# Delay-independent Fuzzy Hyperbolic Guaranteed Cost Control Design for a Class of Nonlinear Continuous-time Systems with Uncertainties[1)]

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**Abstract** This paper develops delay-independent fuzzy hyperbolic guaranteed cost control for nonlinear continuous-time systems with parameter uncertainties. Fuzzy hyperbolic model (FHM) can be used to establish the model for certain unknown complex system. The main advantage of using FHM over Takagi-Sugeno (T-S ) fuzzy model is that no premise structure identification is needed and no completeness design of premise variables space is needed. In addition, an FHM is not only a kind of valid global description but also a kind of nonlinear model in nature. A nonlinear quadratic cost function is developed as a performance measurement of the closed-loop fuzzy system based on FHM. Based on delay-independent Lyapunov functional approach, some sufficient conditions for the existence of such a fuzzy hyperbolic guaranteed cost controller *via* state feedback are provided. These conditions are given in terms of the feasibility of linear matrix inequalities (LMIs). A simulation example is provided to illustrate the design procedure of the proposed method.

**Key words** Fuzzy hyperbolic, guaranteed cost control, linear matrix inequalities, time delays, nonlinear system

## 1 Introduction

Recently, the problem of designing guaranteed cost controllers for uncertain time-delay systems has attracted a number of researchers′ attention[1,2]. The guaranteed cost controller is constructed in such a way that it quadratically stabilizes the uncertain system and also guarantees an upper bound on a given quadratic cost function. Stability criteria for time-delay systems can be classified into two categories. One is delay-dependent criteria, which depend on the size of time delays, the other is delay-independent criterion, which are irrespective of size of time delays (*i.e.*, the time delays are allowed to be arbitrarily large). The delay-independent criteria is considered more conservative in general than the delay-dependent ones, especially when the size of delays is actually small. However, delay-independent criteria are more feasible than delay-dependent ones when the size of time delays is uncertain, unknown or very large[3]. A number of stability analysis and controller systhesis results based on linear matrix inequalities (LMIs) have appeared in the literature where the T-S fuzzy model is used. The stability of the overall fuzzy system is determined by checking a set of LMIs. It is required that a common positive definite matrix $P$ be found to satisfy the set of LMIs. However, this is a difficult problem to solve since such a matrix might not exist in many cases, especially for a lot of fuzzy rules needed to approximate highly nonlinear complex systems. In order to overcome the difficulty, this paper studies delay-independent fuzzy hyperbolic guaranteed cost control for a class of nonlinear continuous-time systems with parameter uncertainties.

Recently, a new continuous-time fuzzy model, called the fuzzy hyperbolic model (FHM), has been proposed in [4], [5], and [6]. As same as the T-S fuzzy model, FHM can be used to establish the model for a certain unknown complex system. Besides the advantage that the FHM is a global model, comparing it with T-S model, we know that no premise structure identification is needed and no completeness

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design of premise variables space is needed when an FHM is used. Thus, nonlinear system FHM can obtain the best representation using FHM. Also FHM can be seen as a neural network model, so we can learn the model parameter by back-propagation (BP) algorithm. In this paper, first, an FHM is used to represent the system. Secondly, delay-independent fuzzy guaranteed cost controller *via* state feedback design based on FHM, called delay-independent fuzzy hyperbolic guaranteed cost controller (DI-FHGCC), is addressed. Thirdly, the DI-FHGCC design problem is converted into a feasible problem of linear matrix inequality (LMI), which makes the prescribed attenuation level as small as possible, subject to some LMI constraints. The LMI feasible problem can be efficiently solved by the convex optimization techniques with global convergence, such as the interior point algorithm[7].

## 2 Preliminaries

In this section we review some necessary preliminaries for the FHM.

**Definition 1**[4,5]**.** Given a plant with $n$ input variables $\boldsymbol{x} = (x_1(t), \cdots, x_n(t))^{\mathrm{T}}$ and $n$ output variables $\dot{\boldsymbol{x}} = (\dot{x}_1(t), \cdots, \dot{x}_n(t))^{\mathrm{T}}$. If each output variable corresponds to a group of fuzzy rules which satisfies the following three conditions:

1) For each output variable $\dot{x}_l, l = 1, 2, \cdots, n$, the corresponding group of fuzzy rules has the following form:

$R^j$: IF $x_1$ is $F_{x_1}$ and $x_2$ is $F_{x_2}$, $\cdots$, and $x_n$ is $F_{x_n}$, THEN $\dot{x}_l = \pm c_{F_{x_1}} \pm c_{F_{x_2}} \pm \cdots \pm c_{F_{x_n}}$, $j = 1, \cdots, 2^n$, where $F_{x_i} (i = 1, \cdot, n)$ are fuzzy sets of $x_i$, which include $P_{x_i}$ (positive) and $N_{x_i}$ (negative), and $c^{\pm}_{F_{x_i}} (i = 1, \cdots, n)$ are $2n$ real constants corresponding to $F_{x_i}$;

2) The constant terms $c^{\pm}_{F_{x_i}}$ in the THEN-part correspond to $F_{x_i}$ in the IF-part. That is, if the linguisitc value of $F_{x_i}$ term in the IF-part is $P_{x_i}$, $c^{+}_{F_{x_i}}$ must appear in the THEN-part; if the linguistic value of $F_{x_i}$ term in the IF-part is $N_{x_i}$, $c^{-}_{F_{x_i}}$ must appear in the THEN-part; if there is no $F_{x_i}$ in the IF-part, $c^{\pm}_{F_{x_i}}$ does not appear in the THEN-part.

3) There are $2^n$ fuzzy rules in each rule base; that is, there are a total of $2^n$ input variable combinations of all the possible $P_{x_i}$ and $N_{x_i}$ in the IF-part; then we call this group of fuzzy rules "hyperbolic type fuzzy rule base (HFRB)". To describe a plant with $n$ output variables, we will need $n$ HFRBs.

**Lemma 1**[4,5]**.** Given $n$ HFRBs, if we define the membership function of $P_{x_i}$ and $N_{x_i}$ as:

$$\mu_{P_{x_i}}(x_i) = e^{-\frac{1}{2}(x_i - k_i)^2}, \mu_{N_{x_i}}(x_i) = e^{-\frac{1}{2}(x_i + k_i)^2} \tag{1}$$

where $i = 1, \cdots, n$ and $k_i$ are positive constants. Denoting $c^{+}_{F_{x_i}}$ by $c_{P_{x_i}}$ and $c^{-}_{F_{x_i}}$ by $c_{N_{x_i}}$, we can derive the following model:

$$\dot{x}_l = f(x) = \sum_{i=1}^{n_l} \frac{c_{P_{x_i}} e^{k_i x_i} + c_{N_{x_i}} e^{-k_i x_i}}{e^{k_i x_i} + e^{-k_i x_i}} = \sum_{i=1}^{n_l} p_i + \sum_{i=1}^{n_l} q_i \frac{e^{k_i x_i} - e^{-k_i x_i}}{e^{k_i x_i} + e^{-k_i x_i}} = \sum_{i=1}^{n_l} p_i + \sum_{i=1}^{n_l} q_i \tanh(k_i, x_i) \tag{2}$$

where $p_i = (c_{P_{x_i}} + c_{N_{x_i}})/2$ and $q_i = (c_{P_{x_i}} - c_{N_{x_i}})/2$. Therefore, the whole system has the following form:

$$\dot{\boldsymbol{x}} = \boldsymbol{P} + A\mathbf{tanh}(kx) \tag{3}$$

where $\boldsymbol{P}$ is a constant vector, $A$ is a constant matrix, and $\tanh(kx)$ is defined by

$$\mathbf{tanh}(kx) = [\tanh(k_1 x_1) \quad \tanh(k_2, x_2) \quad \cdots \quad \tanh(k_n x_n)]^{\mathrm{T}}$$

We will call (3) a fuzzy hyperbolic model (FHM).

From Definition 1, if we set $c_{P_{x_i}}$ and $c_{N_{x_i}}$ negative to each other, we can obtain a homogeneous FHM:

$$\dot{\boldsymbol{x}} = A\mathbf{tanh}(kx) \tag{4}$$

Since the difference between (3) and (4) is only the constant vector term in (3), there is essentially no difference between the control of (3) and (4). In this paper, we will design a fuzzy $H_\infty$ guaranteed cost controller based on FHM described in (4).

