Stability Analysis of T-S Fuzzy Control Systems Based on Parameter-dependent Lyapunov Function1)

DING Bao-Cang SUN He-Xu QIAO Yu-E

(*School of Electricity and Automation, Hebei University of Technology, Tianjin* 300130)

(E-mail: dingbc@jsmail.hebut.edu.cn)

Abstract For discrete-time T-S fuzzy systems, the stability and controller design method are investigated based on parameter-dependent Lyapunov function (PDLF). T-S fuzzy systems differ from non-fuzzy systems with polytopic description or multi-model description in that the weighting coefficients have respective meanings. They, however, have stability aspect in common. By adopting a stability condition for polytopic systems obtained *via* PDLF, and combining the properties of T-S fuzzy systems, new results are given in this paper. An example shows that by applying the new results, the stability conditions that can be distinguished are less conservative.

Key words T-S fuzzy system, parameter-dependent Lyapunov function, polytopic description, stability

1 Introduction

In recent years, T-S fuzzy control method has been receiving increasing attention. The main reason is that this method does not apply the accurate model but rather a rule-based model set. If any possible system response is included in the responses for all sub-models in the set, then studying the stability of the original system will turn to be that of an artificial system based on the model set. Especially, if all the sub-models are linear, which is true in many investigations[1~5], T-S method will bring much convenience for design and analysis.

The earlier investigations for stability of T-S fuzzy systems (such as [2], [4], *etc.*) gave the common Lyapunov function for all sub-models in the model set, which has large conservativeness. [1] and [3] gave fuzzy Lyapunov functions for continuous and discrete models, respectively. If the common Lyapunov function is quadratic, then the corresponding fuzzy Lyapunov function is a linear combination of a set of quadratic functions, with the time-varying combining coefficients calculated by the grade of membership. Actually, the results in [1] and [3] were simultaneously suitable to time-varying and uncertain coefficients, so are equivalent to the investigations in the non-fuzzy systems such as systems with polytopic description and multi-model description[6,7]. Due to this equivalence, the stability conditions of discrete systems given in [7] are less conservative than those in [3]. This paper further studies the stability properties of discrete fuzzy systems to obtain less conservative stability results and controller design method.

2 Discrete T-S fuzzy control systems and some existing stability results

Consider a discrete T-S fuzzy system, with its ith rule described as:

$$R^i: \text{ IF } z_1(k) \text{ is } M_1^i \text{ and } \ldots \text{ and } z_p(k) \text{ is } M_p^i, \text{ THEN } \boldsymbol{x}(k+1) = A_i\boldsymbol{x}(k) + B_i\boldsymbol{u}(k) \tag{1}$$

where $i = 1, 2, \cdots, r$ with r being the number of rules; $\boldsymbol{z}(k) = [z_1(k), \cdots, z_p(k)]^{\mathrm{T}}$ is the premise variable of the system; $\boldsymbol{x}(k) \in \Re^n$, $\boldsymbol{u}(k) \in \Re^m$ are measurable state and input respectively; A_i, B_i are matrices with proper dimensions. The linear state-space equation corresponding to R^i is called sub-model i. Denote $\boldsymbol{x}(k) = \boldsymbol{x}_k, \boldsymbol{u}(k) = \boldsymbol{u}_k, \boldsymbol{z}(k) = \boldsymbol{z}_k$. Then the overall model of the system is represented as:

$$\boldsymbol{x}_{k+1} = \sum_{i=1}^{r} \omega_i(\boldsymbol{z}_k)(A_i\boldsymbol{x}_k + B_i\boldsymbol{u}_k) = A(\boldsymbol{z}_k)\boldsymbol{x}_k + B(\boldsymbol{z}_k)\boldsymbol{u}_k$$

$$A(\boldsymbol{z}_k) = \sum_{i=1}^{r} \omega_i(\boldsymbol{z}_k)A_i, \quad B(\boldsymbol{z}_k) = \sum_{i=1}^{r} \omega_i(\boldsymbol{z}_k)B_i \tag{2}$$

1) Supported by National Natural Science Foundation of P. R. China (60074004)
Received June 7, 2004; in revised form January 19, 2005

$$\omega_i(\boldsymbol{z}_k) = \prod_{j=1}^{p} M_j^i(z_j(k)) \Big/ \sum_{i=1}^{r} \prod_{j=1}^{p} M_j^i(z_j(k)), \quad 0 \leqslant \omega_i(\boldsymbol{z}_k) \leqslant 1, \quad \sum_{i=1}^{r} \omega_i(\boldsymbol{z}_k) = 1 \tag{3}$$

where $M_j^i(z_j(k))$ is the grade of membership of $z_j(k)$ in M_j^i. For the open-loop system

