Guaranteed Cost Control for Discrete-time Singular Large-scale Systems with Parameter Uncertainty[1)]

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Abstract The problem of optimal guaranteed cost control for discrete-time singular large-scale systems with a quadratic cost function is considered in this paper. The system under discussion is subject to norm bounded time-invariant parameter uncertainty in all the matrices of model. The problem we address is to design a state feedback controller such that the closed-loop system not only is robustly stable but also guarantees an adequate level of performance for all admissible uncertainties. A sufficient condition for the existence of guaranteed cost controllers is presented in terms of linear matrix inequalities (LMIs), and a desired state feedback controller is obtained *via* convex optimization. An illustrative example is given to demonstrate the effectiveness of the proposed approach.

Key words Discrete-time singular large-scale system, linear matrix inequality (LMI), parameter uncertainty, guaranteed cost control

1 Introduction

For the past years, the problems of robust stability and robust stabilization for state-space systems with parameter uncertainties have attracted a lot of attention and significant advances have been made on these topics[1~4]. However, in practical application, it is also very interesting to construct a control system, which not only is stable but also ensures an adequate level of performance. To this end, a design approach called guaranteed cost control has been presented, in which an upper bound on the closed-loop value of quadratic cost function is guaranteed by using fixed Lyapunov function. Based on this, many researchers work on the guaranteed cost control problem for uncertain systems. For example, in [5], a guaranteed cost controller was designed for uncertain continuous-time systems by using Riccati equation approach; the corresponding results for uncertain discrete-time systems were reported in [6]. Furthermore, results in [5] were extended to continuous delay systems[7~9] *via* a linear matrix inequality (LMI) approach and Riccati equation approach, respectively. Also, the results in [6] were generalized for discrete delay systems[10,11] by an LMI approach and algebraic matrix inequalities approach.

On the other hand, the control theory based on singular systems has extensively studied for many years since singular system models have much more applications than state-space systems in physical systems. Many notions and results in state-space systems have been extended to singular systems[12]. Very recently, a lot of progress about robust stabilization and H_∞ control for singular systems has been reported in [13]. It should be pointed out that the robust stability problem for singular systems is much complicated than that for state-space systems because it requires considering not only stability and robustness, but also regularity and impulse immunity (for continuous singular systems) and causality (for discrete-time singular systems) simultaneously. The H_∞ control problem and robust stabilization for singular systems were investigated in [14,15]. Similar to the case for state-space systems, in practical applications, parameter uncertainty in discrete-time singular systems is unavoidable. However, for discrete-time singular large-scale systems with parameter uncertainty, seldom results on the problem of guaranteed cost control have been reported so far. Obviously, such problems are very complicated.

In this paper, we investigate the problem of guaranteed cost control for uncertain discrete-time singular large-scale systems. The parameter uncertainties are time invariant and unknown but norm-bounded. For the guaranteed cost control problem, the objective is the design of memoryless state

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feedback controllers so that, for all admissible uncertainties, the close-loop system is regular, causal, stable as well as guaranteed cost. Sufficient conditions for the existence of guaranteed cost controllers are obtained in terms of strict LMIs. Then, the parameterization of the required state feedback gains is also given.

Notation. Throughout this paper, for symmetric matrices X and Y, the notation $X \geqslant Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrix with appropriate dimension. The superscript "T" represent the transpose. Matrices, if not explicitly stated, are assumed to have compatible dimensions. $\Phi_i^{\mathrm{T}} \in R^{n_i \times (n_i - r_i)}$ denotes a matrix with the properties of $E_i^{\mathrm{T}} \Phi_i^{\mathrm{T}} = 0$ and $\mathrm{rank}\Phi_i^{\mathrm{T}} = n_i - r_i$. $((M)_{ij})$ denotes an $n \times n$ dimensional matrix, which has the form of

$$((M)_{ij}) = \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ M & \cdots & & & & M \\ M & \cdots & & M & \cdots & M \\ M & \cdots & & & & M \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}_{n \times n}$$

Here $M \in R^{n_i \times n_j}$, 0 are zero matrixes with appropriate dimension and $\sum\limits_{i=1}^{N} n_i = n$.

