Locally Distributed Feedback Stabilization of Nonuniform Euler-Bernoulli Beam[1)]

YAN Qing-Xu[1,2] HOU Shui-Hung[3] FENG De-Xing[4] QI Jian-Ling[2]

[1](*Key Laboratory of Lithosphere Tectonocs and Exploration, Ministry of Education, Beijing* 100083)

[2](*Information Engineering School, China Geosciences University, Beijing* 100083)

[3](*Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong, China*)

[4](*Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Science, Beijing* 100080)

(E-mail: mahoush@polyu.edu.hk)

Abstract In this article, we study the locally distributed feedback stabilization problem of a nonuniform Euler-Bernoulli beam. Firstly, using the semi-group theory, we establish the well-posedness of the associated closed loop system. Then by proving the uniqueness of the solution to a related ordinary differential equation, we derive the asymptotic stability of the closed loop system. Finally, by means of the piecewise multiplier method, we prove that, by either one distributed force feedback or a distributed moment feedback control, the closed loop system can be exponentially stabilized.

Key words Nonuniform Euler-Bernoulli beam, linear locally distributed feedback control, linear semigroup, exponential stability, piecewise multiplier method.

1 Introduction

Consider the locally distributed feedback stabilization problem of the elastic system governed by the nonuniform Euler-Bernoulli beam

$$\begin{cases} \rho\ddot{y}(x,t)+(EIy'')''(x,t)+u_1(x,t)+u_2(x,t)=0, \quad 0<x<\ell,\ t>0 \\ y(0,t)=y'(0,t)=y(\ell,t)=y'(\ell,t)=0, \quad t\geqslant 0 \\ y(x,0)=y_0(x), \quad \dot{y}(x,0)=y_1(x), \quad x\in(0,\ell) \end{cases} \tag{1}$$

where and henceforth, the prime and the dot always denote the derivatives with respect to space variable x and time variable t, respectively. $u_1(x,t)$ and $u_2(x,t)$ are the locally distributed feedback controls. All the other notations, symbols, variables, functions *etc.* are the same as those appeared in [1,2].

Recently, the feedback stabilization problem of elastic beam has attracted many researchers′ attention and a lot of interesting results have been obtained[1~8]. With the rapid advances of the new materials such as the piezoceramic patches and piezoelectric ones, some researchers′ interests have been converted from the the boundary feedback stabilization problem of the elastic system to the locally or globally distributed feedback control problem of the elastic system. In [1, 2], the authors investigated the asymptotic behavior of Euler-Bernoulli beam and, via Huang′s stability criterion[9], proved that the energy of the Euler-Bernoulli beam with Kelvin-Voigt damping is exponentially stable. Shi *et al* obtained the exponential decay of Timoshenko beam with locally distributed feedbacks by using the technique of frequency domain multiplier method[6~7]. In [8], it was pointed out that with only one linear distributed feedback control of low order, one cannot generally achieve the exponential stability of the Timoshenko beam. Thus, it is natural to ask whether one linear locally distributed force feedback or moment control is enough for the Euler-Bernoulli beam to decay exponentially. This question stimulates us to study the asymptotic behavior of the nonuniform Euler-Bernoulli beam (1) with locally distributed feedback controls

$$\begin{cases} u_1(x,t)=\alpha b(x)\dot{y}(x,t) \\ u_2(x,t)=-\beta\left(c(x)\dot{y}'(x,t)\right)' \end{cases} \tag{2}$$

1)Supported by National Natural Science Foundation of P. R. China (60334040, 60225003), China Geosciences University (Beijing)(200304) and the Research Committee of the Hong Kong Polytechnic University

Received May 19, 2005; in revised form June 22, 2005

where $\alpha, \beta \geqslant 0$, and $\alpha + \beta > 0$. It should be noted that the locally distributed feedback control used in this paper is of lower order of the state variables compared with the feedback control type appeared in [1,2]. Therefore, from the point of view of the engineering, the feedback form discussed in this paper can be easily applied to the practical system.

