Robust Output Feedback Control for Uncertain Discrete Systems with Time Delays[1)]

LIU Bi-Yu[1,2] GUI Wei-Hua[1]

[1](*School of Information Science and Engineering, Central South University, Changsha* 410083)

[2](*School of Mathematical Science and Computational Technology, Central South University, Changsha* 410083)

(E-mail: biyuliu@hotmail.com)

Abstract Based on design of an observer, the issue of dynamic output feedback control is studied for uncertain discrete systems with delays. A comparison theorem is given for nonlinear uncertain discrete systems with multiple time delays. Based on the comparison theorem with some inequalities, some delay-independent sufficient conditions for the robust stabilization of the systems are presented by means of output feedback.

Key words Uncertain discrete systems, time delays, dynamic output feedback, comparison theorem

1 Introduction

It is well known that state-feedback control is one of the important tools in control engineering. To achieve the state-feedback, the complete controllability of the states is necessary. However, in practice, it is difficult to measure the states of the system completely. So, dynamic output-feedback control is commonly used, since the output can be observed dynamically and easily. On the other hand, the limitation of measurements causes uncertainties in the systems. Thus, it leads to the robust control method.

Recently, there have been many contributions on observer-based robust output-feedback control of continuous systems[1~3]. [1,2] presented the results by using Lyapunov stability approach. [3] designed the observer-based output-feedback controllers for uncertain systems by using analytic theory and comparison principle. Since study of discrete systems has its own characteristic and difficulties, one could not see as many stability or stabilization results on discrete systems as on continuous systems[4]. Up to now, some detail analysis on observer-based robust output-feedback control of nonlinear continuous systems and stability analysis on discrete systems were presented[3,4]. The problem is worthwhile and difficult. In this paper, we will establish a comparison principle of discrete systems. Then, we will present a delay-independent sufficient condition to guarantee robust output-feedback stabilization of uncertain discrete systems with multiple time delays by using this comparison principle. Our method is easy to be understood and used in practice. It bypasses the difficulty of selecting Lyapunov functions.

2 System statement and lemmas

Consider a class of uncertain discrete-time systems with multiple time delays represented by

$$\begin{aligned} \boldsymbol{x}(k+1) &= A\boldsymbol{x}(k) + \Delta A(\boldsymbol{x}(k).k) + \sum_{i=1}^{n} A_i\boldsymbol{x}(k-\tau_i) + \\ &\quad \sum_{i=1}^{n} \Delta A_i(\boldsymbol{x}(k-\tau_i),k) + B\boldsymbol{u}(k) + \Delta B(\boldsymbol{u}(k),k) \\ \boldsymbol{y}(k) &= C\boldsymbol{x}(k) + \Delta C(\boldsymbol{x}(k),k), \quad k>0 \\ \boldsymbol{x}(k) &= \boldsymbol{\phi}(k), \quad k=-\tau,-\tau+1,\cdots,0 \end{aligned} \tag{1}$$

where $\boldsymbol{x}(k)\in R^n$ is the state, $\boldsymbol{u}(k)\in R^m$ is the control input, and $\boldsymbol{y}(k)\in R^p$ is the output. Positive integers τ_i $(i=1,2,\cdots,n)$ represent the amount of delay units in the state, while τ denotes the

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maximum of the set $\{\tau_i : i = 1, \cdots, n\}$. Function $\boldsymbol{\phi}(k)$ is a given initial vector. A, A_i $(i = 1, 2, \cdots, n), B$ and C are known real constant matrices of appropriate dimensions. Also $\Delta A(\boldsymbol{x}(k), k)$, $\Delta A_i(\boldsymbol{x}(k - \tau_i), k)(i = 1, 2, \cdots, n)$, $\Delta B(\boldsymbol{u}(k), k)$ and $\Delta C(\boldsymbol{x}(k), k)$ are the time-varying parameter uncertainties with the following upper norm bounds:

$$\begin{aligned} &\|\Delta A(\boldsymbol{x}(k), k)\| \leqslant \alpha\|\boldsymbol{x}(k)\|, \quad \|\Delta A_i(\boldsymbol{x}(k-\tau_i), k)\| \leqslant \alpha_i\|\boldsymbol{x}(k-\tau_i)\| \\ &\|\Delta B(\boldsymbol{u}(k), k) \leqslant \beta\|\boldsymbol{u}(k)\|, \quad \|\Delta C(\boldsymbol{x}(k), k)\| \leqslant \gamma\|\boldsymbol{x}(k)\| \end{aligned} \tag{2}$$

where α, α_i, β and γ are some nonnegative constants. It is assumed that pairs (A, B) and (A, C) are controllable and observable, respectively.