$$\boldsymbol{x}_{k+1} = \sum_{i=1}^{r} \omega_i(\boldsymbol{z}_k) A_i \boldsymbol{x}_k = A(\boldsymbol{z}_k)\boldsymbol{x}_k \tag{4}$$

the common Lyapunov function has the form $V(\boldsymbol{x}_k) = \boldsymbol{x}_k^{\mathrm{T}} P \boldsymbol{x}_k$, where P is a symmetric positive matrix[2,4], while the fuzzy Lyapunov function has the form[3]:

$$V(\boldsymbol{x}_k, \boldsymbol{z}_k) = \boldsymbol{x}_k^{\mathrm{T}} P(\boldsymbol{z}_k) \boldsymbol{x}_k, \quad P(\boldsymbol{z}_k) = \sum_{i=1}^{r} \omega_i(\boldsymbol{z}_k) P_i, \quad P_i = P_i^{\mathrm{T}} > 0 \tag{5}$$

For open and closed-loop fuzzy systems, [3] obtains stability results which are less conservative than [2,4].

Theorem 1[3]. If there exist symmetric positive matrices P_i such that

$$A_i^{\mathrm{T}} P_l A_i - P_i < 0, \quad \forall i, l = 1, 2, \cdots, r \tag{6}$$

then system (4) is globally asymptotically stable.

Theorem 2[3]. For system (1)~(3), the following control law is adopted

$$\boldsymbol{u}_k = \sum_{i=1}^{r} \omega_i(\boldsymbol{z}_k) F_i \boldsymbol{x}_k \tag{7}$$

If there exist symmetric positive matrices P_i such that

$$\begin{bmatrix} P_i & G_{ii}^{\mathrm{T}} P_l \\ P_l G_{ii} & P_l \end{bmatrix} > 0, \ i, l = 1, 2, \cdots, r; \quad \begin{bmatrix} (P_i + P_j)/2 & M_{ij}^{\mathrm{T}} P_l \\ P_l M_{ij} & P_l \end{bmatrix} \geqslant 0, \ i < j \leqslant r \tag{8}$$

where $M_{ij} = (G_{ij} + G_{ji})/2, G_{ij} = A_i + B_i F_j$, then the closed-loop system is globally asymptotically stable.

3 New stability results for discrete T-S fuzzy control systems

Consider the following dynamic uncertain discrete system

$$\boldsymbol{x}_{k+1} = A(\xi_k)\boldsymbol{x}_k, \quad A(\xi_k) = \sum_{i=1}^{r} \xi_i(k) A_i, \quad 0 \leqslant \xi_i(k) \leqslant 1, \quad \sum_{i=1}^{r} \xi_i(k) = 1 \tag{9}$$

where $\xi_i(k)$ are utterly unknown, time-varying bounded parameters; other notations are the same as in (4). Apparently, there is essential difference between $\xi_i(k)$ and $\omega_i(\boldsymbol{z}_k)$. For (9), [7] gave the following stability result.

Lemma 1[7]. If there exist properly dimensional symmetric positive matrices P_i and matrices T_i such that

$$\begin{bmatrix} T_i + T_i^{\mathrm{T}} - P_i^{-1} & T_i^{\mathrm{T}} A_i^{\mathrm{T}} \\ A_i^{\mathrm{T}} T_i & P_l^{-1} \end{bmatrix} > 0, \quad \forall i, l = 1, 2, \cdots, r \tag{10}$$

then system (9) is globally asymptotically stable, while the Lyapunov function can be taken as $V(\boldsymbol{x}_k, \xi_k) = \sum_{i=1}^{r} \xi_i(k) \boldsymbol{x}_k^{\mathrm{T}} P_i \boldsymbol{x}_k$.

Condition (10) can be applied to discrete T-S fuzzy systems as follows.

Theorem 3. If there exist properly dimensional symmetric positive matrices P_i and matrices T_i such that (10) is satisfied, then the T-S fuzzy system (4) is globally asymptotically stable, with (5) as the corresponding Lyapunov function.

Let $T_i = P_i$, then (10) is equivalent to (6)[8]. Hence, compared with (6), (10) represents a less conservative condition although it requires more computation. Applying (10), the closed-loop stability of T-S fuzzy control system under (7) can be further analyzed. For this, denote the closed-loop system as

$$\boldsymbol{x}_{k+1} = \sum_{i=1}^{r}\sum_{j=1}^{r}\omega_i(\boldsymbol{z}_k)\omega_j(\boldsymbol{z}_k)(A_i + B_iF_j)\boldsymbol{x}_k = \sum_{i=1}^{r}\sum_{j=1}^{r}\omega_i(\boldsymbol{z}_k)\omega_j(\boldsymbol{z}_k)G_{ij}\boldsymbol{x}_k = G(\boldsymbol{z}_k, \boldsymbol{z}_k)\boldsymbol{x}_k \tag{11}$$

By comparing (11) with (4), the following result can be obtained directly applying Theorem 3.