This paper is arranged as follows. In Section 2, by virtue of semigroup theory of linear operators[11], we prove the well-posedness of the corresponding closed loop system (1) and (2). In Section 3, by proving the uniqueness of the solution to a certain related ordinary differential equation, we derive the asymptotic stability of the closed loop system. Finally, in Section 4, by applying the piecewise exponential frequency–domain multiplier method used by Liu *et al*[1,5], we show that the closed loop system is exponentially stable when $\alpha + \beta > 0$.

2 The well-posedness of the closed loop system

In this section, we prove the well–posedness of the closed loop system (1) and (2). Obviously, we can rewrite (1) and (2) as

$$\begin{cases} \rho\ddot{y}(x,t) + (EIy'')''(x,t) + \alpha b(x)\dot{y} - \beta\left(c(x)\dot{y}'\right)' = 0, \quad 0 < x < \ell,\ t > 0 \\ y(0,t) = y'(0,t) = y(\ell,t) = y'(\ell,t) = 0, \quad t \geqslant 0 \\ y(x,0) = y_0(x), \quad \dot{y}(x,0) = y_1(x), \quad x \in (0,\ell) \end{cases} \tag{3}$$

In this paper, we shall always assume that the coefficients of the system and the feedback gain coefficients $b(x)$, and $c(x)$ satisfy the following assumption

Assumption S:

$$0 < \rho(x), \quad EI(x) \in C^2[0,\ell]]$$
$$b(x),\ c(x) \in L^\infty(0,\ell), \quad b(x), \quad c(x) \geqslant 0, \quad x \in [0,\ell]$$
$$b(x), \quad c(x) \geqslant \delta, \quad x \in [c,d]$$

where δ is positive number, and $0 \leqslant c < d \leqslant \ell$.

Now we incorporate the closed loop system (1)$\sim$(2) into a certain function space. To this end, we define a product Hilbert space $\mathcal{H}$ as

$$\mathcal{H} = V^2 \times L^2(0,\ell)$$

where $V^n = \{\varphi \in H^n(0,\ell) | \varphi(0) = \varphi'(0) = 0\}$, $\forall n > 1$. The inner product in $\mathcal{H}$ is defined as

$$(Y_1, Y_2)_{\mathcal{H}} = \int_0^\ell \left(EIy_1''\bar{y}_2'' + \rho z_1 \bar{z}_2\right) \mathrm{d}x$$

for $Y_k = [y_k, z_k]^\tau \in \mathcal{H},\ k = 1,2$.

We then define a linear operator $\mathcal{A}$ on $\mathcal{H}$ by

$$\mathcal{A}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} z \\ -\frac{1}{\rho}(EIy'')'' - \frac{1}{\rho}\left(\alpha b(x)z - \beta(c(x)z')'\right) \end{bmatrix}, \quad \begin{bmatrix} y \\ z \end{bmatrix} \in \mathcal{D}(\mathcal{A})$$

$$\mathcal{D}(\mathcal{A}) = \left\{ [y,z]^\tau \in \mathcal{H} \;\middle|\; y \in V^4, z \in V^2, y(\ell) = y'(\ell) = 0 \right\}$$

Thus the closed loop system (3) can be written as the following linear evolution equation in $\mathcal{H}$:

$$\frac{\mathrm{d}Y(t)}{\mathrm{d}t} = \mathcal{A}Y(t) \tag{4}$$

where $Y(t) = [y(\cdot,t), \dot{y}(\cdot,t)]^\tau$.

Lemma 1. Assume $\alpha, \beta \geqslant 0$. Then $\mathcal{A}$ generates a C_0 semigroup $T(t)$ of contractions in $\mathcal{H}$. Moreover, $\mathcal{A}$ has compact resolvent and $0 \in \rho(\mathcal{A})$.