Lemma 1. Consider an initial-value problem of a discrete delay system

$$\begin{aligned} &\boldsymbol{v}(k+1) = A(k)\boldsymbol{v}(k) + \boldsymbol{g}(k, \boldsymbol{v}(k-\tau)) \\ &\boldsymbol{v}(s) = \boldsymbol{\phi}(s), \quad s = -\tau, -\tau+1, \cdots, 0 \end{aligned} \tag{3}$$

It follows that the formula of the solution is

$$\boldsymbol{v}(k) = \Omega(k, 0)\boldsymbol{\phi}(0) + \sum_{t=0}^{k-1} \Omega(k, t+1)\boldsymbol{g}(t, \boldsymbol{v}(t-\tau)) \tag{4}$$

where $\Omega(k, s) = \prod_{t=s}^{k-1} A(t)$ and $\Omega(k, k) = 1$ as $k = s$. And when $k < 0$, we define $\sum_{t=0}^{k-1} \Omega(k, t+1)\boldsymbol{g}(t, \boldsymbol{v}(t-\tau)) = 0$.

Lemma 2. Let $f(k+1) \leqslant af(k) + b \sup_{k-\tau\leqslant\sigma\leqslant k} f(\sigma)$ for $k \geqslant 0$ and $f(k)$ be a nonnegative function. If $a > 0, b > 0$, and $a + b < 1$, then there exists a $\lambda > 0$ such that

$$f(k) \leqslant \sup_{-\tau\leqslant\sigma\leqslant 0} f(\sigma)e^{-\lambda k} \text{ for } k \geqslant 0 \tag{5}$$

where τ is a positive integer, and λ is a unique positive root of the equation $a + be^{\lambda\tau} = e^{-\lambda}$.

Analogous to the proof of the Theorem 1 in [5], we can prove this Halanay type inequality easily.

Lemma 3[6]. (Comparison theorem) Suppose that a vector-valued function

$$\boldsymbol{v}(k) : J \to R^n, J = \{0, \pm 1, \pm 2, \cdots\} \text{ satisfies } \boldsymbol{v}(k) \leqslant \boldsymbol{g}(k, \boldsymbol{v}(k-\tau))$$

where $\boldsymbol{g}(k, \boldsymbol{v}) : J \times R^n \to R^n$ is non-decreasing with respect to $\boldsymbol{v}$, and τ is a positive integer. Then the solution $\omega(k)$ to the comparison equation $\omega(k) = \boldsymbol{g}(k, \omega(k-\tau))$, with $\boldsymbol{v}(k) \leqslant \omega(k), k \in \{-\tau, -\tau+1, \cdots, -1, 0\}$ satisfies $\boldsymbol{v}(k) \leqslant \omega(k)$ for all integers $k \geqslant -\tau$.

3 The design of robust output feedback controller

For the uncertain discrete-time systems with multiple time delays (1), we propose the following Luenberger-like dynamic output feedback controller

$$\begin{aligned} &\hat{\boldsymbol{x}}(k+1) = A\hat{\boldsymbol{x}}(k) + B\boldsymbol{u}(k) + L(\boldsymbol{y}(k) - C\hat{\boldsymbol{x}}(k)) \\ &\boldsymbol{u}(k) = K\hat{\boldsymbol{x}}(k) \end{aligned} \tag{6}$$

where $L \in R^{n\times p}$, $K \in R^{m\times n}$ are two constant matrices to be determined, $\hat{\boldsymbol{x}}(k) \in R^n$ is the state of the observer.

Let the difference of $\boldsymbol{x}(k)$ and $\hat{\boldsymbol{x}}(k)$ be $\boldsymbol{e}(k)$, that is, $\boldsymbol{e}(k) = \boldsymbol{x}(k) - \hat{\boldsymbol{x}}(k)$. Then the closed-loop systems of (1) and (6) are in the form as follows.

