Matrix Approximation with Constraints of Matrix Inequalities and Applications in Robust Control1)

NIAN Xiao-Hong[1,2] YANG Ying[2] HUANG Lin[2]

[1](*School of Information Science and Engineering, Central South University, Changsha* 410075)

[2](*Department of Mechanics and Engineering Science, Peking University, Beijing* 100871)

(E-mail: xhnian@mech.pku.edu.cn)

Abstract This paper deals with a general class of optimization problems in robust control with given expectation value of performance index, which can be transformed into the problem of matrix approximation with matrix inequalities constraints. Theorem for existence and the uniqueness of the solutions of matrix approximation with linear matrix inequalities (LMIs) constraints is presented and algorithms for solving the problems of matrix approximation with matrix inequalities constraints are given based on the fact that such problems can be converted into the generalized eigenvalue problem (GEVP) of LMI. Examples are given to illustrate the main results of the paper.

Key words Robust control, matrix approximation, uncertainty, LMI, GEVP

1 Introduction

The main assignment of the control theory is to design a controller for a given object by taking into account the limitation of practical realization to ensure that the whole system meet the performance index determined by the practical requirement. In the classical control theory, there are developed theories for stability and optimal control of linear systems. However, in practical systems there exist unavoidable uncertainties and disturbances and it is hard to use a precise mathematical model to describe a practical control object. In the past few years, robust control issues dealing with system parameter uncertainties and external disturbances have attracted more and more research interests[1,2].

When studying an uncertain control object in the engineering practice, the controller design is always implemented according to the performance index of an uncertainty-free system or an expected performance index which meets the practical requirement, so that the whole system performance approximates the performance index of the uncertainty-free system or the expected performance index as close as possible. In the optimal control theory, although the optimal controller can ensure the minimum of the performance index, it does not have good robustness. On the other hand, in the engineering practice, not all system designs pursue the minimal value of the performance index. In some cases, the system performance index is determined by the practical requirement and the controller should be designed in such a way that the system performance can approximate the given index as close as possible. These problems can be reduced to the matrix approximation with the matrix inequality constraints. In recent years, many advances in linear matrix inequalities and algorithms for solving nonlinear matrix inequalities have been reported. However, there are few studies on applications of matrix approximation with LMI constraints in control system design. Therefore, studies on problem of matrix approximation with LMI constraints has not only theoretical interests but also applied significance.

In this paper, we start from some basic problems in the robust control theory and study those problems using the method of matrix approximation with LMI constraints. Furthermore, we study the feasibility of the matrix approximation problem and present the optimization algorithm for the problem of matrix approximation with LMI constraints or some specific nonlinear matrix inequalities constraints.

2 Matrix approximation with constraints of LMI

Consider the following problem of matrix approximation with LMI constraints.

Problem 1. $\min\limits_{P, M_1, \cdots, M_r} \|P - P_0\|$, s.t. $L(P, M_1, \cdots, M_r) \leqslant 0$

1) Supported by National Natural Science Foundation of P. R. China (60474029, 60404007, and 60334030), and the Natural Science Foundation of Hunan Province (00JJY1009)
Received December 10, 2003; in revised form May 23, 2004

where $\| \cdot \|$ denotes the $F-$norm($\| A \| = [trace(A^{\mathrm{T}}A)]^{\frac{1}{2}}$) or $2-$norm ($\| A \| = [\lambda_{\max}(A^{\mathrm{T}}A)]^{\frac{1}{2}}$), $L(P, M_1, \cdots, M_r) \leqslant 0$ is an LMI with respect to matrix variable $P \in R^{n\times n}$. $M_i \in \Gamma_i (i = 1, 2, \cdots, r)$ are unknown real parameter matrices of appropriate dimensions. Γ_i is a bounded convex set and $P_0 \in R^{n\times n}$ is a given real matrix.

First let us introduce a fundamental theorem which gives the existence conditions of a solution to Problem 1.

Theorem 1. (**The existence and uniqueness theorem**) There exists a minimal solution to optimization Problem 1 if there exists a feasible solution to $L(P, M_1, \cdots, M_r) \leqslant 0$. Furthermore, if the matrix norm is $F-$ norm, there exists a unique minimal solution to Problem 1.

