

Sliding Mode Control Design for a Class of Systems with Mismatched Uncertainties¹⁾

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Abstract A new sliding mode controller design method is proposed for a class of system with mismatched uncertainties such that the dynamic function restricted on the sliding surface is completely insensitive to the uncertainties. A sufficient and necessary condition which the system possessing this sliding mode controller should satisfy is explicitly presented. The issue of chattering free is also explored. It is concluded that this class of mismatched term does not bring any chattering problem. Finally, a numerical example illustrates the developed method.

Key words Sliding mode control (SMC), uncertain system, mismatched uncertainty

1 Introduction

In recent years, sliding mode control (SMC) as a powerful control strategy has been widely studied and applied to the linear system with matched uncertainties^[1]. But for the system with mismatched uncertainties only a few contributions have been made^[2~6].

In this paper the method presented in [2] is explored further and then extended to a class of more general systems. A new sufficient and necessary condition is proposed for the existence of such a sliding surface on which the sliding motion is immune to the mismatched uncertainties. The new condition explicitly indicates the property the mismatched uncertainties should possess. At last, a numerical example demonstrates the efficacy of the presented results.

The notations throughout the paper are standard. A^* and A^T denote the conjugate transpose and transpose matrix of matrix A , respectively. $\|A\|$ denotes the spectral norm of matrix A . $M > 0$ means that M is a symmetric positive-definite matrix. M^\perp represents the orthogonal complement matrix of full column rank of matrix M .

2 Problem statement

Consider the following uncertain system

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t) + \boldsymbol{f}(t) + \Phi(t)\boldsymbol{z}(t), \quad \boldsymbol{z}(t) = C\boldsymbol{x}(t) \quad (1)$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is state vector, $\boldsymbol{u}(t) \in \mathbb{R}^m$ is input vector, $\boldsymbol{f}(t) \in \mathbb{R}^n$ denotes the lumped matched uncertain terms, $\boldsymbol{z}(t) \in \mathbb{R}^q$ is a middle-vector which represents the relation between the mismatched uncertainties and system state vector. A , B , and C are known constant matrices with appropriate dimensions. The following assumptions underlie this paper.

A1. Matrices B and C are of full rank with $m \geq q$.

A2. Matrix pair (A, B) is controllable, (A, C) is observable.

A3. There exists an unknown but bounded function $F(t) \in \mathbb{R}^m$ such that $\boldsymbol{f}(t) = BF(t)$, where $\|F(t)\| \leq k_1$ with a known scalar $k_1 \geq 0$.

A4. $\Phi(t)\boldsymbol{z}(t)$ represents the mismatched uncertainties, where $\Phi(t) \in \mathbb{R}^{n \times q}$ is an unknown but bounded function satisfying $\|\Phi(t)\| \leq k_2$ with a known scalar $k_2 \geq 0$.

Define the sliding surface as^[7]

$$\boldsymbol{s} = B^T X^{-1} \boldsymbol{x} = 0 \quad (2)$$

¹⁾ Supported by the National Outstanding Youth Science Foundation of China (60025308) and the New Century 151 Talent Project of Zhejiang Province
Received May 21, 2004; in revised form January 16, 2005

where $\mathbf{s} \in \mathfrak{R}^m$ is the sliding surface function, $X \in \mathfrak{R}^{n \times n} > 0$ is a design parameter to be determined later. The generality of (2) will be proved in the next section. Introduce a coordinates transformation $\begin{bmatrix} \mathbf{v} \\ \mathbf{s} \end{bmatrix} = T\mathbf{x}$, with $T = \begin{bmatrix} B^{\perp T} \\ B^T X^{-1} \end{bmatrix}$. Then in new coordinates, the dynamic function of system (1) is

$$\begin{bmatrix} \dot{\mathbf{v}} \\ \dot{\mathbf{s}} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{s} \end{bmatrix} + \begin{bmatrix} 0 \\ B^T X^{-1} B \end{bmatrix} (\mathbf{u}(t) + F(t)) + \begin{bmatrix} B^{\perp T} \\ B^T X^{-1} \end{bmatrix} \Phi(t)\mathbf{z}(t) \quad (3)$$