## 3 Fuzzy hyperbolic guaranteed cost control design *via* State-feedback

The FHM for the nonlinear time-delay systems with parameter uncertainty is proposed as the following form:

$$\begin{aligned}&\dot{\boldsymbol{x}}(t)=(A+\Delta A)\mathbf{tanh}(kx)+(A_d+\Delta A_d)\mathbf{tanh}(kx(t-h(t)))+(B+\Delta B)\boldsymbol{u}(t),\quad t>0\\&\boldsymbol{x}(t)=\boldsymbol{\varphi}(t),\quad t\in[-h_{\max},0]\end{aligned}\tag{5}$$

where $\boldsymbol{x}(t)=[x_1(t),x_2(t),\cdots,x_n(t)]^{\mathrm{T}}\in R^{n\times 1}$ denotes the state vector; $A\in R^{n\times n}$ and $B\in R^{n\times p}$ are system matrices and input matrix, respectively; $\Delta A$, $\Delta A_d$ and $\Delta B$ denote parameter uncertainties; the real valued functional $h(t)$ is the bounded time delay and satisfies $0\leqslant\dot{h}(t)\leqslant\beta<1$ and $\beta$ is a known constant. The initial condition $\boldsymbol{\varphi}(t)$ is given by initial vector function that is continuous for $-h_{\max}\leqslant t\leqslant 0$.

We assume that $[\Delta A\quad\Delta A_d\quad\Delta B]=MF(t)[N_1\quad N_2\quad N_3]$, where $M,N_1,N_2$ and $N_3$ are known real constant matrices of appropriate dimension, and $F(t)$ is an unknown matrix function and satisfies $F^{\mathrm{T}}(t)F(t)\leqslant I$.

**Definition 2.** Consider system (5) with the following cost function

$$J=\int_0^{\infty}[\mathbf{tanh}^{\mathrm{T}}(kx)Q\mathbf{tanh}(kx)+\boldsymbol{u}^{\mathrm{T}}(t)R\boldsymbol{u}(t)]\mathrm{d}t\tag{6}$$

and

$$\boldsymbol{u}(t)=K\mathbf{tanh}(kx(t))\tag{7}$$

where $Q$ and $R$ are symmetric, positive-definite matrices; $K$ is the feedback gain. The controller is called fuzzy hyperbolic guaranteed cost controller (FHGCC) if there exist a fuzzy hyperbolic control $\boldsymbol{u}(t)$ as in (7) and a scalar $J_0$ such that the closed-loop system is asymptotically stable and the closed-loop value of the cost function (6) satisfies $J\leqslant J_0$. $J_0$ is said to be a guaranteed cost and control law $\boldsymbol{u}(t)$ is said to be a fuzzy hyperbolic guaranteed cost control law for system (5).

With the control law (7) the overall closed-loop system can be written as:

$$\begin{aligned}&\dot{\boldsymbol{x}}=(A+\Delta A+(B+\Delta B)K)\mathbf{tanh}(kx)+(A_d+\Delta A_d)\mathbf{tanh}(kx(t-h(t)))\\&\boldsymbol{x}(t)=\boldsymbol{\varphi}(t),\quad t\in[-h_{\max},0]\end{aligned}\tag{8}$$

**Lemma 2[7].** Given matrices $M$, $E$ and $F$ of appropriate dimensions satisfying $F^{\mathrm{T}}F\leqslant I$. For any $\varepsilon>0$, the following result holds

$$MFE+E^{\mathrm{T}}F^{\mathrm{T}}M^{\mathrm{T}}\leqslant\varepsilon MM^{\mathrm{T}}+\frac{1}{\varepsilon}E^{\mathrm{T}}E$$

Then, we get the following result.

**Theorem 1.** For the nonlinear system (5) and associated cost function (6), if there exist a positive scalar $\varepsilon>0$, a positive definite diagonal matrix $X=\mathrm{diag}\{\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_n\}\in R^{n\times n}>0$ and a positive definite matrix $S\in R^{n\times n}>0$ such that the matrix inequality

$$\begin{bmatrix}\Theta+\varepsilon MM^{\mathrm{T}} & * & * & * & * & *\\ SA_d^{\mathrm{T}} & -S & * & * & * & *\\ N_1X+N_3F & N_2S & -\varepsilon I & * & * & *\\ X & 0 & 0 & -(1-\beta)S & * & *\\ X & 0 & 0 & 0 & -Q^{-1} & *\\ F & 0 & 0 & 0 & 0 & -R^{-1}\end{bmatrix}<0\tag{9}$$

holds, then, the control law, $\boldsymbol{u}(t)=K\mathbf{tanh}(kx(t))$ is a fuzzy hyperbolic guaranteed cost control law and

$$J_0=2\sum_{i=1}^{n}\frac{p_i}{k_i}\ln(\cosh k_ix_i(0))+\frac{1}{1-\beta}\int_{-h(0)}^{0}\mathbf{tanh}^{\mathrm{T}}(kx(s))S^{-1}\mathbf{tanh}(kx(s))\mathrm{d}s$$

where $\Theta = AX + XA^{\mathrm{T}} + BF + F^{\mathrm{T}}B^{\mathrm{T}}$, $K = FX^{-1}$, $p_i = \bar{x}_i^{-1}$, and $*$ denotes the entries induced by symmtry.