Proof. For any $Y = [y,z]^\tau \in D(\mathcal{A})$, integrating by parts and referring to the boundary conditions of $Y \in D(\mathcal{A})$, we have

$$\mathrm{Re}(\mathcal{A}Y, Y)_{\mathcal{H}} = -\alpha \int_0^\ell b|z|^2 \mathrm{d}x - \beta \int_0^\ell c|z'|^2 \mathrm{d}x \leqslant 0$$

which implies the dissipativity of $\mathcal{A}$.

In order to prove the maximal dissipativity of $\mathcal{A}$ and $0 \in \rho(\mathcal{A})$, it suffices to show that $\forall \tilde{Y} = [\tilde{y}, \tilde{z}]^\tau \in \mathcal{H}$, there exists a unique $Y = [y, z]^\tau \in D(\mathcal{A})$ satisfying $\mathcal{A}Y = \tilde{Y}$, *i.e.*, $z = \tilde{y}$, and

$$\begin{cases} (EIy'')'' = \beta(c\tilde{y}')' - \alpha b\tilde{y} - \rho\tilde{z} \triangleq f, \quad x \in (0, \ell) \\ y(0) = y'(0) = y(\ell) = y'(\ell) = 0 \end{cases} \tag{5}$$

In fact, we can easily derive that (5) admits the general solution

$$\begin{cases} y'(x) = \int_0^x \frac{\mathrm{d}t}{EI(t)} \int_0^t \mathrm{d}s \int_0^s f(\tau)\mathrm{d}\tau + \int_0^x \frac{s}{EI(s)}\mathrm{d}s \left(EIy''(0)\right)' + EI(0)y''(0) \int_0^x \frac{1}{EI(s)}\mathrm{d}s \\ y(x) = \int_0^x \mathrm{d}t \int_0^t \frac{\mathrm{d}s}{EI(s)} \int_0^s \mathrm{d}\tau \int_0^\tau f(\sigma)\mathrm{d}\sigma + \int_0^x \mathrm{d}t \int_0^t \frac{s}{EI(s)}\mathrm{d}s \left(EIy''(0)\right)' + \\ \qquad \int_0^x \mathrm{d}t \int_0^t \frac{1}{EI(s)}\mathrm{d}s EI(0)y''(0) \end{cases} \tag{6}$$

Since $y(\ell) = y'(\ell) = 0$, we have

$$\begin{cases} \int_0^\ell \frac{s}{EI(s)}\mathrm{d}s \left(EIy''(0)\right)' + \int_0^\ell \frac{1}{EI(s)}\mathrm{d}s EI(0)y''(0) = -\int_0^\ell \frac{\mathrm{d}t}{EI(t)} \int_0^t \mathrm{d}s \int_0^s f(\tau)\mathrm{d}\tau \\ \int_0^\ell \mathrm{d}t \int_0^t \frac{s}{EI(s)}\mathrm{d}s \left(EIy''(0)\right)' + \int_0^\ell \mathrm{d}t \int_0^t \frac{1}{EI(s)}\mathrm{d}s EI(0)y''(0) = \\ \qquad -\int_0^\ell \mathrm{d}t \int_0^t \frac{\mathrm{d}s}{EI(s)} \int_0^s \mathrm{d}\tau \int_0^\tau f(\sigma)\mathrm{d}\sigma \end{cases} \tag{7}$$

Taking both $\left(EIy''(0)\right)'$ and $EI(0)y''(0)$ as unknowns in (7)and using Cauchy-Schwartz inequality, we can easily check that the coefficient determinant of (7) is not zero and therefore, $\left(EIy''(0)\right)'$, $EI(0)y''(0)$ and hence $y''(0)$, $y'''(0)$ are uniquely determined by $\tilde{Y}$. Thus, referring to $y(0) = y'(0) = 0$ and the general theory of ordinary differential equations, we deduce that $y(x)$ and hence Y is uniquely determined by $\tilde{Y}$. Thus we have shown the maximality of $\mathcal{A}$ and that $0 \in \rho(\mathcal{A})$.