Proof. Denote the solution set to $L(P, M_1, \cdots, M_r) \leqslant 0$ as

$$\Gamma = \{(P, M_1, \cdots, M_r) |\ L(P, M_1, \cdots, M_r) \leqslant 0\}$$

Since there exists a feasible solution to $L(P, M_1, \cdots, M_r) \leqslant 0$, we have $\Gamma \neq \emptyset$. Denote

$$\Omega_1 = \{P \mid \ (P, M_1, \cdots, M_r) \in \Gamma\}$$

Then, $\Omega_1 \subset R^{n\times n}$ is a non-null closed subset. For any $P_1 \in \Omega_1$, $\Omega_2 = \{P \mid \| P - P_0 \| \leqslant \| P_1 - P_0 \|\} \subset R^{n\times n}$ is bounded closed convex set. Thus, $\Omega = \Omega_1 \bigcap \Omega_2 \subset R^{n\times n}$ is a non-null bounded closed convex set.

Consider the matrix function $f(P) = \| P - P_0 \|$. Since the matrix norm $\| \cdot \|$ is the continuous convex function of matrix variable, Problem 1 has minimal solution.

In particular, if $f(P) = \| P - P_0 \|_F$, then $f(P)$ is a strictly convex continuous function of the matrix variable P. This deduces that for any $\lambda \in (0, 1)$, $P_1, P_2 \in R^{n\times n}$, $P_1 \neq P_2$, the following inequality holds

$$f(\lambda P_1 + (1 + \lambda)P_2) < \lambda f(P_1) + (1 - \lambda) f(P_2)$$

Since the strictly convex set has a unique minimal solution on a bounded close convex set, the theorem is proved. □

Remark 1. In general, for strict LMI $L(P, M_1, \cdots, M_r) < 0$, $\Gamma = \{(P, M_1, \cdots, M_r) \mid \ L(P, M_1, \cdots, M_r) < 0\}$ is an open set. Thus $\Omega_1 = \{P \mid \ (P, M_1, \cdots, M_r) \in \Gamma\}$ is an open set. $f(P)$ have infimum but does not necessarily have minimum. In this case, we can replace Ω_1, with the set $\Omega^* = \{P \mid \ L(P, M_1, \cdots, M_r) \leqslant -\varepsilon I\}$ (where $\varepsilon > 0$ is a sufficiently small real number). Although this method provides an approximation of the minimum of $f(P)$, it is suitable for finding the feasible solution a the sub-optimal problem. Moreover, we can use $\varepsilon \to 0$ to approximate the infimum.

Proposition 1. 1) If there exist matrices $M_1, M_2, \cdots, M_r$ such that $L(P_0, M_1, \cdots, M_r) \leqslant 0$, the optimal solution of Problem 1 is P_0 and $\min\limits_{P, M_1, \cdots, M_r} \|P - P_0\| = 0$.

2) If there do not exist matrices $M_1, M_2, \cdots, M_r$ such that $L(P_0, M_1, \cdots, M_r) \leqslant 0$, then $\min\limits_{P, M_1, \cdots, M_r} \|P - P_0\| > 0$.

Now, let us discuss the algorithm for the eigenvalue problem with LMI constraints.

When the matrix norm is 2-norm(the matrix norm is assumed to be 2-norm in the following discussion), Problem 1 is equivalent to $\min\limits_{P, M_1, \cdots, M_r} \lambda$ s.t. $\|P - P_0\| \leqslant \lambda$ *and* $L(P, M_1, \cdots, M_r) \leqslant 0$ and the matrix inequality $\|P - P_0\| \leqslant \lambda$ is equivalent to

$$(P - P_0)^{\mathrm{T}}(P - P_0) \leqslant \lambda^2 I \tag{1}$$

Since inequality (1) is equivalent to

$$\begin{bmatrix} \lambda I & (P - P_0)^{\mathrm{T}} \\ P - P_0 & \lambda I \end{bmatrix} \geqslant 0 \tag{2}$$

the optimization Problem 1 can be reduced to the following eigenvalue problem.

Problem 2. $\min\limits_{P, M_1, \cdots, M_r} \lambda$, s.t. $\begin{bmatrix} O & (P - P_0)^{\mathrm{T}} \\ P - P_0 & 0 \end{bmatrix} \leqslant \begin{bmatrix} \lambda I & O \\ O & \lambda I \end{bmatrix}$, $L(P, M_1, \cdots, M_r) \leqslant 0$ which is a standard generalized eigenvalue problem and can be solved using the gevp solver in LMI toolbox[3,4].