Theorem 4. For system (1)∼(3) adopting control law (7), if there exist properly dimensional symmetric positive matrices P_{ij} and matrices T_{ij} such that

$$\begin{bmatrix} T_{ij} + T_{ij}^{\mathrm{T}} - P_{ij}^{-1} & T_{ij}^{\mathrm{T}}G_{ij}^{\mathrm{T}} \\ G_{ij}T_{ij} & P_{ls}^{-1} \end{bmatrix} > 0, \quad \forall i,j,l,s = 1,2,\cdots,r \tag{12}$$

then the closed-loop system is globally asymptotically stable, with the corresponding Lyapunov function as follows

$$V(\boldsymbol{x}_k, \boldsymbol{z}_k, \boldsymbol{z}_k) = \boldsymbol{x}_k^{\mathrm{T}}P(\boldsymbol{z}_k, \boldsymbol{z}_k)\boldsymbol{x}_k = \sum_{i=1}^{r}\sum_{j=1}^{r}\omega_i(\boldsymbol{z}_k)\omega_j(\boldsymbol{z}_k)\boldsymbol{x}_k^{\mathrm{T}}P_{ij}\boldsymbol{x}_k \tag{13}$$

Similar to the open-loop case, Theorem 4 introduces more free variables. Hence, although it increases computational burden, Theorem 4 gives less conservative conditions than Theorem 2.

Corollary 1. For system (1)∼(3) adopting control law (7), if there exist properly dimensional symmetric positive matrices P_i and matrices T_{ij} such that

$$\begin{bmatrix} T_{ij} + T_{ij}^{\mathrm{T}} - [(P_i + P_j)/2]^{-1} & T_{ij}^{\mathrm{T}}G_{ij}^{\mathrm{T}} \\ G_{ij}T_{ij} & [(P_l + P_s)/2]^{-1} \end{bmatrix} > 0, \quad \forall i,j,l,s = 1,2,\cdots,r \tag{14}$$

then the closed-loop system is globally asymptotically stable, with the corresponding Lyapunov function as follows

$$V(\boldsymbol{x}_k, \boldsymbol{z}_k, \boldsymbol{z}_k) = \sum_{i=1}^{r}\sum_{j=1}^{r}\omega_i(\boldsymbol{z}_k)\omega_j(\boldsymbol{z}_k)\boldsymbol{x}_k^{\mathrm{T}}(P_i + P_j)\boldsymbol{x}_k \tag{15}$$

Proof. In (12), (13), let $P_{ij} = (P_i + P_j)/2,\ i,j = 1,2,\cdots,r$. Then (14), (15) can be obtained. □

Remark 1. (10) gives linear matrix inequalities (LMIs) with $2r$ variables $T_i, P_i^{-1}, i = 1,2,\cdots,r$, (12) gives LMIs with $2r^2$ variables $T_{ij}, P_{ij}^{-1}, i,j = 1,2,\cdots,r$, and (14) gives inequalities with $(r^2 + r)$ variables $T_{ij}, P_i,\ i,j = 1,2,\cdots,r$. (14), which does not give LMIs, presents stability conditions more conservative than (12). This shows that although it only needs r positive matrices, applying (15) as Lyapunov function brings inconvenience.

All the above discussions are for the case with F_i given, $i = 1,2,\cdots,r$. If F_i are not known and need to be solved, then (8) and (12) in this paper and (11)∼(13) in [3] are not LMIs any more, since they do not satisfy the superposition principle. To solve this problem, let $T_{ij} = T_j$ in (12) and the following conclusion can be obtained:

Corollary 2. For system (1)∼(3) adopting control law (7), if there exist properly dimensional symmetric positive matrices P_{ij} and matrices T_j, Y_j such that

$$\begin{bmatrix} T_j + T_j^{\mathrm{T}} - P_{ij}^{-1} & Y_j^{\mathrm{T}}B_i^{\mathrm{T}} + T_j^{\mathrm{T}}A_i^{\mathrm{T}} \\ A_iT_j + B_iY_j & P_{ls}^{-1} \end{bmatrix} > 0, \quad \forall i,j,l,s = 1,2,\cdots,r \tag{16}$$

then the closed-loop system is globally asymptotically stable with the corresponding fuzzy controller feedback gains $F_j = Y_jT_j^{-1}$.