The compactness of the resolvent of $\mathcal{A}$ is easily derived by using the Sobolev embedding theorem. Hence, the proof is completed by virtue of Lummer-Phillips theorem[11]. □

According to the C_0 semigroup theory[11], we get

Theorem 1. Assume $\alpha, \beta \geqslant 0$. Then for any $Y_0 \in \mathcal{H}$, (4) admits a unique weak solution $Y(t) = T(t)Y_0$, where $T(t)$ is the C_0 semigroup of contractions generated by $\mathcal{A}$. Moreover, if $Y_0 \in \mathcal{D}(\mathcal{A})$, then $Y(t) = T(t)Y_0$ is a strong solution to (4).

3 Asymptotic decay of the closed loop system

We now discuss the asymptotic stability of the closed loop system (4) under the assumption **S**. The energy corresponding to the solution of the closed loop system (4) is

$$E(t) = \frac{1}{2}\left[\int_0^\ell EI|y''|^2\mathrm{d}x + \int_0^\ell \rho|z|^2\mathrm{d}x\right] \tag{8}$$

where $Y(t) = [y(\cdot, t), \dot{y}(\cdot, t)]^\tau$ is the solution to (4). It is easy to check that

$$\dot{E}(t) = -\alpha \int_0^\ell b(x)|z|^2\mathrm{d}x - \beta \int_0^\ell c(x)|z'|^2\mathrm{d}x \tag{9}$$

Let $i\,IR$ denote the imaginary axis.

Lemma 2. Suppose that assumption **S** holds true and $\alpha + \beta > 0$. Then $i\,IR \subset \rho(\mathcal{A})$, the resolvent set of $\mathcal{A}$.

Proof. Since $\mathcal{A}$ has compact resolvent, it suffices to prove that $i\,IR \cap \sigma_p(\mathcal{A}) = \emptyset$, where $\sigma_p(\mathcal{A})$ is the set of the point spectrum of $\mathcal{A}$. Assuming the contrary, then there should exist an eigenvalue

$i\lambda \in i\,I\!R$ of $\mathcal{A}$. Obviously $\lambda \neq 0$. Let $\boldsymbol{\Psi} = [y, z]^{\tau} \in D(\mathcal{A})$ be an eigenfunction of $\mathcal{A}$ corresponding to $i\lambda$. From

$$0 = \mathrm{Re}((\mathcal{A} - i\lambda)\,\boldsymbol{\Psi},\, \boldsymbol{\Psi})_{\mathcal{H}} = \mathrm{Re}(\mathcal{A}\,\boldsymbol{\Psi},\, \boldsymbol{\Psi})_{\mathcal{H}} = -\alpha \int_0^{\ell} b(x)|z|^2 \mathrm{d}x - \beta \int_0^{\ell} c(x)|z'|^2 \mathrm{d}x$$

the assumption $\alpha + \beta > 0$, the property of $b(x)$, $c(x)$ and $(\mathcal{A} - i\lambda)\,\boldsymbol{\Psi} = 0$, it follows that y satisfies either

$$\begin{cases} (EIy'')'' - \rho\lambda^2 y = 0 \\ y(0) = y'(0) = y(\ell) = y'(\ell) = 0 \\ y(x) \equiv 0, \quad \forall x \in [c, d] \end{cases} \tag{10}$$

or

$$\begin{cases} (EIy'')'' - \rho\lambda^2 y = 0 \\ y(0) = y'(0) = y(\ell) = y'(\ell) = 0 \\ y'(x) \equiv 0, \quad \forall x \in [c, d] \end{cases} \tag{11}$$

corresponding to $\alpha > 0$, $\beta = 0$ and $\alpha = 0$, $\beta > 0$ respectively. If y satisfies (10), we can easily deduce that $y(c) = y'(c) = y''(c) = y'''(c) = y(d) = y'(d) = y''(d) = y'''(d) = 0$. Therefore, (10) admits a unique solution $y \equiv 0$ on $[0, \ell]$ and thus, $\boldsymbol{\Psi} = [y, z]^{\tau} = [y, i\lambda y]^{\tau} \equiv [0, 0]^{\tau}$ on $[0, \ell]$, a contradiction to that $\boldsymbol{\Psi}$ is an eigenfunction.