3 Applications in control problems

3.1 Robust stability

Consider the following uncertain system

$$\dot{\boldsymbol{x}}(t) = [A + \Delta A(t)]\boldsymbol{x}(t) \tag{3}$$

where $A \in R^{n\times n}$ is a real matrix, $\Delta A(t) \in R^{n\times n}$ is the time-varying uncertain matrix.

Define the performance index function as

$$J(x_0) = \int_0^{\infty} \boldsymbol{x}^{\mathrm{T}}(t)Q\boldsymbol{x}(t)\mathrm{d}t \tag{4}$$

where $Q \in R^{n\times n}$ is a positive definite matrix.

For the uncertainty-free case of System (3), suppose P_0 is the unique positive definite solution of Lyapunov equation

$$A^{\mathrm{T}}P + PA + Q = 0 \tag{5}$$

Since Q can be selected properly, we choose $J_0(\boldsymbol{x}_0) = \boldsymbol{x}_0^{\mathrm{T}}P_0\boldsymbol{x}_0$ as the performance index which describes the disturbance process of the system. But $x^{\mathrm{T}}P_0x$ cannot definitely be used as a Lyapunov function for system family (3). Then our problem is to find a positive definite matrix P such that the system is robustly stable and the corresponding $\boldsymbol{x}_0^{\mathrm{T}}P\boldsymbol{x}_0$ approximates the system performance index $\boldsymbol{x}_0^{\mathrm{T}}P_0\boldsymbol{x}_0$ as close as possible, *i.e.*, to find positive definite matrices P, Q satisfying

$$(A + \Delta A)^{\mathrm{T}}P + P(A + \Delta A) + Q \leqslant 0 \tag{6}$$

and minimizing $\| P - P_0 \|$. In dynamic systems, since the components of x can have different attenuation request, we use a positive definite matrix W to weight each component. But $\boldsymbol{x}^{\mathrm{T}}W\boldsymbol{x}$ cannot be used as a Lyapunov function even for an uncertainty-free system. Thus we turn to find a positive definite matrix P satisfying (6) and minimizing $\| P - W \|$.

Consider the following optimization problem.

Problem 3. $\min\limits_{P} \ \| P - P_0 \|$, s.t. (6).

If the uncertain parameters belong to an interval matrix, *i.e.*, $\Delta A(t) \in G[R, S] \triangleq \{\Delta A \mid R \leqslant \Delta A(t) \leqslant S\}$ and we denote $H[R, S] = \{A_1, A_2, \ldots, A_k,\ k = 2^{n^2}\}$ as the vertex set of the interval matrix $G[R, S]$, then Problem 3 is recast into

$$\begin{aligned} &\min_{P} \ \| P - P_0 \| \\ &\text{s.t. } (A + A_i)^{\mathrm{T}}P + P(A + A_i) + Q \leqslant 0,\ i = 1, 2, \ldots, k \end{aligned}$$

which is a standard matrix approximation problem with the LMI constraints. From the results of the above section, this problem can be reduced to the following generalized eigenvalue problem

$$\begin{aligned} &\min_{P} \ \lambda \\ &\text{s.t. } \begin{bmatrix} O & P - P_0 \\ P - P_0 & O \end{bmatrix} \leqslant \begin{bmatrix} \lambda I & O \\ O & \lambda I \end{bmatrix} \\ &\quad (A + A_i)^{\mathrm{T}}P + P(A + A_i) + Q \leqslant 0,\ i = 1, 2, \ldots, k \end{aligned}$$

If the uncertain parameters satisfy

$$\Delta A(t) = DF(t)E,\ F^{\mathrm{T}}(t)F(t) \leqslant I \tag{7}$$

the matrix inequality (6) is equivalent to that there exists a real number $\varepsilon > 0$ such that the Riccati inequality (see [7])

$$A^{\mathrm{T}}P + PA + \varepsilon^{-1}PDD^{\mathrm{T}}P + \varepsilon E^{\mathrm{T}}E + Q \leqslant 0 \tag{8}$$

holds. Then Problem 3 becomes

$$\min_{P} \ \| P - P_0 \|, \ \text{s.t. (8)}$$

which can be solved through the following generalized eigenvalue problem

$$\begin{aligned}&\min_{\varepsilon,P}\ \lambda\\ &\text{s.t.}\ \begin{bmatrix} O & P-P_0\\ P-P_0 & O\end{bmatrix}\leqslant\begin{bmatrix}\lambda I & O\\ O & \lambda I\end{bmatrix}\\ &\quad\ \begin{bmatrix}A^{\mathrm{T}}P+PA+\varepsilon E^{\mathrm{T}}E+Q & PD\\ D^{\mathrm{T}}P & -\varepsilon I\end{bmatrix}\leqslant 0\end{aligned}$$