2 Definitions and problem formulation

Consider linear discrete-time singular large-scale systems with parameter uncertainties described by

$$E_i \boldsymbol{x}_i(k+1) = [A_{ii} + \Delta A_{ii}]\boldsymbol{x}_i(k) + \sum_{j=1, j\neq i}^{N} [A_{ij} + \Delta A_{ij}]\boldsymbol{x}_j(k) + [B_i + \Delta B_i]\boldsymbol{u}_i(k) \tag{1}$$

$(i = 1, 2, \cdots, N)$, where $\boldsymbol{x}_i(k) \in R^{n_i}$ is the state, $\boldsymbol{u}_i(k) \in R^{m_i}$ is the control input. The matrix $E_i \in R^{n_i \times n_i}$ may be singular, we shall assume that $\mathrm{rank}E_i = r_i \leqslant n_i \cdot \sum\limits_{i=1}^{N} n_i = n$, $\sum\limits_{i=1}^{N} r_i = r \leqslant n$, A_{ii}, A_{ij}, B_i are known real constant matrices with appropriate dimensions. $\Delta A_{ii}, \Delta A_{ij}, \Delta B_i$ are time-invariant matrices representing norm-bounded parameter uncertainties, and are assumed to be of the form:

$$|\Delta A_{ij}| \prec D_{ij}, \ |\Delta B_i| \prec H_i, \quad i, j = 1, 2, \cdots, N \tag{2}$$

Here $|F| \prec \bar{F}$ means $|f_{ij}| \leqslant \bar{f}_{ij}$ for every element of $F = (f_{ij}), \bar{F} = (\bar{f}_{ij})$ and D_{ij}, H_i are known real constant non-negative matrices with the same dimensions as $\Delta A_{ij}, \Delta B_i$. ΔA_{ij} and ΔB_i are said to be admissible if (2) holds.

Associated with system (1) is the following cost function

$$J = \sum_{i=1}^{N} \sum_{k=0}^{\infty} [\boldsymbol{x}_i^{\mathrm{T}}(k) Q_i \boldsymbol{x}_i(k) + \boldsymbol{u}_i^{\mathrm{T}}(k) R_i \boldsymbol{u}_i(k)] \tag{3}$$

where $Q_i > 0, R_i > 0$.

Now consider the following memoryless linear state feedback controller

$$\boldsymbol{u}_i(k) = K_i \boldsymbol{x}_i(k) \tag{4}$$

Then the resulting closed-loop system form (1) and (4) can be written as

$$E_i \boldsymbol{x}_i(k+1) = [(A_{ii} + B_i K_i) + (\Delta A_{ii} + \Delta B_i K_i)]\boldsymbol{x}_i(k) + \sum_{j=1, j\neq i}^{N} [A_{ij} + \Delta A_{ij}]\boldsymbol{x}_j(k) \tag{5}$$

Definition 1. Consider the uncertain discrete-time singular large-scale system (1) and cost function (3). A state feedback controller in the form (4) is said to be a guaranteed cost controller, if there exists a positive scalar J^* so that the resulting closed-loop system (5) is robustly stable and cost function (3) $J \leqslant J^*$ for all admissible uncertainties ΔA_{ij} and ΔB_i. In this case, J^* is said to be a guaranteed cost.

In this paper we study guaranteed cost control problem for uncertain discrete-time singular large-scale system (1). First, we study the applicable sufficient conditions for the guaranteed cost controller of the discrete-time singular large-scale systems in the sense of Definition 1 for all admissible uncertainties. Then we further investigate the optimal guaranteed cost control problem to find a memoryless state feedback controller for the given discrete-time singular large-scale system so that the resulting closed-loop is robustly stable and the upper bound on the closed-loop cost function (3) is minimized for all admissible uncertainties. In this case, for simplicity, system (1) is said to be optimal guaranteed cost control.