In the case that y is the solution to system (11), by means of Feng *et al.* [12], we can still derive the contradiction $\boldsymbol{\Psi} \equiv \mathbf{0}$. □

Based on the results of Huang[10], we get the main result in this section:

Theorem 2. Suppose that assumption **S** holds true and $\alpha + \beta > 0$. Then for any initial state $Y_0 \in \mathcal{H}$, the energy $E(t)$ given by (8) decays asymptotically, *i.e.*,

$$\lim_{t \to +\infty} E(t) = 0$$

4 Exponential decay of the closed loop system

In this section, we demonstrate the exponential stability of the closed loop system (4) under the condition that $\alpha + \beta > 0$. The main result in this section is the following:

Theorem 3. Suppose that assumption **S** is verified and one of the following conditions holds true:

1) $\alpha > 0$, $\beta \geqslant 0$

2) $\alpha = 0$, $\beta > 0$ and $c = 0$ (or $d = \ell$)

Then the energy $E(t)$ of the closed loop system (4) decays exponentially, *i.e.*, for every $Y_0 \in \mathcal{H}$, there exist positive constants C and ω, independent of Y_0, such that

$$E(t) \leqslant C e^{-\omega t} \|Y_0\|^2$$

Proof. We prove here only the case $\alpha > 0$, and $\beta = 0$, as the proofs for the other cases are similar. According to Huang[9], it follows from Theorems 1 and Lemma 2 that we need only to prove that

$$\sup_{\lambda \in i\,I\!R} \|(\lambda - \mathcal{A})^{-1}\| < +\infty \tag{12}$$

Let us proceed by contradiction. Suppose that (12) does not hold true. Then there are $\lambda_n \in i\,I\!R$ and $Z_n = [y_n, z_n]^{\tau} \in \mathcal{D}(\mathcal{A})$, $n = 1, 2, \cdots$, such that

$$\begin{cases} \|\boldsymbol{Z}_n\|_{\mathcal{H}} = 1 \\ |\lambda_n| \to +\infty, \quad (\text{as} \;\; n \to \infty) \\ \|(\lambda_n - \mathcal{A})\boldsymbol{Z}_n\|_{\mathcal{H}} \triangleq \|\tilde{\boldsymbol{Z}}_n\|_{\mathcal{H}} = o(1) \end{cases} \tag{13}$$

where $\tilde{\boldsymbol{Z}}_n = (\lambda_n - \mathcal{A})\boldsymbol{Z}_n = [\tilde{y}_n, \tilde{z}_n]^{\tau}$, the notation $a_n = o(1)$ means that the sequence $a_n \to 0$ as $n \to \infty$.

From (13) we have

$$\text{Re}(\tilde{\boldsymbol{Z}}_n, \boldsymbol{Z}_n)_{\mathcal{H}} = \alpha \int_0^\ell b(x)|z_n|^2 \, \text{d}x = o(1) \tag{14}$$

We also have

$$-iI_m(\tilde{\boldsymbol{Z}}_n, \boldsymbol{Z}_n)_{\mathcal{H}} = \lambda_n \left[\int_0^\ell EI|y_n''|^2 \text{d}x - \int_0^\ell \rho |z_n|^2 \text{d}x \right] = o(1) \tag{15}$$