where $A_{11} = B^{\perp T} A X B^{\perp} (B^{\perp T} X B^{\perp})^{-1}$, $A_{12} = B^{\perp T} A B (B^T X^{-1} B)^{-1}$, $A_{21} = B^T X^{-1} A X B^{\perp} (B^{\perp T} X B^{\perp})^{-1}$, $A_{22} = B^T X^{-1} A B (B^T X^{-1} B)^{-1}$. Note that

$$\mathbf{z}(t) = C[XB^{\perp} (B^{\perp T} X B^{\perp})^{-1} \quad B(B^T X^{-1} B)^{-1}] \begin{bmatrix} \mathbf{v} \\ \mathbf{s} \end{bmatrix}$$

Then by $\mathbf{s} = 0$, one can get the dynamic function of the sliding motion as

$$\dot{\mathbf{v}} = B^{\perp T} A X B^{\perp} (B^{\perp T} X B^{\perp})^{-1} \mathbf{v} + B^{\perp T} \Phi(t) C X B^{\perp} (B^{\perp T} X^{-1} B^{\perp})^{-1} \mathbf{v} \quad (4)$$

Since $\Phi(t)$ is unknown time-varying function and dissatisfies the matched condition, *i.e.*, $B^{\perp T} \Phi(t) \neq 0$, the dynamics of sliding motion is usually influenced by it. In the previous reports^[4,5,7], the robust control theory was utilized to design the corresponding sliding surface such that system (4) is asymptotically stable. It is well-known that the robust design method will unavoidably induce some conservatism in the result. However, if the sliding surface (2) satisfies

$$C X B^{\perp} = 0 \quad (5)$$

then the dynamic function of sliding motion can be simplified as

$$\dot{\mathbf{v}} = B^{\perp T} A X B^{\perp} (B^{\perp T} X B^{\perp})^{-1} \mathbf{v} \quad (6)$$

It is obvious that in case of (5), the dynamics of the closed-loop is immune to the mismatched uncertainty $\Phi(t)\mathbf{z}(t)$. Now the objectives can be formulated as:

- 1) design a sliding surface (2) satisfying equation (5) such that the system (6) is asymptotically stable;
- 2) design a control law such that the sliding surface (2) can be reached in a finite time.

3 Main results

3.1 Sliding surface design

First we prove the generality of the sliding surface definition (2).

Lemma 1. Given any sliding surface $S\mathbf{x} = 0$ where $S \in \mathfrak{R}^{m \times n}$ is a sliding matrix, there exists a matrix $X \in \mathfrak{R}^{n \times n} > 0$ such that sliding motion on the sliding surface $B^T X^{-1} \mathbf{x} = 0$ has the same dynamics with that on the sliding surface $S\mathbf{x} = 0$.

Proof. Note that in sliding mode control matrix $S B$ is non-singular^[1], or else the equivalent control will be not unique. From this, it is known that matrix multiplication $\begin{bmatrix} B^{\perp T} \\ B^T \end{bmatrix} [B^{\perp} \quad S^T] = \begin{bmatrix} I & B^{\perp T} S^T \\ 0 & B^T S^T \end{bmatrix}$ is non-singular. Further it follows that matrix $[B^{\perp} \quad S^T]$ is non-singular. So the matrix $X > 0$ can be chosen as $X = (B^{\perp} X_1 B^{\perp T} + S^T X_2 S)^{-1}$ with $X_1 \in \mathfrak{R}^{(n-m) \times (n-m)} > 0$ and $X_2 \in \mathfrak{R}^{m \times m} > 0$. Select $N = B^T S^T X_2$ such that $B^T X^{-1} = N S$. This completes the proof. \square

Theorem 2. For system (1) with the assumptions (A1~A4), the following propositions are equivalent:

- (P1) there exists a sliding surface such that the sliding motion is asymptotically stable and is completely immune to the mismatched uncertainty.
- (P2) the triple $\{A, B, C\}$ satisfy the following conditions

$$\text{rank}(CB) = \text{rank}(C) \quad (7)$$

$$\text{rank} \begin{bmatrix} \lambda I - A & B \\ C & 0 \end{bmatrix} = n + q, \quad \forall \text{Re}(\lambda) > 0 \tag{8}$$

(P3) there are symmetric matrices $W_1 \in \mathfrak{R}^{(n-q) \times (n-q)}$, $W_2 \in \mathfrak{R}^{m \times m}$ and scalar $\varepsilon > 0$ such that the following LMIs hold

$$AC^\perp W_1 C^{\perp T} + C^\perp W_1 C^{\perp T} A^T - \varepsilon BB^T > 0 \tag{9}$$

$$C^\perp W_1 C^{\perp T} + BW_2 B^T > 0 \tag{10}$$

Proof. By Lemma 1, (P1) implies that there exists $X > 0$ such that the second term of (4) is zero and system (6) is asymptotically stable. Since $\Phi(t)$ is any time-varying function, it can be derived that equation (5) holds. The equivalence between (P1) and (P3) can be proved along the similar line in [2] and hence omitted here. In the sequel, we only prove (P1) \iff (P2).

(P1) \implies (P2): It follows from (5) that there is a matrix $M \in \mathfrak{R}^{q \times m}$ satisfying $CX = MB^T$. So matrix $MB^T C^T = CXC^T$ is of full rank. This implies the equation (7) holds.

On the other hand, since system (6) is asymptotically stable, there exists a $Q_0 > 0$ such that

$$B^{\perp T} A X B^\perp (B^{\perp T} X B^\perp)^{-1} Q_0 + Q_0 (B^{\perp T} X B^\perp)^{-1} B^{\perp T} X A^T B^\perp < 0 \tag{11}$$

Define a matrix $Q > 0$ as

$$Q = [X B^\perp (B^{\perp T} X B^\perp)^{-1} \quad B] \begin{bmatrix} Q_0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} (B^{\perp T} X B^\perp)^{-1} B^{\perp T} X \\ B^T \end{bmatrix} \tag{12}$$

It can be verified that the inequality (11) is equivalent to $B^{\perp T} (AQ + QA^T) B^\perp < 0$. From Finsler Lemma^[8], it follows that

$$AQ + QA^T - \varepsilon B^T B < 0 \tag{13}$$

where $\varepsilon > 0$. Assume that equation (8) is not true, *i.e.*, there are a complex λ and a non-zero vector $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ such that $\text{Re}(\lambda) > 0$ and

$$\begin{bmatrix} \lambda I - A^T & C^T \\ B^T & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \tag{14}$$

which means that $A^T v_1 = \lambda v_1 + C^T v_2$ and $B^T v_1 = 0$. Thus, multiplying both sides of (13) by v_1^* and v_1 respectively yields $v^* A Q v_1 + v_1^* Q A^T v_1 - \varepsilon v_1^* B B^T v_1 < 0$. Note that $Q C^T = B B^T C^T$; then

$$(\lambda + \lambda^*) v_1^* Q v_1 < 0 \tag{15}$$

Since $v^* Q v_1 > 0$, it follows from (15) that $\text{Re}(\lambda) > 0$, which yields a contradiction. So equation (13) must hold.