We conclude this section by presenting three preliminary results, which will be used in the proof of our main results in the following sections.

Proposition 1[2]**.** Let matrix $\Delta A \in R^{n\times m}$ satisfy $|\Delta A| \prec D$. Then $\Delta A^{\mathrm{T}}\Delta A \leqslant \varGamma(D)$.

Here

$$\varGamma(D) = \begin{cases} \|D^{\mathrm{T}}D\|I, & \|D^{\mathrm{T}}D\|I \leqslant m\mathrm{diag}(D^{\mathrm{T}}D) \\ m\mathrm{diag}(D^{\mathrm{T}}D), & \text{else} \end{cases} \tag{6}$$

Proposition 2[14]. $E\boldsymbol{x}(k+1) = A\boldsymbol{x}(k)$ is admissible if and only if there exists a positive definite matrix P and a symmetric matrix $S \in R^{(n-r)\times(n-r)}$ such that

$$A^{\mathrm{T}}(P - \varPhi^{\mathrm{T}}S\varPhi)A - E^{\mathrm{T}}PE < 0 \tag{7}$$

where $\varPhi^{\mathrm{T}} \in R^{n\times(n-r)}$ denotes a matrix with the properties of $E^{\mathrm{T}}\varPhi^{\mathrm{T}} = 0$ and $\mathrm{rank}\varPhi^{\mathrm{T}} = n - r$.

Proposition 3. Suppose that a symmetric matrix W is invertible, A and ΔA are matrices with appropriate dimensions and there exists a constant $\varepsilon > 0$ such that $\varepsilon I - W > 0$. Then

$$(A+\Delta A)^{\mathrm{T}}W(A+\Delta A) \leqslant A^{\mathrm{T}}[W + W(\varepsilon I - W)^{-1}W]A + \varepsilon\Delta A^{\mathrm{T}}\Delta A$$

Proof. The proof is very similar to the proof of Lemma 5 in [15].

3 Design of guaranteed cost controller

In this section, we shall give a sufficient condition for the guaranteed cost controller. Then a solution to guaranteed cost control problem for uncertain discrete-time singular large-scale system (1) is proposed, where an LMI approach will be developed.

Theorem 1. The uncertain discrete-time singular large-scale system (1) with $B_i = 0$ and $\Delta B_i = 0$ is robustly stable if there exist positive definite matrices P_i, symmetric matrices S_i and a scalar $\varepsilon > 0$ such that

$$\begin{bmatrix} A_{ii}^{\mathrm{T}}(P_i - \varPhi_i^{\mathrm{T}}S_i\varPhi_i)A_{ii} - E_i^{\mathrm{T}}P_iE_i + \bar{Q}_i & A_{ii}^{\mathrm{T}}(P_i - \varPhi_i^{\mathrm{T}}S_i\varPhi_i) \\ (P_i - \varPhi_i^{\mathrm{T}}S_i\varPhi_i)A_{ii} & P_i - \varPhi_i^{\mathrm{T}}S_i\varPhi_i - \varepsilon I \end{bmatrix} < 0 \tag{8}$$

where $\bar{Q}_i = (2N-1)\varepsilon[\varGamma(D_{ii}) + \sum\limits_{j=1,j\neq i}^{N}(\varGamma(D_{ji}) + A_{ji}^{\mathrm{T}}A_{ji})],\ i = 1, 2, \cdots, N.$