It follows from (13) $\sim$ (15), the definition of $\mathcal{D}(\mathcal{A})$ and the condition of Theorem 3 that

$$\begin{aligned} & y_n(0) = y_n'(0) = z_n(0) = z'(0) = y_n(\ell) = y_n'(\ell) = z_n(\ell) = z_n'(\ell) = 0 \\ & \int_0^\ell b|z_n|^2 \text{d}x = o(1), \quad \int_c^d |z_n|^2 \text{d}x = o(1), \quad \int_c^d |\lambda_n y_n|^2 \text{d}x = o(1) \\ & \int_0^\ell \rho |z_n|^2 \text{d}x - \frac{1}{2} = o(1), \int_0^\ell EI|y_n''|^2 \text{d}x - \frac{1}{2} = o(1) \\ & \int_0^\ell \rho |\lambda_n y_n|^2 \text{d}x - \frac{1}{2} = o(1) \end{aligned} \tag{16}$$

Since $\lambda_n \boldsymbol{Z}_n - \mathcal{A}\boldsymbol{Z}_n = \tilde{\boldsymbol{Z}}_n$, we have

$$\rho \lambda_n^2 y_n + (EIy_n'')'' + \alpha b z_n = \rho(\tilde{z}_n + \lambda_n \tilde{y}_n) \tag{17}$$

Now we proceed the proof by the technique of the piecewise multiplier method used in Liu *et al.*[1,5]. Define a real function $q_0(\cdot) \in C^3[0, \ell]$ by

$$\begin{cases} 0 \leqslant q_0(x) \leqslant 1, & \forall x \in [0, \ell] \\ q_0(x) = 0, & \forall x \in [0, c] \cup [d, \ell] \\ q_0(x) = 1, & \forall x \in [c_0, d_0] \end{cases}$$

where $\ell > d > d_0 > c_0 > c > 0$. Using (16) and integrating by parts, we obtain

$$o(1) = \int_c^d y_n \left(q_0 \bar{y}_n' \right)' \text{d}x = -\int_c^d q_0 |y_n'|^2 \text{d}x$$

Thus, it follows from the definition of $q_0(x)$ that

$$\int_{c_0}^{d_0} |y_n'|^2 \text{d}x = o(1) \tag{18}$$

Define another real function $q(\cdot) \in C^3[0, \ell]$ by

$$\begin{cases} 0 \leqslant q(x) \leqslant 1, & \forall x \in [0, \ell], \\ q(x) = 0, & \forall x \in [0, c_0] \cup [d_0, \ell], \\ q(x) = 1, & \forall x \in [c_1, d_1] \end{cases}$$

where $\ell > d_0 > d_1 > c_1 > c_0 > 0$.

Multiplying both sides of (17) by $q\bar{y}_n$, integrating from 0 to ℓ, referring to (16), (18), and integrating by parts if necessary, we get

$$\begin{aligned} & \int_0^\ell \lambda_n^2 \rho q |y_n|^2 \text{d}x + \int_0^\ell (EIy_n'')'' q \bar{y}_n \text{d}x = \int_{c_0}^{d_0} \lambda_n^2 \rho q |y_n|^2 + \int_{c_0}^{d_0} EIq|y_n''|^2 \text{d}x + o(1) = \\ & \quad \int_0^\ell \left(\rho \tilde{z}_n + \rho \lambda_n \tilde{y}_n - \alpha b z_n \right) q \bar{y}_n \text{d}x = o(1) \end{aligned}$$

By virtue of (16) and referring to the definition of q, we have

$$\int_{c_1}^{d_1} |y_n''|^2 \text{d}x = o(1) \tag{19}$$

Set $q_1(x) = \gamma_1(x)(e^{M_1 x} - 1)$ with $\gamma_1(x) \in C^4[0, \ell]$ satisfying

$$0 \leqslant \gamma_1(x) \leqslant 1, \ \forall x \in [0, \ell]$$

$$\gamma_1(x) = 1, \ \forall x \in [0, c_1] \cup [d_1, \ell]$$

$$\gamma_1(x) = 0, \quad \forall x \in [c_2, d_2]$$

where $c_1 < c_2 < d_2 < d_1$.

