# Robust Control of Uncertain Markov Jump Singularly Perturbed Systems[1)]

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**Abstract** In this paper, we study the robust control for uncertain Markov jump linear singularly perturbed systems (MJLSPS), whose transition probability matrix is unknown. An improved heuristic algorithm is proposed to solve the nonlinear matrix inequalities. The results of this paper can apply not only to standard, but also to nonstandard MJLSPS. Moreover, the proposed approach is independent of the perturbation parameter and therefore avoids the ill-conditioned numerical problems.

**Key words** Singular perturbations, Markov jump parameters, matrix inequality, robust control

## 1 Introduction

Recently, the singular perturbation technique has been a strong tool to study multiple-time-scale systems[1]. On the other hand, Markov jump system has been noticed for many years[2]. In [3] the bounded real property was utilized to study the $H_\infty$ control for Markov jump linear singularly perturbed systems (MJLSPS), which result in a set of coupled Riccati equations. A set of coupled matrix inequlity condition was constructed in [4], and an iterative algorithm was given to solve it. However, the initial values can be obtained only under some conservative conditions. In this paper, the results in [4] are generalized to uncertain cases. Furthermore, a more relaxed algorithm is also proposed.

## 2 Problem formulations

Consider the following uncertain MJLSPS:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \tilde{A}_{11}(r(t))\boldsymbol{x}_1(t) + \tilde{A}_{12}(r(t))\boldsymbol{x}_2(t) + \tilde{B}_1(r(t))\boldsymbol{u}(t) + D_1(r(t))\boldsymbol{w}(t) \\ \varepsilon \cdot \dot{\boldsymbol{x}}_2(t) = \tilde{A}_{21}(r(t))\boldsymbol{x}_1(t) + \tilde{A}_{22}(r(t))\boldsymbol{x}_2(t) + \tilde{B}_2(r(t))\boldsymbol{u}(t) + D_2(r(t))\boldsymbol{w}(t) \\ \boldsymbol{z}(t) = G_1(r(t))\boldsymbol{x}_1(t) + G_2(r(t))\boldsymbol{x}_2(t) + H(r(t))\boldsymbol{u}(t) + L(r(t))\boldsymbol{w}(t) \end{cases} \tag{1}$$

where $\boldsymbol{x}_1(t) \in R^{n_1}$ and $\boldsymbol{x}_2(t) \in R^{n_2}$ are the slow, fast state variables, $\boldsymbol{u}(t) \in R^m$ is the control input, $\boldsymbol{w}(t) \in R^q$ is the external disturbance, $\boldsymbol{z}(t) \in R^p$ is the output vector. $\varepsilon$ is the singular perturbation parameter which satisfies $0 < \varepsilon \ll 1$. $\tilde{A}_{11}(r(t))$, $\tilde{A}_{12}(r(t))$, $\tilde{A}_{21}(r(t))$, $\tilde{A}_{22}(r(t))$, $\tilde{B}_1(r(t))$, $\tilde{B}_2(r(t))$, $D_1(r(t))$, $D_2(r(t))$, $G_1(r(t))$, $G_2(r(t))$, $H(r(t))$ and $L(r(t))$ are the functions of the stochastically jumping process $\{r(t)\}$, where $r(t)$ is a Markov jump process taking values in the finite set $S = \{1, 2, \cdots, s\}$. Denote $\Pi = [\pi_{ij}]$ as the transition matrix, where $i, j = 1, 2, \cdots, s$. Then the trnasition probability is $\Pr\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \pi_{ij}\Delta + o(\Delta), i \neq j \\ 1 + \pi_{ii}\Delta + o(\Delta), i = j \end{cases}$, where $\Delta > 0$, $\pi_{ij} \geqslant 0$, $i \neq j$. For every $i$, we have $\sum_{j=1}^{s} \pi_{ij} = 0$. In this paper, we assume that $\Pi$ is unknown, but can be represented as a polutope, *i.e.*, $\Pi = \sum_{l=1}^{s} \mu_l \Pi_l$, where $\Pi_l = [\pi_{ij}^l]$ is a known transition matrix and