Proof. Define

$$X = \mathrm{diag}(X_1, X_2, \cdots, X_N),\ E = \mathrm{diag}(E_1, E_2, \cdots, E_N),\ X_i = P_i - \varPhi_i^{\mathrm{T}}S_i\varPhi_i$$
$$\varPhi = \mathrm{diag}(\varPhi_1, \varPhi_2, \cdots, \varPhi_N),\ S = \mathrm{diag}(S_1, S_2, \cdots, S_N),\ P = \mathrm{diag}(P_1, P_2, \cdots, P_N) > 0$$
$$\Delta\bar{Q}_i = (2N-1)\varepsilon[\Delta A_{ii}^{\mathrm{T}}\Delta A_{ii} + \sum_{j=1,j\neq i}^{N}(\Delta A_{ji}^{\mathrm{T}}\Delta A_{ji} + A_{ji}^{\mathrm{T}}A_{ji})], (\varepsilon I - X)^{-1} = \sum_{i=1}^{N}(((\varepsilon I - X_i)^{-1})_{ii})$$
$$A + \Delta A = \bar{A} + \sum_{i=1}^{N}((\Delta A_{ii})_{ii}) + \sum_{i,j=1,j\neq i}^{N}((A_{ij} + \Delta A_{ij})_{ij}),\ \bar{A} = \sum_{i=1}^{N}((A_{ii})_{ii})$$

Then $X = P - \varPhi^{\mathrm{T}}S\varPhi, E^{\mathrm{T}}\varPhi^{\mathrm{T}} = 0, \mathrm{rank}\varPhi^{\mathrm{T}} = n - r$.

By Schur complements, it is easy to show that (8) is equivalent to

$$A_{ii}^{\mathrm{T}}X_iA_{ii} - E_i^{\mathrm{T}}P_iE_i + \bar{Q}_i + A_{ii}^{\mathrm{T}}X_i(\varepsilon I - X_i)^{-1}X_iA_{ii} < 0 \tag{9}$$

with

$$\varepsilon I - X_i > 0 \tag{10}$$

Note that (9) and Proposition 3. We have

$$\varepsilon I - X > 0 \tag{11}$$

$$\begin{aligned}&(A+\Delta A)^{\mathrm{T}}X(A+\Delta A) - E^{\mathrm{T}}PE \leqslant \bar{A}^{\mathrm{T}}[X + X(\varepsilon I - X)^{-1}X]\bar{A} - E^{\mathrm{T}}PE+\\&\varepsilon[\sum_{i=1}^{N}((\Delta A_{ii})_{ii}) + \sum_{i,j=1,j\neq i}^{N}((A_{ij}+\Delta A_{ij})_{ij})]^{\mathrm{T}}[\sum_{i=1}^{N}((\Delta A_{ii})_{ii}) + \sum_{i,j=1,j\neq i}^{N}((A_{ij}+\Delta A_{ij})_{ij})] \leqslant\\&\sum_{i=1}^{N}((A_{ii}^{\mathrm{T}}X_iA_{ii} - E_i^{\mathrm{T}}P_iE_i + A_{ii}^{\mathrm{T}}X_i(\varepsilon I - X_i)^{-1}X_iA_{ii} + \Delta\bar{Q}_i)_{ii})\end{aligned} \tag{12}$$

This inequality together with (9) implies

$$(A+\Delta A)^{\mathrm{T}}(P - \Phi^{\mathrm{T}}S\Phi)(A+\Delta A) - E^{\mathrm{T}}PE < 0 \tag{13}$$

Finally, by Proposition 2, the desired result follows immediately.

Remark 1. In the case that $N = 1$, *i.e.*, when (1) reduces to a discrete-time singular system, it is easy to show that Theorem 1 coincides with Theorem 1 in [9]. Therefore, Theorem 1 can be regarded as an extension of existing results in discrete-time singular system.

Now we are in a position to present a solution to the guaranteed cost control problem for uncertain discrete-time singular large-scale systems.