**Proof.** Choose the following Lyapunov function candidate for system (8)

$$V(t) = 2\sum_{i=1}^{n}\frac{p_i}{k_i}\ln(\cosh k_i x_i) + \frac{1}{1-\beta}\int_{t-h(t)}^{t}\mathbf{tanh}^{\mathrm{T}}(kx(s))H\mathbf{tanh}(kx(s))\mathrm{d}s \tag{10}$$

where $x_i$ is the $i$th element of $x$, and $k_i$ is the $i$th diagonal element of $k$; $H$ is a positive-definite matrix. Here, $k_i > 0$, $p_i > 0$. Because $\cosh(k_i x_i) = e^{k_i x_i} + e^{-k_i x_i}/2 \geqslant (e^{k_i x_i})^{\frac{1}{2}} = 1$ and $k_i > 0$, $p_i > 0$, we know $V(t) > 0$ for all $x$ and $V(t) \to \infty$ as $\|x\|_2 \to \infty$.

Along the trajectories of system (8), the time derivative of $V(t)$ is given by

$$\begin{aligned}\dot{V} =& 2\sum_{i=1}^{n} p_i\mathbf{tanh}(k\boldsymbol{x})\dot{\boldsymbol{x}} + \alpha\mathbf{tanh}^{\mathrm{T}}(k\boldsymbol{x})H\mathbf{tanh}(k\boldsymbol{x}) - \alpha(1-\dot{h}(t))\mathbf{tanh}^{\mathrm{T}}(kx_h)H\mathbf{tanh}(kx_h) = \\ & 2\mathbf{tanh}^{\mathrm{T}}(k\boldsymbol{x})P\dot{\boldsymbol{x}} + \alpha\mathbf{tanh}^{\mathrm{T}}(k\boldsymbol{x})H\mathbf{tanh}(k\boldsymbol{x}) - \alpha(1-\dot{h}(t))\mathbf{tanh}^{\mathrm{T}}(kx_h)H\mathbf{tanh}(kx_h) \leqslant \\ & \begin{bmatrix}\mathbf{tanh}(k\boldsymbol{x})\\ \mathbf{tanh}(kx_h)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}P\bar{A}+\bar{A}^{\mathrm{T}}P+\alpha H & P\bar{A}_d\\ \bar{A}_d^{\mathrm{T}}P & -H\end{bmatrix}\begin{bmatrix}\mathbf{tanh}(k\boldsymbol{x})\\ \mathbf{tanh}(kx_h)\end{bmatrix} + \mathbf{tanh}^{\mathrm{T}}(k\boldsymbol{x})Q\mathbf{tanh}(k\boldsymbol{x}) + \\ & \boldsymbol{u}^{\mathrm{T}}(t)R\boldsymbol{u}(t) - \mathbf{tanh}^{\mathrm{T}}(k\boldsymbol{x})Q\mathbf{tanh}(k\boldsymbol{x}) - \boldsymbol{u}^{\mathrm{T}}(t)R\boldsymbol{u}(t) = \\ & \xi^{\mathrm{T}}\Theta\xi - \mathbf{tanh}^{\mathrm{T}}(k\boldsymbol{x})Q\mathbf{tanh}(kx) - \boldsymbol{u}^{\mathrm{T}}R\boldsymbol{u}(t)\end{aligned} \tag{11}$$

where

$$\xi = [\mathbf{tanh}^{\mathrm{T}}(k\boldsymbol{x}) \quad \mathbf{tanh}^{\mathrm{T}}(kx_h)]^{\mathrm{T}},\ k = \mathrm{diag}(k_1, k_2, \cdots, k_n),\ P = \mathrm{diag}(p_1, p_2, \cdots, p_n) \in R^{n\times n}$$

$$\bar{A} = (A+\Delta A) + (B+\Delta B)K,\ \bar{A}_d = A_d + \Delta A_d,\ \alpha = \frac{1}{1-\beta}$$

$$\boldsymbol{x}(t-h(t)) \triangleq x_h,\ \Theta = \begin{bmatrix}P\bar{A}+\bar{A}^{\mathrm{T}}P+\alpha H+Q+K^{\mathrm{T}}RK & P\bar{A}_d\\ \bar{A}_d^{\mathrm{T}}P & -H\end{bmatrix}$$