(P2) \implies (P1): Without loss of generality, we assume that the input matrix has the form of $B = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$. Partition matrix C as $C = [C_1 \quad C_2]$ with $C_2 \in \mathfrak{R}^{q \times m}$. From (7), C_2 is of full row rank. Define matrices $L_1 = C_1^T (C_2 C_2^T)^{-1} C_2$ and $L_2 = \begin{bmatrix} C_2^{\perp T} \\ C_2 \end{bmatrix}$. Also partition matrix A as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ with $A_{11} \in \mathfrak{R}^{(n-m) \times (n-m)}$. Then a simple manipulation yields

$$\begin{aligned} \begin{bmatrix} \lambda I_n - A^T & C^T \\ B^T & 0 \end{bmatrix} &\sim \left[\begin{array}{c|c|c} \lambda I_{n-m} - A_{11}^T & -A_{21}^T & C_1^T \\ \hline -A_{12}^T & \lambda I_m - A_{22}^T & C_2^T \\ \hline 0 & I_m & 0 \end{array} \right] \\ &\sim \left[\begin{array}{c|c} \lambda I_{n-m} - A_{11}^T & C_1^T \\ \hline -A_{12}^T & C_2^T \end{array} \right] \sim \left[\begin{array}{c|c} \lambda I_{n-m} - A_{11}^T + L_1 A_{12}^T & 0 \\ \hline -A_{12}^T & C_2^T \end{array} \right] \\ &\sim \left[\begin{array}{c|c} \lambda I_{n-m} - A_{11}^T + L_1 A_{12}^T & 0 \\ \hline C_2^{\perp T} A_{12}^T & 0 \\ \hline C_2 A_{12}^T & C_2 C_2^T \end{array} \right] \sim \begin{bmatrix} \lambda I_{n-m} - A_{11}^T + L_1 A_{12}^T \\ C_2^{\perp T} A_{12}^T \end{bmatrix} \end{aligned} \tag{16}$$

By the condition (8), (16) means that the matrix pair $(A_{11}^T - L_1 A_{12}^T, C_2^{\perp T} A_{12}^T)$ is detectable. So there exist $X_1 \in \mathbb{R}^{(n-m) \times (n-m)} > 0$ and $K \in \mathbb{R}^{(n-m) \times (m-q)}$ such that

$$(A_{11} - A_{12} L_1^T - A_{12} C_2^{\perp} K^T) X_1 + X_1 (A_{11}^T - L_1 A_{12}^T - K C_2^{\perp T} A_{12}^T) < 0 \tag{17}$$

The symmetric positive definite matrix X of sliding surface definition (2) is chosen as

$$X = \begin{bmatrix} X_1 & X_1(-L_1 - K C_2^{\perp T}) \\ (-L_1^T - C_2^{\perp} K^T) X_1 & X_2 + (-L_1^T - C_2^{\perp} K^T) X_1 (-L_1 - K C_2^{\perp T}) \end{bmatrix} \tag{18}$$

where $X_2 \in \mathbb{R}^{m \times m} > 0$. Without loss of generality, set $B^{\perp T} = [I_{n-m} \ 0]$. Then it can be easily testified that equation (5) holds. By substituting (18) into system (6), one obtains

$$\dot{v} = (A_{11} - A_{12} L_1^T - A_{12} C_2^{\perp} K^T) v \tag{19}$$

Form (17), it follows that system (19) is asymptotically stable. This completes the proof. \square

3.2 Synthesis of control law

Theorem 3. The sliding surface (2) of system (1) comes from LMIs (9) and (10). Then under the following control law

$$u = (B^T X^{-1} B)^{-1} \left(B^T X^{-1} A x + k_1 \|B^T X^{-1}\| \|CBW_2\| s + (k_2 \|B^T X^{-1} B\| + \eta) \frac{s}{\|s\|} \right) \tag{20}$$

the trajectories of the system can in a finite time enter into and subsequently remain on the sliding surface, where η is a positive scalar to adjust the convergent rate.