3.2 Quadratic robust sub-optimal control

Consider the sub-optimal control problem of the following uncertain linear system[2]

$$\begin{cases}\dot{\boldsymbol{x}}(t)=[A+\Delta A(t)]\boldsymbol{x}(t)+B\boldsymbol{u}(t)\\ \boldsymbol{y}(t)=C\boldsymbol{x}(t)\end{cases}\tag{9}$$

with the performance index function

$$J(\boldsymbol{u},\boldsymbol{x}_0)=\int_0^{\infty}[\boldsymbol{x}^{\mathrm{T}}(t)C^{\mathrm{T}}C\boldsymbol{x}(t)+\boldsymbol{u}^{\mathrm{T}}(t)R\boldsymbol{u}(t)]\mathrm{d}t\tag{10}$$

where $\boldsymbol{x}(t)\in R^n$ is state vector, $\boldsymbol{u}(t)\in R^m$ is the control input, $A\in R^{n\times n}, B\in R^{n\times m}, C\in R^{l\times n}$ are given matrices, $\Delta A\in R^{n\times n}$ are time-varying uncertain parameter matrices satisfying (7).

Without loss of generality, we suppose (A,B) is stabilizable and (A,C) is observable.

For System (9), the system performance index depends on $\Delta A(t)$. Even if $\Delta A(t)$ is time-invariant, the system cannot be guaranteed to be stabilizable or observable. However, on the assumption that the system is robustly stable, the index (10) is still valid. Thus, here we only discuss the robust sub-optimal control problem with the index (10). Specifically, we will study the following problems:

Problem 4. Given the expected upper bound of the index (10) $J_0(x_0)=x_0^{\mathrm{T}}P_0x_0$, design a state feedback controller to robustly stabilize System (9) and minimize

$$|J(u,x_0)-J_0(x_0)|$$

Definition 1. If there exists state feedback control law $u^*(t)=K^*\boldsymbol{x}(t)$ such that System (9) is robustly stable and

$$|J(u^*,x_0)-J_0(x_0)|=\min_u|J(u,x_0)-J_0(x_0)|$$

then we call $u^*(t)=K^*\boldsymbol{x}(t)$ the robust sub-optimal objective control.

As is known, for the optimal state feedback controller of the uncertainty-free linear system[8]

$$\begin{cases}\dot{\boldsymbol{x}}(t)=A\boldsymbol{x}(t)+B\boldsymbol{u}(t)\\ \boldsymbol{y}(t)=C\boldsymbol{x}(t)\end{cases}\tag{11}$$

the index function (10) given by

$$\boldsymbol{u}(t)=-R^{-1}B^{\mathrm{T}}P_0\boldsymbol{x}(t)\tag{12}$$

where $A\in R^{n\times n}, B\in R^{n\times m}, C\in R^{l\times n}$ are given matrices, $R\in R^{m\times m}$ is a positive definite matrix and P_0 is a positive definite solution to the following Riccati equation

$$A^{\mathrm{T}}P+PA-PBR^{-1}B^{\mathrm{T}}P+C^{\mathrm{T}}C=0\tag{13}$$

Then the optimal value of the performance index for the uncertainty-free system is $J(u,x_0)=x_0^{\mathrm{T}}P_0x_0$.

Remark 2. It can be seen from Definition 1 that when we take the performance index of the optimal control for uncertainty-free system as the control objective, the robust optimal objective control is the general robust optimal control. When we take the minimization of the performance index as the control objective, *i.e.*, $P_0=0$, the robust sub-optimal objective control is reduced to the guaranteed cost control[9~13]. So robust optimal (sub-optimal) objective control is a more general concept.

In the following we discuss the design problem of the robust sub-optimal objective controller. First we give a basic result.