$$\begin{aligned} &\boldsymbol{z}(k+1) = \tilde{A}\boldsymbol{z}(k) + \sum_{i=1}^{n} \tilde{A}_i\boldsymbol{z}(k-\tau_i) + \Delta\boldsymbol{E}(\boldsymbol{z}(k), k) + \sum_{i=1}^{n} \Delta\tilde{A}_i(\boldsymbol{z}(k-\tau_i), k) \\ &\boldsymbol{z}(k) = \tilde{\phi}(k) = \begin{pmatrix} \phi(k) \\ 0 \end{pmatrix}, \quad k = -\tau, -\tau+1, \cdots, -1, 0 \end{aligned} \tag{7}$$

where

$$\boldsymbol{z}(k) = \begin{pmatrix} \hat{\boldsymbol{x}}(k) \\ \boldsymbol{e}(k) \end{pmatrix}, \ \tilde{A} = \begin{pmatrix} A+BK & LC \\ 0 & A-LC \end{pmatrix}, \ \tilde{A}_i = \begin{pmatrix} 0 & 0 \\ A_i & A_i \end{pmatrix}$$

$$\Delta \boldsymbol{E}(\boldsymbol{z}(k),k)=\begin{pmatrix} L\Delta C(\boldsymbol{x}(k),k) \\ \Delta A(\boldsymbol{x}(k),k)+\Delta B(\boldsymbol{u}(k),k)-L\Delta C(\boldsymbol{x}(k),k)\end{pmatrix}$$

$$\sum_{i=1}^{n}\Delta\tilde{A}_i(\boldsymbol{z}(k-\tau_i),k)=\begin{pmatrix} 0 \\ \sum_{i=1}^{n}\Delta A_i(\boldsymbol{x}(k-\tau_i),k)\end{pmatrix}$$

Assume that the control parameter matrices of (6) are chosen to make matrix $\tilde{A}$ stable (that is, all the eigenvalues of $\tilde{A}$ are inside the unit disc) . Under this condition we have the following delay-independent exponential stability criterion for the closed-loop system (7) with multiple time delays.

Theorem 1. Consider system (7) and assume that $\tilde{A}$ is a stable matrix and satisfies

$$\|\tilde{A}^k\|\leqslant r\eta^k \tag{8}$$

for some real numbers $r>0, 0<\eta<1$ and $k\geqslant 0$. If the inequality

$$\eta+r\Omega+r\sum_{i=1}^{n}(\|\tilde{A}_i\|+\alpha_i)<1 \tag{9}$$

holds, where $\Omega=2\|L\|\gamma+\alpha+\beta\|K\|$ satisfies $\|\Delta\boldsymbol{E}(\boldsymbol{z}(k),k)\|\leqslant\Omega\|\boldsymbol{z}(k)\|$, then the uncertain discrete-time system (7) is exponentially stable.

Proof. From Lemma 1, we represent the solution of system (7) satisfying the initial condition $\boldsymbol{z}(k)=\tilde{\phi}(k), k=-\tau,-\tau+1,\cdots,0$ as

$$\boldsymbol{z}(k,\tilde{\phi}(k))=\tilde{A}^k\tilde{\phi}(0)+\sum_{s=0}^{k-1}\tilde{A}^{k-1-s}\left[\sum_{i=1}^{n}\tilde{A}_i\boldsymbol{z}(s-\tau_i)+\Delta\boldsymbol{E}(\boldsymbol{z}(s),s)+\sum_{i=1}^{n}\Delta\tilde{A}_i(\boldsymbol{z}(s-\tau_i),s)\right]$$

Since $\|A+B\|\leqslant\|A\|+\|B\|$ and $\|AB\|\leqslant\|A\|\|B\|$, performing the norm operation on both sides of the above equation leads to

$$\|\boldsymbol{z}(k,\tilde{\phi}(k))\|\leqslant r\eta^k\|\tilde{\phi}(0)\|+\sum_{s=0}^{k-1}r\eta^{k-1-s}\left[\Omega\|\boldsymbol{z}(s)\|+\sum_{i=1}^{n}(\|\tilde{A}_i\|+\alpha_i)\|\boldsymbol{z}(s-\tau_i)\|\right]$$

We consider the following scalar difference system:

$$P(k)=r\eta^k\|\tilde{\phi}(0)\|+\sum_{s=0}^{k-1}r\eta^{k-1-s}\left[\Omega P(s)+\sum_{i=1}^{n}(\|\tilde{A}_i\|+\alpha_i)P(s-\tau_i)\right] \tag{10}$$

Using the comparison theorem of Lemma 3, one obtains $\|\boldsymbol{z}(k,\tilde{\phi}(k))\|\leqslant P(k)$, for $k\geqslant-\tau$, and $\sup\limits_{k-\tau\leqslant\sigma\leqslant k}\|\boldsymbol{z}(\sigma)\|\leqslant\sup\limits_{k-\tau\leqslant\sigma\leqslant k}P(\sigma)$, $\sup\limits_{-\tau\leqslant\sigma\leqslant 0}\|\boldsymbol{z}(\sigma)\|\leqslant\sup\limits_{-\tau\leqslant\sigma\leqslant 0}P(\sigma)$.