Let $\Theta < 0$. Then $\dot{V} \leqslant -\mathbf{tanh}^{\mathrm{T}}(k\boldsymbol{x})Q\mathbf{tanh}(k\boldsymbol{x}) - u^{\mathrm{T}}R\boldsymbol{u}(t) \leqslant -\lambda_m(Q)\|\mathbf{tanh}(k\boldsymbol{x})\|^2 < 0$. Thus, the closed-loop system is asymptotically stable. Furthermore, integrating $\dot{V} \leqslant -\mathbf{tanh}^{\mathrm{T}}(k\boldsymbol{x})Q\mathbf{tanh}(k\boldsymbol{x}) - \boldsymbol{u}^{\mathrm{T}}(t)R\boldsymbol{u}(t)$ from time 0 to $t_f$ yields

$$\int_0^{t_f}[\mathbf{tanh}^{\mathrm{T}}(k\boldsymbol{x})Q\mathbf{tanh}(k\boldsymbol{x}) + \boldsymbol{u}^{\mathrm{T}}(t)R\boldsymbol{u}(t)]\mathrm{d}t \leqslant V(0) - V(t_f) \tag{12}$$

Since $V(t) \geqslant 0$ and $\dot{V} < 0$, $\lim_{t_f\to\infty} V(t_f) = \mu$ which is a non-negative constant. When $t_f \to \infty$, (12) becomes

$$\begin{aligned}&\int_0^{t_f}[\mathbf{tanh}^{\mathrm{T}}(k\boldsymbol{x})Q\mathbf{tanh}(k\boldsymbol{x}) + \boldsymbol{u}^{\mathrm{T}}(t)R\boldsymbol{u}(t)]\mathrm{d}t \leqslant V(0) = \\ & 2\sum_{i=1}^{n}\frac{p_i}{k_i}\ln(\cosh k_i\varphi_i(0)) + \alpha\int_{-h(0)}^{0}\mathbf{tanh}^{\mathrm{T}}(k\varphi(s))H\mathbf{tanh}(k\varphi(s))\mathrm{d}s\end{aligned} \tag{13}$$

$\Theta$ can be rewritten as the following form:

$$\Omega + \begin{bmatrix}PM\\ 0\end{bmatrix}F(t)[N_1+N_3K \quad N_2] + [N_1+N_3K \quad N_2]^{\mathrm{T}}F^{\mathrm{T}}(t)\begin{bmatrix}PM\\ 0\end{bmatrix}^{\mathrm{T}} < 0 \tag{14}$$

where $\Omega = \begin{bmatrix}P(A+BK)+(A+BK)^{\mathrm{T}}P+\alpha H+Q+K^{\mathrm{T}}RK & PA_d\\ A_d^{\mathrm{T}}P & -H\end{bmatrix}$.

Comparing inequality (14) with Lemma 2, we can obtain

$$\begin{aligned}&\begin{bmatrix}PM\\ 0\end{bmatrix}F(t)[N_1+N_3K \quad N_2] + [N_1+N_3K \quad N_2]^{\mathrm{T}}F^{\mathrm{T}}(t)\begin{bmatrix}PM\\ 0\end{bmatrix}^{\mathrm{T}} \leqslant \\ & \varepsilon\begin{bmatrix}PM\\ 0\end{bmatrix}\begin{bmatrix}PM\\ 0\end{bmatrix}^{\mathrm{T}} + \varepsilon^{-1}[N_1+N_3K \quad N_2]^{\mathrm{T}}[N_1+N_3K \quad N_2]\end{aligned} \tag{15}$$

Thus, the necessary and sufficient condition for inequality (14) to hold is that there exists a positive constant $\varepsilon > 0$ such that

$$\Omega + \varepsilon \begin{bmatrix} PM \\ 0 \end{bmatrix} \begin{bmatrix} PM \\ 0 \end{bmatrix}^{\mathrm{T}} + \varepsilon^{-1}[N_1 + N_3K \quad N_2]^{\mathrm{T}}[N_1 + N_3K \quad N_2] < 0 \tag{16}$$

