s$, and $\Phi^{\mathrm{T}} = e^{A^{\mathrm{T}}\mathrm{T}}$ holds. Similar to the proof of Theorem 5, this theorem can be proven utilizing Corollary 2. □

Theorem 7. $\forall A \in R^{n\times n}$, let $\lambda_1, \lambda_2, \cdots, \lambda_s$ be the latent roots of A which are different from each other. Suppose that the continuous system $\{A, B, C, D\}$ is controllable and observable. Then, the necessary and sufficient condition for the corresponding discrete system $\{\Phi, G, C, D\}$ being also controllable and observable is: for an arbitrary non-zero integer k, both inequality (25) and inequality (26) hold at the same time.

Proof. Combining Theorem 5 and Theorem 6, we can reach Theorem 7. □

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WANG Cheng-Hong Received his master degree from China University of Mining and Technology in 1988, and his Ph. D. degree from the Institute of Automation, Chinese Academy of Sciences in 1997. Since 1999, he has been with Division III of Information Science Department, National Natural Science Foundation of China, where he works as a professor and director. His research interests include control theory, reliability theory, complex system theory, and operational research.

SONG Su Received his master and Ph. D. degrees from Harbin Engineering University in 1992 and 1997, respectively. He works as a professor in Division III of Information Science Department, National Natural Science Foundation of China. His research interests include adaptive control and complex system theory.

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