For the comparison difference system (10), we have

$$P(k+1)=\eta P(k)+r\left[\Omega P(k)+\sum_{i=1}^{n}(\|\tilde{A}_i\|+\alpha_i)P(k-\tau_i)\right]\leqslant(\eta+r\Omega)P(k)+r\sum_{i=1}^{n}(\|\tilde{A}_i\|+\alpha_i)\sup_{k-\tau\leqslant\sigma\leqslant k}P(\sigma)$$

It follows from Lemma 2 and (9) that there exists a positive number λ such that

$$P(k)\leqslant\sup_{-\tau\leqslant\sigma\leqslant 0}P(\sigma)e^{-\lambda k}$$

and the inequality holds for $\|\boldsymbol{z}(k,\tilde{\phi}(k))\|$ similarly. Hence, the exponentially stable of $P(k)$ implies the stability of $\boldsymbol{z}(k,\tilde{\phi}(k))$. Consequently, system (7) is exponentially stable, if inequality (9) holds. This completes the proof. □

In the following we propose the procedure of the observer design.

1) Adjusting the coefficients L and K in the observer (6) such that all the eigenvalues of matrix $\tilde{A}$ locate inside the unit circle.

2) Keep on placing the eigenvalues of matrix $\tilde{A}$ toward the origin to ensure that the coefficients of the observer are sufficiently small.

3) Check inequality (9). If the inequality is not satisfied, go step (2). If the inequality is satisfied we stop the procedure, and the observer is well defined.

4 Example

Consider the uncertain discrete system (1) with two time-delays, where $A = \begin{pmatrix} -0.1 & -0.2 \\ 0.1 & -0.4 \end{pmatrix}$, $A_1 = \begin{pmatrix} 0.11 & 0.01 \\ -0.01 & 0.12 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0.02 & 0.01 \\ 0.01 & -0.02 \end{pmatrix}$, $B = C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and suppose that $\alpha = 0.12$, $\alpha_1 = 0.02$, $\alpha_2 = 0.01$, $\beta = 0.15$ and $\gamma = 0.1$. The 1-norm is considered here. If we adjust gain matrices of the dynamic controller (6) to $K = \begin{pmatrix} 0.12 & 0.22 \\ -0.11 & 0.33 \end{pmatrix}$, $L = \begin{pmatrix} -0.13 & -0.21 \\ 0.11 & -0.14 \end{pmatrix}$ and place the eigenvalues of $\tilde{A}$ at $0.025 \pm 0.0132i$, -0.2597 and 0.0297, then we can compute that $\Omega = 0.2725$, and $\eta + r\Omega + r\sum_{i=1}^{n}(\|\tilde{A}_i\| + \alpha_i) = 0.89825 < 1$. Therefore, we obtain the following dynamic controller

$$\hat{\boldsymbol{x}}(k+1) = \begin{pmatrix} -0.1 & -0.2 \\ 0.1 & -0.4 \end{pmatrix} \hat{\boldsymbol{x}}(k) + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{u}(k) + \begin{pmatrix} -0.13 & -0.21 \\ 0.11 & -0.14 \end{pmatrix} \left(\boldsymbol{y}(k) - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \hat{\boldsymbol{x}}(k) \right)$$

$$\boldsymbol{u}(k) = \begin{pmatrix} 0.12 & 0.22 \\ -0.11 & 0.33 \end{pmatrix} \hat{\boldsymbol{x}}(k)$$

By using Theorem 1, system (1) is stabilized *via* the above dynamic controller.

5 Conclusion

By using the comparison theorem and the difference inequality we proposed, the stabilization problem of uncertain discrete-time control systems with time delays are discussed. The controller we designed is the dynamic observer of Luenberg-like type. Our method bypasses the difficulty arisen from constructing Lyapunov functions. The numerical example shows the availability of the proposed method.

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LIU Bi-Yu Received her master degree from the Department of Mathematics, Huazhong Normal University in 1988. She is currently a Ph. D. candidate in School of Information Science and Engineering at Central South University. Her research interests include decentralized control and robust control.

GUI Wei-Hua Professor of Central South University. His research interests include theory and application of decentralized control, complex systems control and process control.