Theorem 2. For the uncertain discrete-time singular large-scale system (1), if there exist positive definite matrices P_i, symmetric matrices S_i and scalars $\varepsilon > 0, \beta > 0$ such that

$$\begin{bmatrix} A_{ii}^{\mathrm{T}}X_iA_{ii} - E_i^{\mathrm{T}}P_iE_i + \hat{Q}_i & A_{ii}^{\mathrm{T}}X_i & 0\\ X_iA_{ii} & X_i - \varepsilon I & X_iB_i\\ 0 & B_i^{\mathrm{T}}X_i & -\beta I - 2(2N-1)\varepsilon\Gamma(H_i) - B_i^{\mathrm{T}}X_iB_i - R_i\end{bmatrix} < 0 \tag{14}$$

where

$$\hat{Q}_i = (2N-1)\varepsilon[2\Gamma(D_{ii}) + \sum_{j=1,j\neq i}^{N}(\Gamma(D_{ji}) + A_{ji}^{\mathrm{T}}A_{ji})] + Q_i$$

$$X_i = (P_i - \Phi_i^{\mathrm{T}}S_i\Phi_i), \quad i = 1, 2, \cdots, N$$

Then a guaranteed cost state feedback controller can be chosen by

$$\boldsymbol{u}_i(k) = K_i\boldsymbol{x}_i(k) \tag{15}$$

$$K_i = -[\beta I + 2(2N-1)\varepsilon\Gamma(H_i) + B_i^{\mathrm{T}}X_iB_i + R_i]^{-1}B_i^{\mathrm{T}}X_iA_{ii} \tag{16}$$

In this case, the corresponding guaranteed cost is $J^* = \sum_{i=1}^{N}\boldsymbol{x}_i^{\mathrm{T}}(0)E_i^{\mathrm{T}}P_iE_i\boldsymbol{x}_i(0)$.

Proof. Under the conditions of the theorem, we apply controller (15) to (1) and obtain the close-loop system as follows:

$$E_i\boldsymbol{x}_i(k+1) = [A_{K_i} + \Delta A_{k_i}]\boldsymbol{x}_i(k) + \sum_{j=1,j\neq i}^{N}[A_{ij} + \Delta A_{ij}]\boldsymbol{x}_j(k) \tag{17}$$

Here $A_{K_i} = A_{ii} + B_iK_i, \Delta A_{K_i} = \Delta A_{ii} + \Delta B_iK_i$. Define

$$A_K = \mathrm{diag}(A_{K_1}, A_{K_2}, \cdots, A_{K_N}),\ \Delta A_K = \mathrm{diag}(\Delta A_{K_1}, \Delta A_{K_2}, \cdots, \Delta A_{K_N})$$

$$\Delta\hat{Q}_{K_i} = (2N-1)\varepsilon[\Delta A_{K_i}^{\mathrm{T}}\Delta A_{K_i} + \sum_{j=1,j\neq i}^{N}(\Delta A_{ji}^{\mathrm{T}}\Delta A_{ji} + A_{ji}^{\mathrm{T}}A_{ji})]$$

$$\tilde{A}+\Delta\tilde{A}=A_k+\Delta A_K+\sum_{i,j=1,j\neq i}^{N}((A_{ij}+\Delta A_{ij})_{ij}),\ K=\mathrm{diag}(K_1,K_2,\cdots,K_N)$$
$$Q=\mathrm{diag}(Q_1,Q_2,\cdots,Q_N),\ R=\mathrm{diag}(R_1,R_2,\cdots,R_N)$$

Similar to the proof of Theorem 1, we have

$$(\tilde{A}+\Delta\tilde{A})^{\mathrm{T}}(P-\varPhi^{\mathrm{T}}S\varPhi)(\tilde{A}+\Delta\tilde{A})-E^{\mathrm{T}}PE+Q+K^{\mathrm{T}}RK\leqslant$$
$$\sum_{i=1}^{N}((A_{K_i}^{\mathrm{T}}X_iA_{K_i}-E_i^{\mathrm{T}}P_iE_i+A_{K_i}^{\mathrm{T}}X_i(\varepsilon I-X_i)^{-1}X_iA_{K_i}+\Delta\hat{Q}_{K_i}+Q_i+K_i^{\mathrm{T}}R_iK_i)_{ii}) \tag{18}$$