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$\mu_l$ is the unknown scalar satisfying $\sum_{l=1}^{h} \mu_l = 1$. For simplicity, we denote $\tilde{A}_{11}(r(t)) = \tilde{A}_{11i}$ when $r(t) = i$. The unknown matrix can be represented as $\tilde{A}_{11i} = A_{11i} + \Delta A_{11i}$, $\tilde{A}_{21i} = A_{21i} + \Delta A_{21i}$, $\tilde{A}_{12i} = A_{12i} + \Delta A_{12i}$, $\tilde{A}_{22i} = A_{22i} + \Delta A_{22i}$, $\tilde{B}_{1i} = B_{1i} + \Delta B_{1i}$, $\tilde{B}_{2i} = B_{2i} + \Delta B_{2i}$, where $A_{11i}$, $A_{12i}$, $A_{21i}$, $A_{22i}$, $B_{1i}$ and $B_{2i}$ are known matrices. $\Delta A_{11i}$, $\Delta A_{21i}$, $\Delta A_{21i}$, $\Delta A_{22i}$, $\Delta B_{1i}$ and $\Delta B_{2i}$ are uncertain terms which satisfy $\begin{bmatrix} \Delta A_{11i} & \Delta A_{12i} & \Delta B_{1i} \\ \Delta A_{21i} & \Delta A_{22i} & \Delta B_{2i} \end{bmatrix} = \begin{bmatrix} \Gamma_{1i} \\ \Gamma_{2i} \end{bmatrix} \Upsilon_i(t)[\Theta_{1i} \quad \Theta_{2i} \quad Z_i]$, where $\Gamma_{1i} \in R^{n_1 \times n_f}$, $\Gamma_{2i} \in R^{n_2 \times n_f}$, $\Theta_{1i} \in R^{n_f \times n_1}$, $\Theta_{2i} \in R^{n_f \times n_2}$ and $Z_i \in R^{n_f \times m}$ are known matrices. The uncertain matrix $\Upsilon_i(t) \in R^{n_f \times n_f}$ satisfy $\Upsilon_i^{\mathrm{T}}(t)\Upsilon_i(t) \geqq I_{n_f}$. For $r(t) = i$, $i \in S$, we define $\boldsymbol{x}(t) = \begin{bmatrix} \boldsymbol{x}_1(t) \\ \boldsymbol{x}_2(t) \end{bmatrix}$, $\tilde{A}_i = \begin{bmatrix} \tilde{A}_{11i} & \tilde{A}_{12i} \\ \tilde{A}_{21i} & \tilde{A}_{22i} \end{bmatrix}$, $\tilde{B}_i = \begin{bmatrix} \tilde{B}_{1i} \\ \tilde{B}_{2i} \end{bmatrix}$, $D_i = \begin{bmatrix} D_{1i} \\ D_{2i} \end{bmatrix}$, $G_i = [G_{1i} \quad G_{2i}]$, $A_i = \begin{bmatrix} A_{11i} & A_{12i} \\ A_{21i} & A_{22i} \end{bmatrix}$, $B_i = \begin{bmatrix} B_{1i} \\ B_{2i} \end{bmatrix}$, $\Gamma_i = \begin{bmatrix} \Gamma_{1i} \\ \Gamma_{2i} \end{bmatrix}$, $\Theta_i = [\Theta_{1i} \quad \Theta_{2i}]$, $[\Delta A_i \quad \Delta B_i] = \Gamma_i \Upsilon_i(t)[\Theta_i \quad Z_i]$, $E_\varepsilon = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \varepsilon \cdot I_{n_2} \end{bmatrix}$. It is obvious that $\tilde{A}_i = A_i + \Delta A_i$, $\tilde{B}_i = B_i + \Delta B_i$. Finally, (1) can be rewritten as

$$\begin{cases} E_\varepsilon \dot{\boldsymbol{x}}(t) = \tilde{A}_i \boldsymbol{x}(t) + \tilde{B}_i \boldsymbol{u}(t) + D_i \boldsymbol{w}(t) \\ \boldsymbol{z}(t) = G_i \boldsymbol{x}(t) + H_i \boldsymbol{u}(t) + L_i \boldsymbol{w}(t) \end{cases} \tag{2}$$

## 3 Design of $H_\infty$ controller

Consider the state-feedback controller $\boldsymbol{u}(t) = K(r(t))\boldsymbol{x}(t)$. In this case, the closed-loop system becomes

$$\begin{cases} E_\varepsilon \dot{\boldsymbol{x}}(t) = (\tilde{A}(r(t)) + \tilde{B}(r(t))K(r(t)))\boldsymbol{x}(t) + D(r(t))\boldsymbol{w}(t) \\ \boldsymbol{z}(t) = (G(r(t)) + H(r(t))K(r(t)))\boldsymbol{x}(t) + L(r(t))\boldsymbol{w}(t) \end{cases} \tag{3}$$

**Theorem 1.** If there exist matrices $P_{11i} > 0$, $P_{22i} > 0$, $P_{21i}$ and real number $\alpha_i > 0$ for $i = 1, 2, \cdots, s$ and $l = 1, 2, \cdots, h$, such that the following inequalities hold.

$$\Psi_i^l(\alpha_i, K, P_i) \equiv \begin{bmatrix} \left\{ \begin{matrix} (A_i + B_iK_i)^{\mathrm{T}} P_i + P_i^{\mathrm{T}}(A_i + B_iK_i) + \sum_{j=1}^{s} \pi_{ij}^l E P_j + \\ \alpha_i P_i^{\mathrm{T}} \Gamma_i \Gamma_i^{\mathrm{T}} P_i + \alpha_i^{-1}(\Theta_i + Z_iK_i)^{\mathrm{T}}(\Theta_i + Z_iK_i) \end{matrix} \right\} & * & * \\ D_i^{\mathrm{T}} P & -\gamma^2 I & * \\ G_i + H_iK_i & L_i & -I \end{bmatrix} < 0 \tag{4}$$

where $P_i = \begin{bmatrix} P_{11i} & 0 \\ P_{21i} & P_{22i} \end{bmatrix}$ and $E = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}$, then there exists $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*]$, the closed-loop system (3) is robustly stochstically stable, and for any $T_f > 0$, one has $E\{\int_0^{T_f} z^{\mathrm{T}}(t)\boldsymbol{z}(t)\mathrm{d}t\} < \gamma^2 \int_0^{T_f} w^{\mathrm{T}}(t)\boldsymbol{w}(t)\mathrm{d}t$.