Proof. The derivative of sliding surface respect to time is

$$\dot{s} = B^T X^{-1} A x + B^T X^{-1} B u + B^T X^{-1} B F(t) + B^T X^{-1} \Phi(t) C x \tag{21}$$

Note that

$$\Phi(t) C x = \Phi(t) C X X^{-1} x = \Phi(t) C B W_2 s \tag{22}$$

Substituting the control law (20) into (21) yields $s^T \dot{s} \leq -\eta \|s\|$, which implies that the system will in a finite time enter into and then remain on the sliding surface. This completes the proof. \square

Remark. It can be found from (22) that the mismatched uncertainty $\Phi(t)z(t)$ does not bring the chattering for the system. If system (1) only owns the uncertain term $\Phi(t)z(t)$ and if the condition that the sliding surface is reached in a finite time is approximately replaced by one that the reaching rate of sliding surface is larger than the convergent rate of sliding motion, then the control law (20) can be replaced by

$$u = -(B^T X^{-1} B)^{-1} (B^T X^{-1} A x + (k_1 \|B^T X^{-1}\| \|CBW_2\| + \mu) s) \tag{23}$$

In that case, one has $s^T \dot{s} \leq -\mu \|s\|^2$, which implies the decay rate of the sliding surface is larger than $e^{-\mu t}$. It is obvious that control law (23) is a continuous function and does not result in any chattering problem.

4 Numerical example

Consider the following uncertain system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \sin x_2 & 1 & 0 \\ \sin x_3 & 0 & 1 \\ 2 + \sin x_1 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{24}$$

The initial state is assumed to be $x_0 = [10, -10, 10]^T$. By Theorem 2, one can get

$$W_1 = \begin{bmatrix} 38.8056 & -20.0158 \\ -20.0158 & 5.9298 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 57.2142 & 7.3831 \\ 7.3831 & 121.6529 \end{bmatrix}$$

and the sliding surface is

$$s = 0.01 \times \begin{bmatrix} 2.2397 & -0.1404 & -0.4737 \\ -3.8116 & 1.0876 & 3.6712 \end{bmatrix} x \tag{25}$$

Chose $k_1 = 2$ and $k_2 = 0$. Note that system (24) only owns mismatched uncertain term, two control laws (20) and (23) are used here for comparing. Set $\eta = 0.1$, and the control law (20) is

$$\mathbf{u} = - \begin{bmatrix} 57.214 & 7.383 \\ 7.383 & 92.896 \end{bmatrix} \left(\begin{bmatrix} -0.009 & 0.008 & -0.020 \\ 0.073 & 0.072 & 0.158 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 6.638\mathbf{s} + 0.1 \frac{\mathbf{s}}{\|\mathbf{s}\|} \right) \quad (26)$$

Set $\mu = 100$, and the control (23) is

$$\mathbf{u} = - \begin{bmatrix} 57.214 & 7.383 \\ 7.383 & 92.896 \end{bmatrix} \left(\begin{bmatrix} -0.009 & 0.008 & -0.020 \\ 0.073 & 0.072 & 0.158 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 6.638\mathbf{s} + 100\mathbf{s} \right) \quad (27)$$

The simulation results are shown in Fig. 1 which demonstrates the efficacy of the proposed method.

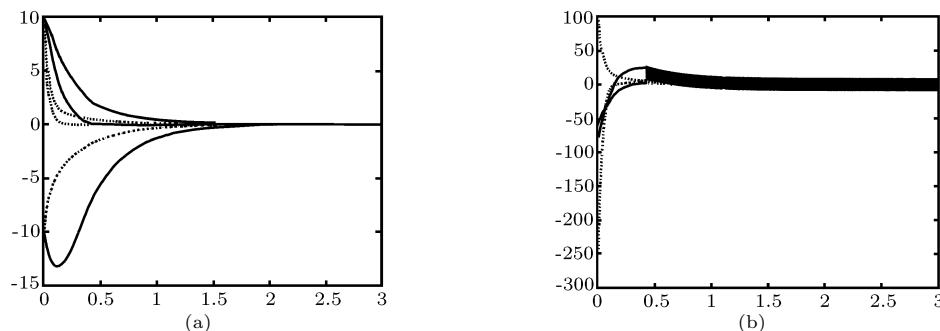


Fig. 1 The histories of system states (a) and control laws (b). (-) solid line: the trajectories under the control law (26); (·) dotted line: the trajectories under the control law (27)

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