Theorem 2. If there exist positive definite matrix P, matrix K and a real number $\varepsilon > 0$ such that

$$A^{\mathrm{T}}P + PA + K^{\mathrm{T}}B^{\mathrm{T}}P + PBK + K^{\mathrm{T}}R^{-1}K + C^{\mathrm{T}}C + \varepsilon PDD^{\mathrm{T}}P + \varepsilon^{-1}E^{\mathrm{T}}E < 0 \tag{14}$$

then the uncertain system (9) is robustly quadratically stable with the state feedback controller $\boldsymbol{u}(t) = K\boldsymbol{x}(t)$ and the system performance index satisfies $J(u, x_0) < x_0^{\mathrm{T}}Px_0$.

Proof. The closed loop system with the state feedback controller $\boldsymbol{u}(t) = K\boldsymbol{x}(t)$ is

$$\dot{\boldsymbol{x}}(t) = [A + \Delta A(t) + BK]\boldsymbol{x}(t) \tag{15}$$

Define the Lyapunov function as $V(\boldsymbol{x}(t)) = \boldsymbol{x}^{\mathrm{T}}(t)P\boldsymbol{x}(t)$; then the derivative of $V(\boldsymbol{x}(t))$ along the trajectory of System (15) is

$$\begin{aligned}\dot{V}(x) =& \boldsymbol{x}^{\mathrm{T}}(t)[A^{\mathrm{T}}P + PA + K^{\mathrm{T}}B^{\mathrm{T}}P + PBK + \Delta A^{\mathrm{T}}P + P\Delta A]\boldsymbol{x}(t) = \boldsymbol{x}^{\mathrm{T}}(t)[A^{\mathrm{T}}P + PA + K^{\mathrm{T}}B^{\mathrm{T}}P + \\ & PBK + C^{\mathrm{T}}C + K^{\mathrm{T}}RK + \Delta A^{\mathrm{T}}P + P\Delta A]\boldsymbol{x}(t) - \boldsymbol{x}^{\mathrm{T}}(t)[C^{\mathrm{T}}C + K^{\mathrm{T}}RK]\boldsymbol{x}(t)\end{aligned}$$

Thus when

$$A^{\mathrm{T}}P + PA + K^{\mathrm{T}}B^{\mathrm{T}}P + PBK + C^{\mathrm{T}}C + K^{\mathrm{T}}RK + \Delta A^{\mathrm{T}}P + P\Delta A < 0 \tag{16}$$

we have

$$\dot{V}(x) \leqslant -\boldsymbol{x}^{\mathrm{T}}(t)[C^{\mathrm{T}}C + K^{\mathrm{T}}RK]\boldsymbol{x}(t) < 0 \tag{17}$$

and the closed loop system (15) is robustly stable. Inequality (16) is equivalent to that there exists $\varepsilon > 0$ such that (14) holds (see [7]).

Integrating both sides of (17) from 0 to T yields

$$V(x(T)) - V(x_0) \leqslant -\int_0^{\mathrm{T}} [C^{\mathrm{T}}C + K^{\mathrm{T}}RK]\mathrm{d}t \tag{18}$$

Since the closed loop system (15) is asymptotically stable, we have

$$\lim_{t\longrightarrow\infty} x(T) = 0, \lim_{t\longrightarrow\infty} V(x(T)) = 0$$

Then

$$J(u, x_0) = \int_0^{\infty} [\boldsymbol{x}^{\mathrm{T}}(t)C^{\mathrm{T}}C\boldsymbol{x}(t) + u^{\mathrm{T}}(t)R\boldsymbol{u}(t)]\mathrm{d}t \leqslant V(x_0) = x_0^{\mathrm{T}}Px(0)$$

which completes the proof. □

From Theorem 2 we know

$$|J(u, x_0) - J_0(x_0)| \leqslant |x_0^{\mathrm{T}}Px_0 - x_0^{\mathrm{T}}P_0x_0| = |x_0^{\mathrm{T}}(P - P_0)x_0| \leqslant \| x_0 \|^2 \| P - P_0 \|$$

Then Problem 4 can be reduced to the optimization problem of $\min\limits_{P,K} \| P - P_0 \|$ s.t.(14).