(16) gives LMIs with $2r + r^2$ variables $T_j, Y_j, P_{ij}^{-1},\ i,j = 1,2,\cdots,r$. By the results in this paper, judging the stability is equivalent to judging the feasibility of the LMIs. For example, adopting Corollary 2, judging the stability is reduced to: if there exist corresponding P_{ij}, T_j, Y_j such that (16) is satisfied; that is, if the following problem has a feasible solution, then the corresponding $F_j = Y_jT_j^{-1}$ exist that globally asymptotically stabilizes the closed-loop system:

$$\text{find} \quad P_{ij}, T_j, Y_j, \quad \text{s.t. (16)} \tag{17}$$

4 A comparison example

We directly consider the T-S fuzzy model in [3]. The fuzzy system (1)∼(3) consists of the following two rules:

$$R^1: \text{ IF } z(k) \text{ is about } 0, \text{ THEN } \boldsymbol{x}(k+1) = A_1\boldsymbol{x}(k) + B_1u(k)$$
$$R^2: \text{ IF } z(k) \text{ is about } \pi \text{ or } -\pi, \text{ THEN } \boldsymbol{x}(k+1) = A_2\boldsymbol{x}(k) + B_2u(k)$$

where $z(k) = x_2(k) - \frac{2}{11}x_1(k)$,

$$A_1 = \begin{bmatrix} 15/11 & 0 & 0 \\ -4/11 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ A_2 = \begin{bmatrix} 15/11 & 0 & 0 \\ -4/11 & 1 & 0 \\ \frac{0.04}{11\pi} - \frac{2}{11}\theta & -\frac{0.02}{\pi} + \theta & 1 \end{bmatrix}, \ \theta \in [-20, 20], \ \boldsymbol{B}_1 = \boldsymbol{B}_2 = \begin{bmatrix} -5/7 \\ 0 \\ 0 \end{bmatrix} \quad (18)$$

For $\theta = 3, 19$, the stability cannot be judged by the three methods listed in the simulation part of [3]. However, by Corollary 2 in this paper the stability can be judged, that is to solve (17) to obtain the feasible solution, which shows that the closed-loop system is stable:

For $\theta = 3$, $\boldsymbol{F}_1 = [4.8775, -9.5916, -2.0007]$ and $\boldsymbol{F}_2 = [4.8779, -9.5938, -2.0011]$;

For $\theta = 19$, $\boldsymbol{F}_1 = [4.5282, -8.3837, -0.2674]$ and $\boldsymbol{F}_2 = [4.5469, -8.4487, -0.2695]$.

This shows that the results in this paper make the stability judging conditions for T-S fuzzy control systems more relaxed.

5 Conclusions

New stability results for T-S fuzzy control systems are obtained applying parameter-dependent Lyapunov function, and the stability judging conditions for T-S fuzzy control systems are more relaxed. Further investigations include stability synthesis of optimization-based control (optimal control, predictive control[5], *etc.*) for T-S fuzzy model. With the stability conditions in (16), there is much freedom for choosing the control law. Other investigations include stability, domain of attraction for constrained system, for which the results in this paper can serve as a basis.

References

1 Tanaka K, Hori T, Wang H O. A fuzzy Lyapunov approach to fuzzy control system design. In: Proceedings of American Control Conference, Arlington: 2001. 4790∼4795

2 Lee K R, Jeung E T, Park H B. Robust fuzzy H_∞ control for uncertain nonlinear systems via state feedback: An LMI approach. Fuzzy Sets and Systems, 2001, **120**(1): 123∼134

3 Wang Y, Zhang Q L, Sun Z Q, Sun F C. Analysis and design of discrete fuzzy system with fuzzy Lyapunov approach. *Acta Automatica Sinica*, 2004, **30**(2): 255∼260

4 Yoneyama J, Masahiro N, Hitoshi K, Akria I. H_∞ control for Takagi-Sugeno fuzzy systems. *International Journal of Systems Science*, 2001, **32**(7): 915∼924

5 Li S Y, Li N. Fuzzy Predictive Control of Complex Systems and Applications. Beijing: Science and Technology Press, 2003

6 Feron E, Apkarian P, Gahinet P. Analysis and synthesis of robust control systems *via* parameter-dependent Lyapunov functions. *IEEE Transactions on Automatic Control*, 1996, **41**: 1041∼1046

7 Daafouz J, Bernussou J. Parameter dependent Lyapunov functions for discrete time systems with time varying parameter uncertainties. *Systems and Control Letters*, 2001, **43**: 355∼359

8 De Oliveira M C, Bernussou J, Geromel J C. A new discrete-time robust stability condition. *Systems and Control Letters*, 1999, **37**: 261∼265

DING Bao-Cang Received his master degree from China University of Petroleum in 2000 and Ph. D. degree from Shanghai Jiaotong University in 2003. Now, he is an associate professor of Hebei University of Technology. His research interests include predictive control, fuzzy control, H_∞ control and their applications in process industry and motor systems.

SUN He-Xu Professor in the Institute of Automation at Hebei University of Technology. His research interests include theory and application of predictive control, fuzzy control.

QIAO Yu-E Graduate student in the Institute of Automation at Hebei University of Technology.