Now, pre-and post-multiply (16) by $\mathrm{diag}(P^{-1}, H^{-1})$. (16) becomes

$$\begin{bmatrix} AX + XA^{\mathrm{T}} + BF + F^{\mathrm{T}}B^{\mathrm{T}} + \alpha XHX + XQX + F^{\mathrm{T}}RF & A_dS \\ SA_d^{\mathrm{T}} & -S \end{bmatrix} + \varepsilon \begin{bmatrix} M \\ 0 \end{bmatrix} \begin{bmatrix} M \\ 0 \end{bmatrix}^{\mathrm{T}} + \\ \varepsilon^{-1}[N_1X + N_3F \quad N_2S]^{\mathrm{T}}[N_1X + N_3F \quad N_2S] < 0 \tag{17}$$

where $X = P^{-1}$, $H = S^{-1}$, $F = KX$. According to Schur complement, (17) can become LMI (9) in the theorem. When LMI (9) is feasible, the guaranteed cost controller designed ensures the closed-loop system to be asymptotically stable and an upper bound of the closed-loop cost function given by

$$J_0 = 2\sum_{i=1}^{n} \frac{p_i}{k_i}\ln(\cosh k_i\varphi_i(0)) + \frac{1}{1-\beta}\int_{-h(0)}^{0} \mathbf{tanh}^{\mathrm{T}}(k\varphi(s))H\mathbf{tanh}(k\varphi(s))\mathrm{d}s = \\ 2\sum_{i=1}^{n} \frac{\bar{x}_i^{-1}}{k_i}\ln(\cosh k_i\varphi_i(0)) + \frac{1}{1-\beta}\int_{-h(0)}^{0} \mathbf{tanh}^{\mathrm{T}}(k\varphi(s)0)S^{-1}\mathbf{tanh}(k\varphi(s))\mathrm{d}s$$

Therefore, this completes the proof. □

In fact, any feasible solution to (9) yields a suitable robust guaranteed cost controller. A better robust guaranteed cost control law minimizes the upper bound $J_0$. Then, we can obtain Theorem 2.

**Theorem 2.** Consider the nonlinear system (5) and its associated cost function (6). If the optimization problem

$$\min_{\varepsilon,X,V,S,K} \delta + \alpha \cdot Tr(V) \tag{18}$$

subject to 1) LMIs (9), 2) $\begin{bmatrix} -V & \Phi^{\mathrm{T}} \\ \Phi & -S \end{bmatrix} < 0$ has a solution $\hat{\varepsilon}, \hat{X}, \hat{V}, \hat{S}, \hat{K}$, where

$$\Phi\Phi^{\mathrm{T}} = \int_{-h_0}^{0} \mathbf{tanh}(k\varphi(s))\mathbf{tanh}^{\mathrm{T}}(k\varphi(s))\mathrm{d}s$$

$Tr(\cdot)$ denotes the trace of the matrix $(\cdot)$, $\delta = 2\sum_{i=1}^{n} \frac{\bar{x}_i^{-1}}{k_i}\ln(\cosh k_i\varphi_i(0))$, $h_0 = h(0)$ then, the corresponding guaranteed cost control law, $\boldsymbol{u}(t) = K\mathbf{tanh}(k\boldsymbol{x}(t))$ is an optimal guaranteed cost control. Under this control law the closed-loop cost function (6) is minimized.

**Proof.** By Theorem 2, the control law constructed in terms of any feasible solution $\varepsilon, X, M, S, K$ of (9) is a guaranteed cost control law. According to Schur complement, the condition 2) is equivalent to $\Phi^{\mathrm{T}}S^{-1}\Phi < V$.

Since $tr(AB) = tr(BA)$, we have

$$\int_{-h_0}^{0} \mathbf{tanh}^{\mathrm{T}}(k\varphi(s))S^{-1}\mathbf{tanh}(k\varphi(s))\mathrm{d}s = \int_{-h_0}^{0} Tr[\mathbf{tanh}^{\mathrm{T}}(k\varphi(s))S^{-1}\mathbf{tanh}(k\varphi(s))]\mathrm{d}s = \\ Tr[\Phi\Phi^{\mathrm{T}}S^{-1}] = Tr[\Phi^{\mathrm{T}}S^{-1}\Phi] < Tr(V)$$

So it follows that

$$J_0 = 2\sum_{i=1}^{n} \frac{\bar{x}_i^{-1}}{k_i}\ln(\cosh k_i\varphi_i(0)) + \frac{1}{1-\beta}\int_{-h_0}^{0} \mathbf{tanh}^{\mathrm{T}}(k\varphi(s))H\mathbf{tanh}(k\varphi(s))\mathrm{d}s \leqslant \Delta + \alpha Tr(V)$$

Therefore, the guaranteed cost controller subject to (18) is an optimal guaranteed cost control. Under this controller the closed-loop cost function (6) is minimized.