Note

$$\begin{bmatrix} A_{K_i}^{\mathrm{T}}X_iA_{K_i}-E_i^{\mathrm{T}}P_iE_i+\Delta Q_{K_i}+K_i^{\mathrm{T}}R_iK_i & A_{K_i}^{\mathrm{T}}X_i \\ X_iA_{K_i} & X_i-\varepsilon I\end{bmatrix}=$$
$$\begin{bmatrix} A_{K_i}^{\mathrm{T}}X_iA_{K_i}+(2N-1)\varepsilon\Delta A_{K_i}^{\mathrm{T}}\Delta A_{K_i}+K_i^{\mathrm{T}}R_iK_i & A_{K_i}^{\mathrm{T}}X_i \\ X_iA_{K_i} & X_i\end{bmatrix}+$$
$$\begin{bmatrix} -E_i^{\mathrm{T}}P_iE_i+(2N-1)\varepsilon\sum_{j=1,j\neq i}^{N}(\Delta A_{ji}^{\mathrm{T}}\Delta A_{ji}+A_{ji}^{\mathrm{T}}A_{ji})+Q_i & 0 \\ 0 & -\varepsilon I\end{bmatrix} \tag{19}$$

and

$$\begin{bmatrix} A_{K_i}^{\mathrm{T}}X_iA_{K_i}+(2N-1)\varepsilon\Delta A_{K_i}^{\mathrm{T}}\Delta A_{K_i}+K_i^{\mathrm{T}}R_iK_i & A_{K_i}^{\mathrm{T}}X_i \\ X_iA_{K_i} & X_i\end{bmatrix}=$$
$$\begin{bmatrix} A_{K_i}^{\mathrm{T}} \\ I\end{bmatrix}X_i(A_{K_i}\quad I)+(2N-1)\varepsilon\begin{bmatrix}\Delta A_{K_i}^{\mathrm{T}} \\ 0\end{bmatrix}(\Delta A_{K_i}\quad 0)+\begin{bmatrix}K_i^{\mathrm{T}} \\ 0\end{bmatrix}R_i(K_i\quad 0)\leqslant$$
$$J_i+\left[\begin{pmatrix}K_i^{\mathrm{T}} \\ 0\end{pmatrix}+\begin{pmatrix}A_{ii}^{\mathrm{T}}X_iB_i \\ X_iB_i\end{pmatrix}V_i^{-1}\right]V_i\left[\begin{pmatrix}K_i^{\mathrm{T}} \\ 0\end{pmatrix}+\begin{pmatrix}A_{ii}^{\mathrm{T}}X_iB_i \\ X_iB_i\end{pmatrix}V_i^{-1}\right]^{\mathrm{T}} \tag{20}$$

where

$$J_i=\begin{bmatrix} A_{ii}^{\mathrm{T}}X_iA_{ii}+2(2N-1)\varepsilon\varGamma(D_{ii}) & A_{ii}^{\mathrm{T}}X_i \\ X_iA_{ii} & X_i\end{bmatrix}$$
$$V_i=\beta I+2(2N-1)\varepsilon\varGamma(H_i)+B_i^{\mathrm{T}}(P_i-\varPhi_i^{\mathrm{T}}S_i\varPhi_i)B_i+R_i$$

Substituting (20) into (19) and considering (15), we have

$$\begin{bmatrix} A_{K_i}^{\mathrm{T}}X_iA_{K_i}-E_i^{\mathrm{T}}P_iE_i+\Delta Q_{K_i}+Q_i+K_i^{\mathrm{T}}R_iK_i & A_{K_i}^{\mathrm{T}}X_i \\ X_iA_{K_i} & X_i-\varepsilon I\end{bmatrix}\leqslant$$
$$\begin{bmatrix}0 \\ X_iB_i\end{bmatrix}V_i^{-1}\begin{bmatrix}0 \\ X_iB_i\end{bmatrix}^{\mathrm{T}}+\begin{bmatrix} A_{ii}^{\mathrm{T}}X_iA_{ii}-E_i^{\mathrm{T}}P_iE_i+\hat{Q} & A_{ii}^{\mathrm{T}}X_i \\ X_iA_{ii} & X_i-\varepsilon I\end{bmatrix} \tag{21}$$

Considering the inequalities (14), (18), and (21), and by Schur complements, we have

$$(\tilde{A}+\Delta\tilde{A})^{\mathrm{T}}(P-\varPhi^{\mathrm{T}}S\varPhi)(\tilde{A}+\Delta\tilde{A})-E^{\mathrm{T}}PE<-(Q+K^{\mathrm{T}}RK)<0$$

From this inequality and Proposition 2,we have that system (17) is robustly stable.