Multiplying both sides of (17) by $q_1 \bar{y}_n'$, integrating from 0 to c_2, referring to (16) and (18), and integrating by parts if necessary, we obtain

$$\int_0^{c_2} \lambda_n^2 \rho y_n q_1 \bar{y}_n' \mathrm{d}x + \int_0^{c_2} (EIy_n'')'' q_1 \bar{y}_n' \mathrm{d}x = \frac{1}{2} \int_0^{c_2} (\rho q_1)' |\lambda_n y_n|^2 \mathrm{d}x +$$
$$\frac{1}{2} \int_0^{c_2} (3q_1' EI - q_1 EI') |y_n''|^2 \mathrm{d}x + o(1) = \int_0^{c_2} (\rho \tilde{z}_n + \rho \lambda_n \tilde{y}_n - \alpha b z_n) \, q_1 \bar{y}_n' \mathrm{d}x = o(1) \tag{20}$$

Here we used the estimate $\int_0^{c_2} |y_n'(x)|^2 \mathrm{d}x = o(1)$ in proving estimate (20). In fact, we can prove estimate $\int_0^{c_2} |y_n'(x)|^2 \mathrm{d}x = o(1)$ as follows:

Define $\bar{q}(\cdot) \in C^3[0, \ell]$ by

$$\begin{cases} 0 \leqslant \bar{q}(x) \leqslant 1, & \forall x \in [0, \ell] \\ \bar{q}(x) = 0, & \forall x \in [c_3, \ell] \\ \bar{q}(x) = 1, & \forall x \in [0, c_2] \end{cases}$$

where $\ell > c_3 > c_2 > 0$. Using (16) and integrating by parts, we obtain

$$o(1) = \int_0^{c_3} y_n \left(\bar{q} \bar{y}_n' \right)' \mathrm{d}x = - \int_0^{c_3} \bar{q} |y_n'|^2 \mathrm{d}x$$

Thus, it follows from the definition of $\bar{q}(x)$ that $\int_0^{c_2} |y_n'(x)|^2 \mathrm{d}x = o(1)$.

It follows from (16), (18) and (19) that

$$\int_0^{c_1} (\rho q_1)' |\lambda_n y_n|^2 \mathrm{d}x + \int_0^{c_1} (3q_1' EI - q_1 EI') |y_n''|^2 \mathrm{d}x = o(1) \tag{21}$$

Set $q_2(x) = \gamma_1(x)(e^{M_1(\ell - x)} - 1)$. Multiplying both sides of (17) by $q_2 \bar{y}_n'$, integrating from d_2 to ℓ and repeating what we have just done as above, we get

$$\int_{d_1}^{\ell} (\rho q_2)' |\lambda_n y_n|^2 \mathrm{d}x + \int_{d_1}^{\ell} (q_2' EI - q_2 EI') |y_n''|^2 \mathrm{d}x = o(1) \tag{22}$$

It follows from (21), (22) and the definition of q_1 and q_2 that

$$\int_0^{c_1} \left(M_1 e^{M_1 x} \rho + (e^{M_1 x} - 1) \rho' \right) |\lambda_n y_n|^2 \mathrm{d}x + \int_0^{c_1} \left(M_1 e^{M_1 x} EI - (e^{M_1 x} - 1) EI' \right) |y_n''|^2 \mathrm{d}x +$$
$$\int_{d_1}^{\ell} \left(M_1 e^{M_1(\ell - x)} \rho - (e^{M_1(\ell - x)} - 1)) \rho' \right) |\lambda_n y_n|^2 \mathrm{d}x +$$
$$\int_{d_1}^{\ell} \left(M_1 e^{M_1(\ell - x)} EI + (e^{M_1(\ell - x)} - 1) EI' \right) |y_n''|^2 \mathrm{d}x = o(1) \tag{23}$$

Hence when M_1 is large enough, we get

$$\int_0^{c_1} |\lambda_n y_n|^2 \mathrm{d}x, \quad \int_0^{c_1} |y_n''|^2 \mathrm{d}x, \quad \int_{d_1}^{\ell} |\lambda_n y_n|^2 \mathrm{d}x, \quad \int_{d_1}^{\ell} |y_n''|^2 \mathrm{d}x = o(1) \tag{24}$$