This completes the proof. □

## 4 Simulation example

Consider the following continuous-time nonlinear system with time delays:

$$\begin{aligned}\dot{x}_1(t) = &-0.1x_1^3(t) - 0.0125x_1(t-d(t)) - 0.02x_2(t) - 0.67x_2^3(t) - \\ &0.1x_2^3(t-d(t)) - 0.005x_2(t-d(t)) - 20\boldsymbol{u}(t)\\ \dot{x}_2(t) = &\, x_1(t)\end{aligned} \tag{19}$$

Suppose that we have 16 HFRBs about $\dot{x}_i (i = 1, 2)$ as follows, respectively:

If $x_1$ is $P_{x_1}$ and $x_2$ is $P_{x_2}$ and $x_1^d$ is $P_{x_1^d}$ and $x_2^d$ is $P_{x_2^d}$, then $\dot{x}_i - b_i u = C_{x_1}^i + C_{x_2}^i + C_{x_1^d}^i + C_{x_2^d}^i$;

If $x_1$ is $N_{x_1}$ and $x_2$ is $P_{x_2}$ and $x_1^d$ is $P_{x_1^d}$ and $x_2^d$ is $P_{x_2^d}$, then $\dot{x}_i - b_i u = -C_{x_1}^i + C_{x_2}^i + C_{x_1^d}^i + C_{x_2^d}^i$;

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If $x_1$ is $N_{x_1}$ and $x_2$ is $N_{x_2}$ and $x_1^d$ is $N_{x_1^d}$ and $x_2^d$ is $N_{x_2^d}$, then $\dot{x}_i - b_i u = -C_{x_1}^i - C_{x_2}^i - C_{x_1^d}^i - C_{x_2^d}^i$;

where $x_1^d$, $x_2^d$ denote $x_1(t-d(t))$, $x_2(t-d(t))$, respectively.

Here, we choose membership functions of $P_{x_i}$, $P_{x_i^d}$, $N_{x_i}$ and $N_{x_i^d}$ as follows:

$$\mu_{P_{x_i}}(\boldsymbol{x}) = e^{-\frac{1}{2}(x_i-k_i)^2},\ \mu_{N_{x_i}}(\boldsymbol{x}) = e^{-\frac{1}{2}(x_i-k_i)^2},\ \mu_{P_{x_i^d}}(\boldsymbol{x}) = e^{-\frac{1}{2}(x_i^d-k_i)^2},\ \mu_{N_{x_i^d}}(\boldsymbol{x}) = e^{-\frac{1}{2}(x_i^d-k_i)^2}$$

Then, we have the following 2-dimensional fuzzy hyperbolic model:

$$\dot{\boldsymbol{x}} = A\mathbf{tanh}(k\boldsymbol{x}) + A_d\mathbf{tanh}(kx_d) + B\boldsymbol{u} \tag{20}$$

where $A = \begin{bmatrix} C_{x_1}^1 & C_{x_2}^1 \\ C_{x_1}^2 & C_{x_2}^2 \end{bmatrix}$, $A_d = \begin{bmatrix} C_{x_1^d}^1 & C_{x_2^d}^1 \\ C_{x_1^d}^2 & C_{x_2^d}^2 \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

Fig. 1 shows the network structure of FHM, where $f_1(\boldsymbol{x}) = \mathbf{tanh}(\boldsymbol{x})$, $f_2(\boldsymbol{x}) = \boldsymbol{x}$, $f_3(\boldsymbol{x}) = \boldsymbol{x}$, and $g = 1$. By neural network BP algorithm[5], we obtain FHM of (20) as in (5):

$$A = \begin{bmatrix} 0.0472 & -7.5773 \\ 2.8487 & -0.0617 \end{bmatrix},\ A_d = \begin{bmatrix} -0.3520 & -0.2173 \\ 0.0329 & -0.0440 \end{bmatrix},\ \boldsymbol{B} = \begin{bmatrix} -20 \\ 0 \end{bmatrix},\ k = \mathrm{diag}\{0.3586, 0.0970\}$$