We define the following Lyapunov functional candidate:

$$V(\boldsymbol{x}(k))=\boldsymbol{x}(k)^{\mathrm{T}}E^{\mathrm{T}}(P-\varPhi^{\mathrm{T}}S\varPhi)E\boldsymbol{x}(k)=\boldsymbol{x}(k)^{\mathrm{T}}E^{\mathrm{T}}PE\boldsymbol{x}(k)\geqslant 0 \tag{22}$$

$V(\boldsymbol{x}(k))$ along the solution of system (17) satisfies

$$\Delta V(\boldsymbol{x}(k))=\boldsymbol{x}(k)^{\mathrm{T}}[(\tilde{A}+\Delta\tilde{A})^{\mathrm{T}}(P-\varPhi^{\mathrm{T}}S\varPhi)(\tilde{A}+\Delta\tilde{A})-E^{\mathrm{T}}PE]\boldsymbol{x}(k)\leqslant$$
$$-\boldsymbol{x}(k)^{\mathrm{T}}[Q+K^{\mathrm{T}}RK]\boldsymbol{x}(k)=-\sum_{i=1}^{N}\boldsymbol{x}_i(k)^{\mathrm{T}}[Q_i+K_i^{\mathrm{T}}R_iK_i]\boldsymbol{x}_i(k) \tag{23}$$

Summing both sides of inequality (23) from zero to infinity gives

$$V(\boldsymbol{x}(\infty)) - V(\boldsymbol{x}(0)) \leqslant -\sum_{i=1}^{N}\sum_{k=0}^{\infty}\boldsymbol{x}_i(k)^{\mathrm{T}}[Q_i + K_i^{\mathrm{T}}R_iK_i]\boldsymbol{x}_i(k)$$

Note that $V(\boldsymbol{x}(\infty)) \geqslant 0$. Hence, it follows from the above inequality that

$$J = \sum_{i=1}^{N}\sum_{k=0}^{\infty}\boldsymbol{x}_i(k)^{\mathrm{T}}[Q_i + K_i^{\mathrm{T}}R_iK_i]\boldsymbol{x}_i(k) \leqslant V(\boldsymbol{x}(0)) = \sum_{i=1}^{N}\boldsymbol{x}_i^{\mathrm{T}}(0)E^{\mathrm{T}}PE\boldsymbol{x}_i(0) \tag{24}$$

Therefore, the proof follows immediately from this inequality and Definition 1.

Remark 2. Theorem 2 presents a sufficient condition for the existence of guaranteed cost state feedback controller for uncertain discrete-time singular large-scale system (1). It is worth noting that the condition for solvability is expressed by using the system matrices of (1); the design procedure involves no decomposition of the system, which may around certain numerical problems arising from decomposition of matrices, and thus makes the design procedure relatively simple and reliable.

It is worth noting that Theorem 2 gives a set of guaranteed cost controller characterized in terms of the solutions to LMIs (14). Each guaranteed cost controller ensures the resulting closed-loop system (5) is robustly stable and an upper bound on the closed-loop cost function given by (24). In view of this, it is desirable to find an optimal guaranteed cost controller, which minimizes the upper bound (24). The problem is dealt with in the following theorem.