It follows from (16) and (19) that

$$\int_0^{\ell} |\lambda_n y_n|^2 \mathrm{d}x \leqslant \int_0^{c_1} |\lambda_n y_n|^2 \mathrm{d}x + \int_c^{d} |\lambda_n y_n|^2 \mathrm{d}x + \int_{d_1}^{\ell} |\lambda_n y_n|^2 \mathrm{d}x = o(1) \tag{25}$$

which is an obvious contraction to $\int_0^{\ell} \rho |\lambda_n y_n|^2 \mathrm{d}x - \frac{1}{2} = o(1)$ listed in (16) and thus we complete the proof of the theorem. □

References

1 Liu Kang-Sheng, Liu Zhuang-Yi. Exponential decay of energy of the Euler-Bernoulli beam with locally distributed Kelvin-Voigt damping. *SIAM Journal of Control and Optimazation*, 1998, **36**(3): 1086∼1098
2 Chen Shu-Ping, Liu Kang-Sheng, Liu Zhuang-Yi. Spectrum and stability for elastic system with global or local Kelvin-voigt damping. *SIAM Journal Applied Mathematics*, 1999, **59**(2): 651∼668
3 Feng D X, Zhang W T. Feedback stabilization of Euler-Bernoulli beam. *Acta Automatica Sinica*, 1996, **22**(2): 135∼144
4 Guo G B, Luo Z H. Initial-boundary value problem and exponential decay for a flexible-beam vibration with gain adaptive direct strain feedback control. *Nonlinear Analysis, Theory, Methods & Applications*, 1996, **27**: 353∼365
5 Liu K, Liu Z. Boundary stabilization of a nonhomogeneous beam with rotatory inertia at the tip, *Journal of Computational and Applied mathematics*, 2000, **114**(1): 1∼10
6 Shi D H, Feng D. X. Exponential decay of Timoshenko beam with locally distributed feedback, In: Proceedings of the 1999 IFAC World Congress, Beijing, Vol. F, 158∼162
7 Shi D H, Feng D X. Optimal energy decay rate of Timoshenko beam with distributed damping. *Australian and Newzealand Industrial and Applied Mathematics* (to appear)
8 Abdelaziz Soufyane, Stabilisation de la Poutre de Timoshenko, *Countro Rendue Acad. Sci. Paris*, t. 328, Serie 1, 1999, 731∼734
9 Huang H F. Characteristic conditions for exponential stability of linear dynamical system in Hilbert space. *Chinese Annals of Differential Equations*, 1985, **1**(1): 43∼53
10 Huang F L. On the asymptotic stability of linear dynamical system in Banach space. *Chinese Science Bulletin*, 1983, **10**: 584∼586
11 Pazy A. Semigroups of Linear Operators and Applications to Partial Differential Equations. New York: Spring-Verlay, 1983
12 Feng De-Xing, Zhu Guang-Tian. Special properties of principal operatiors for a class of elastic viberation problems, *Chinese Science Bulletin, Special Issue*, 1983: 79∼82

YAN Qing-Xu Received his Ph. D. degree from Academy of Mathematics & System Sciences, Chinese Academy of Sciences. Now he is a professor working at China Geosciences University in Beijing. His research interests include distributed parameter control theory.

HOU Shui-Hung Vice Dean of the Department of Applied Mathematics at Hong Kong Polytechnical University. His research interests include the multi-valued function analysis and control theory.

FENG De-Xing Professor of Institute of Systems Science, Academy of Mathematics and System Science, Chinese Academy of Sciences. His research interests include control theory of distributed parameter systems.

QI Jian-Ling Received her Ph. D. degree from School of Earth Physical Exploration and Information Technology. Now she is an associate professor of the Information Engineering School, China Geosciences University. Her research interests include intelligent control and its applications.