we assume the uncertainties $\boldsymbol{M} = [1 \quad 0]^{\mathrm{T}}$, $\boldsymbol{N}_1 = [0.1 \quad 0]$, $\boldsymbol{N}_2 = [0.5 \quad 0]$, $N_3 = [0.1]$, $F(t) = \sin(t)$, $d(t) = 1 + 0.5\cos(0.9t)$. Solve the LMI problem in (18). We obtain $u = [3.2303 \quad -0.2266]\tanh(k\boldsymbol{x})$ and corresponding $J = 81.9481$. Fig. 2 depicts the behavior of the closed-loop system in solid lines based on the FHM for the initial conditions $\boldsymbol{x}(0) = [0.75 \quad -0.5]^{\mathrm{T}}$. Fig. 3 shows that the control input $u$. Simulation result demonstrates the effectiveness of the fuzzy hyperbolic guaranteed control approach.

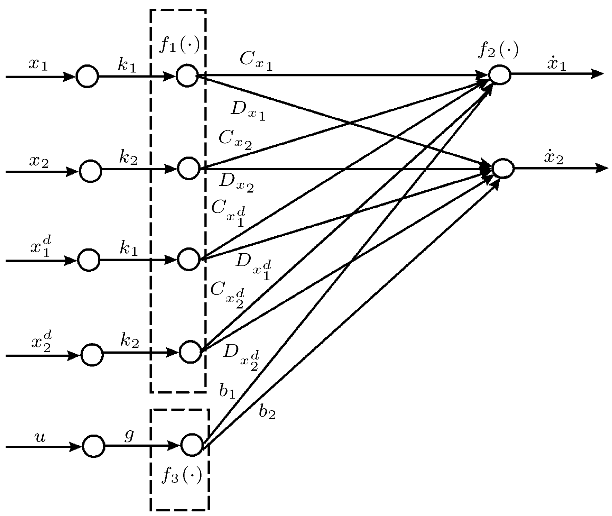


Fig. 1 The network structure of FHM

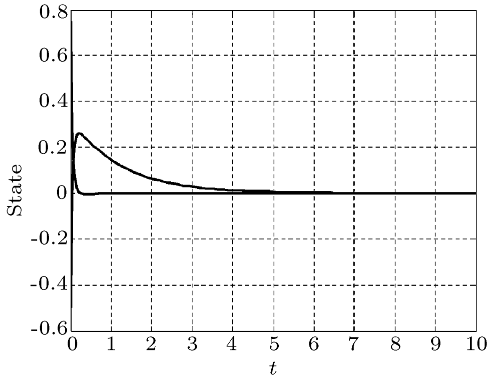


Fig. 2 State response of the closed-loop system

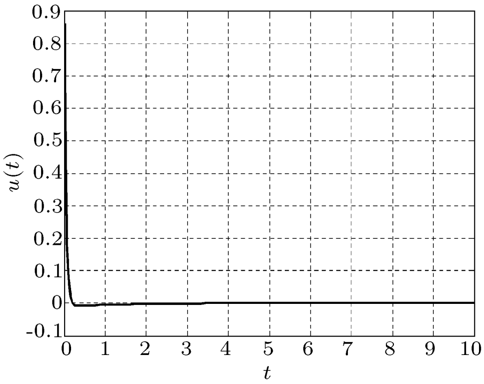


Fig. 3 The response of control input

## 5 Conclusion

In this paper, we propose the delay-independent fuzzy hyperbolic guaranteed cost control for nonlinear uncertain systems with time delay using FHM. The design problem of DI-FHGCC is converted into linear matrix inequalities. The controller designed achieves closed-loop asymptotic stability and provides an upper bound on the closed-loop value of cost function. Simulation example is provided to illustrate the design procedure of the proposed method.

The FHM combines the merits of fuzzy model, neural network model and linear model. This kind of model shows a new way for nonlinear complex system modeling without enough expert experience: First we derive the FHM only if we know some inference relationship between the derivative of state variables and the state variables (input variables); then we use BP (or other neural network learning algorithms) to identify the model parameters. Also, we can describe the fuzzy hyperbolic controller with linguistic information. That is, the controller is also a kind of fuzzy controller. Therefore, comparing fuzzy hyperbolic guaranteed cost control to other guaranteed cost control, we note two important advantages of the former: 1) the fuzzy hyperbolic controller is transparent in the sense that it can be described by a set of If-Then linguistic rules; and 2) the fuzzy hyperbolic controller is bounded.

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