The proof is just like those in [4]. In the following, we propose an iterative approach to solve (4), which is different from the one in [4]. First, we define

$$\Sigma_i^l(\alpha_i, K_i, P_i, \lambda) \equiv$$
$$\begin{bmatrix} \left\{ \begin{matrix} (A_i + B_iK_i)^{\mathrm{T}} P_i + P_i^{\mathrm{T}}(A_i + B_iK_i) + \pi_{ii}^l E P_i + \\ \lambda \sum_{j=1, j\neq i}^{s} \pi_{ij}^l E P_j + \alpha_i P_i^{\mathrm{T}} \Gamma_i \Gamma_i^{\mathrm{T}} P_i + \alpha_i^{-1}(\Theta_i + Z_iK_i)^{\mathrm{T}}(\Theta_i + Z_iK_i) \end{matrix} \right\} & * & * \\ D_i^{\mathrm{T}} P & -\gamma^2 I & * \\ G_i + H_iK_i & L_i & -I \end{bmatrix} < 0 \tag{5}$$

where $\lambda$ is a real number in [0,1]. If $P_i$ is fixed as $P_i^*$, $\Sigma_i^l(\alpha_i, K_i, P_i^*, \lambda)$ can be transformed as LMI, and we denote it as $\overrightarrow{\Sigma}_i^l(\alpha_i, K_i, P_i^*, \lambda) < 0$; if $K_i$ is fixed as $K_l^*$ and $\alpha_i$ is fixed as $\alpha_i^*$, $\Sigma_i^l(\alpha_i^*, K_i^*, P_i, \lambda)$ can also be transformed as LMI, and we denote it as $\overleftarrow{\Sigma}_i^l(\alpha_i^*, K_i^*, P_i, \lambda) < 0$. Then, we can summary the iterative algorithm as follows:

**Step 1.** Let $k = 0$, $\Lambda = 0$, $\lambda = 0$. Compute the initial values $\alpha_i(0), K_i(0)$ and $P_i$ which satisfy $\Sigma_i^l(\alpha_i, K_i, P_i, 0) < 0$.

**Step 2.** Let $k = k+1$, $\lambda_k = k/2^{\Lambda}$ and fix $P_i$ as $P_i(k-1)$. If the LMI $\overrightarrow{\Sigma}^l_i(\alpha_i, K_i, P_i(k-1), \lambda_k) < 0$ upon $\alpha_i$ and $K_i$ is feasible, we denote the solutions as $\alpha_i(k)$ and $K_i(k)$. Let $P_i(k) = P_i(k-1)$, then goto Step 4. Otherwise, goto Step 3.

**Step 3.** Fix $\alpha_i$, $K_i$ as $\alpha_i(k-1)$ and $K_i(k-1)$, respectively. If the LMI $\overleftarrow{\Sigma}^l_i(\alpha_i(k-1), K_i(k-1), P_i, \lambda_k) < 0$ upon $P_i$ is feasible, then we can minimize $\sum\limits_{i=1}^{s} trace(P_i)$ suject to $\overleftarrow{\Sigma}^l_i(\alpha_i(k-1), K_i(k-1), P_i, \lambda_k) < 0$. Denote the corresponding solutions as $P_i(k)$. Let $\alpha_i(k) = \alpha_i(k-1)$ and $K_i(k) = K_i(k-1)$, goto Step 4. Otherwise, Let $\Lambda = \Lambda + 1$. If $\Lambda \leqslant \Lambda_{\max}$ ($\Lambda$ is a prescribed threshold), then let $K = 0$ and return to Step 2. If $\Lambda > \Lambda_{\max}$, this algorithm cannot give feasible solutions, it exits.

**Step 4.** If $k < 2^{\Lambda}$, then return to Step 2. If $k = 2^{\Lambda}$, we obtain the feasible solutions $\alpha_i(k), K_i(k)$ and $P_i(k)$.

**Remark.** In [4], one necessary condition for the initial problem is feasible is that each sub-system has to be stabilizable. This is a rather conservative condition for Markov jump systems. In this paper, the solution space of initial problem $\Sigma^l_i(\alpha_i, K_i, P_i, 0) < 0$ is a subset of that of the original problem $\Sigma^l_i(\alpha_i, K_i, P_i, 1) < 0$. Therefore the above-mentioned problem is avoided. In addition, we do not require the input matrix to be square, which is an assumption of [5].

## 4 Conclusions

This paper proposed some new results based on [4]. A more effective algorithm is proposed, which can eliminate some unneccessry assumptions in [4].

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