By virtue of Schur complement formula, the matrix inequality (14) is equivalent to

$$\begin{bmatrix} A^{\mathrm{T}}P + PA + K^{\mathrm{T}}B^{\mathrm{T}}P + PBK + \varepsilon PDD^{\mathrm{T}}P & K^{\mathrm{T}} & E^{\mathrm{T}} & C^{\mathrm{T}} \\ K & -R & O & O \\ E & O & -\varepsilon I & O \\ C & O & O & -I \end{bmatrix} < 0 \tag{19}$$

Pre- and postmultiplying both sides of the above inequality by $Diag[\ P^{-1} \quad I \quad I \quad I\ ]$ and letting $P^{-1} = X,\ KP = W$, yield

$$\begin{bmatrix} XA^{\mathrm{T}} + AX + W^{\mathrm{T}}B^{\mathrm{T}} + BW + \varepsilon DD^{\mathrm{T}} & W^{\mathrm{T}} & XE^{\mathrm{T}} & XC^{\mathrm{T}} \\ W & -R & O & O \\ EX & O & -\varepsilon I & O \\ CX & O & O & -I \end{bmatrix} < 0 \tag{20}$$

which can be solved through the following eigenvalue problem.

Problem 5. $\min\limits_{\varepsilon,X,W} \lambda$, s.t. $\begin{bmatrix} O & X - X_0 \\ X - X_0 & O \end{bmatrix} < \begin{bmatrix} \lambda I & O \\ O & \lambda I \end{bmatrix}$, (20)

Theorem 3. If there exist real number ε, positive definite matrix P such that the following Riccati inequality

$$A^{\mathrm{T}}P + PA - PBR^{-1}B^{\mathrm{T}}P + C^{\mathrm{T}}C + \varepsilon PDD^{\mathrm{T}}P + \varepsilon^{-1}E^{\mathrm{T}}E < 0 \tag{21}$$

holds, then the uncertain system (9) is robustly quadratically stable with the state feedback controller (12) and the performance index (10) satisfies $J(u, x_0) < x_0^{\mathrm{T}}Px_0$.

Proof. Inequality (21) can be derived by substituting $K = -R^{-1}B^{\mathrm{T}}P$ into (14).

Then Problem 4 can be reduced to the matrix approximation problem of $\min\limits_{P,K} \| P - P_0 \|$ s.t.(21), which can be solved through the following generalized eigenvalue problem.

Problem 6.

$$\begin{aligned} &\min_{\varepsilon, X} \ \lambda \\ &\text{s.t.} \begin{bmatrix} O & X - X_0 \\ X - X_0 & O \end{bmatrix} < \begin{bmatrix} \lambda I & O \\ O & \lambda I \end{bmatrix} \\ &\begin{bmatrix} XA^{\mathrm{T}} + AX - BR^{-1}B^{\mathrm{T}} + \varepsilon DD^{\mathrm{T}} & XE^{\mathrm{T}} & XC^{\mathrm{T}} \\ EX & -\varepsilon I & O \\ CX & O & -I \end{bmatrix} < 0 \end{aligned}$$

4 Numerical examples

Example 1. Consider the system $\dot{\boldsymbol{x}}(t) = A_i\boldsymbol{x}(t)$ $(i = 1, 2, 3)$, where $A_1 = \begin{bmatrix} -1 & 2 \\ 1 & -3 \end{bmatrix}$, $A_2 = \begin{bmatrix} -0.8 & 1.5 \\ 1.3 & -2.7 \end{bmatrix}$, $A_3 = \begin{bmatrix} -1.4 & 0.9 \\ 0.7 & -2.0 \end{bmatrix}$. Our problem is to find a positive definite matrix P which satisfies

$$A_i^{\mathrm{T}}P + PA_i < 0, \ i = 1, 2, 3$$

and minimizes $\| P - P_0 \|$, where P_0 is a given symmetric matrix. This problem can be translated into the following generalized eigenvalue minimization problem

$$\begin{aligned} &\min_{P} \ \lambda \\ &\text{s.t.} \begin{bmatrix} O & P - P_0 \\ P - P_0 & O \end{bmatrix} < \begin{bmatrix} \lambda I & O \\ O & \lambda I \end{bmatrix} \\ &A_i^{\mathrm{T}}P + PA_i < 0, \ i = 1, 2, \dots, k \end{aligned}$$

and the minimal solution can be obtained by solving the above generalized eigenvalue minimization problem using the gevp solver in LMI Toolbox.

Let $P_0 = \begin{bmatrix} 3 & 1.5 \\ 1.5 & 1.5 \end{bmatrix}$; then P_0 is a feasible solution to (1), Thus, the minimal solution $P = P_0$. Let $P_0 = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$. We can obtain the minimal solution $P = \begin{bmatrix} 3.0958 & 1.8814 \\ 1.8814 & 1.9042 \end{bmatrix}$. In this case, $\| P - P_0 \| < 0.1524$.