Theorem 3. Consider the uncertain discrete-time singular large-scale system (1) and cost function (3). Suppose the following optimization problem

$$\min_{\varepsilon,\beta,P_i,S_i}\sum_{i=1}^{N}tr(E_i^{\mathrm{T}}P_iE_i) \tag{25}$$

$$\begin{aligned}\text{s.t. } 1)&\quad \text{LMIs (14)}\\ 2)&\quad P_i > 0\end{aligned}$$

has a solution for $\varepsilon, \beta, P_i, S_i$. Then, the corresponding guaranteed cost controller in the form of (15) and (16) is an optimal guaranteed cost controller in the sense that under this controller the upper bound on the closed-loop cost function (3) is minimized.

Proof. The proof can be carried out by noting the proof of Theorem 2 and using the same argument as in the proof of Theorem 1 in [17].

4 Numerical example

In this section, we give an example to illustrate the effectiveness of the proposed method.

Consider the uncertain discrete-time singular large-scale system (1) with parameters as follows:

$$E_1 = \begin{bmatrix}1 & 0.4 & 0.2\\ 0.5 & 0.2 & 0.1\\ 0.5 & 0.4 & 0.3\end{bmatrix},\ A_{11} = \begin{bmatrix}0.5 & 0.2 & 0.1\\ 0.25 & -0.1 & 0.25\\ 0.5 & 0.2 & 0.27\end{bmatrix},\ D_{11} = \begin{bmatrix}0.002 & 0.001 & 0.001\\ 0.001 & 0.002 & 0.001\\ 0.001 & 0.001 & 0.002\end{bmatrix}$$

$$A_{12} = \begin{bmatrix}0.01 & -0.01\\ 0.01 & -0.01\\ -0.01 & 0.01\end{bmatrix},\ D_{12} = \begin{bmatrix}0.001 & 0.001\\ 0.001 & 0.001\\ 0 & 0.001\end{bmatrix},\ B_1 = \begin{bmatrix}0.1 & 0.05\\ 0.1 & 0\\ 0.05 & 0.1\end{bmatrix},\ H_1 = \begin{bmatrix}0.001 & 0.001\\ 0.001 & 0.001\\ 0 & 0.001\end{bmatrix}$$

$$E_2 = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\ A_{22} = \begin{bmatrix}0.5 & 0.5\\ -0.5 & 1\end{bmatrix},\ D_2 = \begin{bmatrix}0.002 & 0.001\\ 0.001 & 0.002\end{bmatrix},\ \boldsymbol{B}_2 = \begin{bmatrix}0.1\\ 0.05\end{bmatrix}$$

$$A_{21} = \begin{bmatrix}0.01 & -0.01 & 0.01\\ 0 & 0.01 & -0.01\end{bmatrix},\ D_{21} = \begin{bmatrix}0.001 & 0.001 & 0.001\\ 0 & 0.001 & 0.001\end{bmatrix},\ \boldsymbol{H}_2 = \begin{bmatrix}0.001\\ 0.001\end{bmatrix}$$

$$Q_1 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},\ Q_2 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ R_1 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ R_2 = 1$$

Now we try to find a memoryless optimal guaranteed cost state feedback controller for it such that, for all admissible uncertainties, the resulting closed-loop system is admissible. Choose $\boldsymbol{\Phi}_1 = [1 \quad -2 \quad 0]$ and $\boldsymbol{\Phi}_2 = [0 \quad 1]$.

Therefore, by Theorem 3 the corresponding optimal guaranteed cost state feedback controller can be obtained as

$$\boldsymbol{u}_1(t) = \begin{bmatrix} -0.3074 & -0.1407 & -0.0856 \\ -0.1804 & -0.0090 & -0.1237 \end{bmatrix} \boldsymbol{x}_1(t), \quad \boldsymbol{u}_2(t) = [-0.0494 \quad -0.1977]\boldsymbol{x}_2(t)$$

Furthermore, the corresponding closed-loop cost function is $J^* = 10$.

5 Conclusion

In this paper we have considered the problems of robust stability and guaranteed cost control for uncertain discrete-time singular large-scale system with parameter uncertainties. Attention has been focused on the design of memoryless state feedback optimal guaranteed cost controllers. An LMI design approach has been developed. An example has been presented to demonstrate the proposed method.

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