Example 2. Consider the optimal robust control problem (9) with the performance index (10), where $\Delta A(t) = DF(t)E$, $F^{\mathrm{T}}F(t) \leqslant I$, $A = \begin{bmatrix} -2 & -1 \\ 0 & \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $C = [1\ 1]$, $R = 1$, $D = \begin{bmatrix} 0.4 & 1 \\ 1 & 0.4 \end{bmatrix}$, $E = \begin{bmatrix} 0.3 & 0.3 \\ 0.3 & 0.3 \end{bmatrix}$. We can obtain the optimization solution of the uncertainty-free system $P_0 = \begin{bmatrix} 0.2271 & 0.0757 \\ 0.0757 & 2.2271 \end{bmatrix}$. Solving Problem 6, we obtain the minimal solution as $X = \begin{bmatrix} 1.7745 & -0.0069 \\ -0.0069 & 0.2090 \end{bmatrix}$, $\varepsilon = 0.4158$. Thus, $P = \begin{bmatrix} 1.7745 & -0.0069 \\ -0.0069 & 0.2090 \end{bmatrix}$, and the optimization control is $\boldsymbol{u}(t) = [-0.5822\ \ -4.8038]$ $\boldsymbol{x}(t)$. In this case, $\| P - P_0 \| < 2.5595$.

5 Conclusions

In this paper we study some basic robust control issues such as the roust stability, robust optimal control, *etc.*, which can be transformed into the problem of matrix approximation with LMI constraints. The existence conditions for the unique optimal solution to such problems are presented, as well as the algorithms for matrix approximation with matrix inequalities. It is shown that the matrix approximation with LMI constraints or matrix inequalities can be reduced to the generalized eigenvalue problem of LMI.

References

1 Zhou K M, Doyle J C, Glover K. Robust optimal control, Englewood Cliffs: Prentice Hall, 1996

2 Huang Lin. Theoretic Foundation of Stability and Robustness, Beijing: Science Press, 2003

3 Boyd S, Cgaoui L E, Feron E, Balakrishnan V. Linear matrix inequalities in system and control theory, Philadelphia, SIAM, 1994

4 Pascal Gahinet, Nemirovski Nemirovski, Alan Laub, Mahmoud Chilali. The LMI control toolbox, Natick: The Math Works Inc. 1995

5 Jeremy G V, Richard D B. A tutorial on linear and bilinear matrix inequalities, *Journal of Process Control*, 2003, **10**: 363~385

6 Xie L. Output feedback H_∞ control of systems with parameter uncertainty, *International Journal of Control*, 1996, **63**: 741~750

7 Wang You-Yi, Xie Li-Hua, Carlos E, De Souza. Robust control of a class of uncertain nonlinear systems, *Systems Control Letters*, 1992, **19**: 139~149

8 Huang Lin, Zheng Ying-Ping, Zhang Di. The second Lyapunov method and the problem of optimal controller analysis and design, *Acta Automatica Sinica*, 1964, **2**(4): 203~218

9 Petersen I R, Hollot C V. A Riccati equation approach to the stabilization of uncertain linear systems, *Automatica*, 1986, **22**(4): 397~411

10 Marsh C, Wei H. Robustness bounds for systems with Parametric uncertianty, *Automatica*, 1996, **32**(10): 1447~1453

11 Douglas J, Athans M. Robust linear quadratic designs with real parameter uncertainties. *IEEE Transactions on Automatic Control*, 1994, **39**(1): 107~111

12 Moheimani S O R, Peterson I R. Optimal guaranteed cost control of uncertain systems via static and dynamic output feedback. *Automatica*, 1996, **32**(4): 575~579

13 Neto A T, Dion J M. Robustness bounds for LQ regulator, *IEEE Transactions on Automatic Control*, 1992, **37**(9): 1373~1377

NIAN Xiao-Hong Received his master and Ph. D. degrees from Shandong University and Peking University in 1992 and 2004, respectively. He is currently a professor at Central South University. His research interests include robust control, optimal control, differential games, and electric traction control.

YANG Ying Received her Ph. D. degree from Peking University in 2003. She is currently a postdoctor in the Department of Mechanics and Engineering Science at Peking University. Her research interests include robust control and nonlinear systems control theory.

HUANG Lin Academician of Chinese Acadamy of Sciences. He is a professor in the Department of Mechanics and Engineering Science at Peking University. His research interests include stability theory and its applications, robust control, and complex control systems theory.