# Suboptimal Control for Discrete Linear Systems with Time-delay: A Successive Approximation Approach[1)]

TANG Gong-You WANG Hai-Hong
(*College of Information Science and Engineering, Ocean University of China, Qingdao* 266071)
(E-mail: gtang@ouc.edu.cn)

**Abstract** A successive approximation approach for designing optimal controllers is presented for discrete linear time-delay systems with a quadratic performance index. By using the successive approximation approach, the original optimal control problem is transformed into a sequence of nonhomogeneous linear two-point boundary value (TPBV) problems without time-delay and time-advance terms. The optimal control law obtained consists of an accurate feedback terms and a time-delay compensation term which is the limit of the solution sequence of the adjoint equations. By using a finite-step iteration of the time-delay compensation term of the optimal solution sequence, a suboptimal control law is obtained. Simulation examples are employed to test the validity of the proposed approach.

**Key words** Discrete systems, time-delay systems, optimal control, suboptimal control, successive approximation approach

## 1 Introduction

The optimal control for discrete control systems with time-delay is an important task in the control theory. It is difficult to solve the optimal control problem for time-delay systems with the quadratic performance indices, which has attracted the interest of many researchers from the mathematical and control community[1,2]. We may transform the discrete time-delay systems into equivalent higher order systems in mathematics, but two new problems may arise. First, the order of the transformed system would increase by the product of the time-delay and the order of the systems. For some long time-delay or high order systems, the computing work would increase exponentially, *i.e.*, the "dimension disaster" may arise. Thus, the above method can only be applied to some small time-delay or low order systems[3]. Second, after the transformation the system may not maintain its controllability and/or observability, and the Riccati equation fails to satisfy the conditions of the optimal solution. According to the necessary optimality conditions, the optimal control problem for discrete linear time-delay systems with a quadratic performance index may lead to a discrete linear two-point boundary value (TPBV) problem in which both time-delay (in state) and time-advance (in co-state) terms are involved. To find the optimal solution is a very difficult undertaking. To avoid the computational difficulties in solving the optimal control law, the optimal guaranteed-cost control problem[4,5] has recently received considerable attention. But the control algorithm cannot guarantee an optimal level of performance. Therefore, it is more practical to find new approximate methods offering tradeoffs between computation complexity and precision. Most often the TPBV problem is solved by approximate methods leading to a suboptimal solution[6~8].

A successive approximation approach is presented for the optimal control of discrete linear time-delay systems with a quadratic performance index. In the approach presented we consider the time-delay terms in the state equation of the system as external disturbance inputs. A sequence of non-delay discrete linear systems are constructed whose solution sequence uniformly converges to the solution of the original discrete time-delay system. Thus, the time-delay optimal control problem is transformed into solving an optimal control sequence of non-delay systems. By using a finite-step iteration of the time-delay compensation term in the optimal solution sequence, we obtain a suboptimal control law. Simulations show that the algorithm proposed can reduce computation and is easily implemented within the given precision.

1) Supported by National Natural Science Foundation of P. R. China (60074001), and the Natural Science Foundation of Shandong Province (Y2000G02)
Received May 14, 2003; in revised form July 16, 2004

## 2 Problem statement

Consider the discrete linear system with state delay described by

$$\begin{aligned} &\boldsymbol{x}(k+1)=A\boldsymbol{x}(k)+A_1\boldsymbol{x}(k-h)+B\boldsymbol{u}(k), \quad k=0,1,2,\cdots \\ &\boldsymbol{x}(k)=\boldsymbol{\varphi}(k), \quad k=-h,-h+1,\cdots,0 \end{aligned} \tag{1}$$

where $\boldsymbol{x}\in R^n$, $\boldsymbol{u}\in R^r$ are the state vector and the control vector, respectively; $A, A_1$ and $B$ are constant matrices of appropriate dimensions, $\boldsymbol{\varphi}(k)$ is the initial state function, and $h\in N=\{1,2,\cdots\}$ is a positive integer time-delay.

The quadratic performance index of system (1) is selected as

$$J=\frac{1}{2}\boldsymbol{x}^{\mathrm{T}}(N)Q_f\boldsymbol{x}(N)+\frac{1}{2}\sum_{k=0}^{N-1}[\boldsymbol{x}^{\mathrm{T}}(k)Q\boldsymbol{x}(k)+\boldsymbol{u}^{\mathrm{T}}(k)R\boldsymbol{u}(k)] \tag{2}$$

where $R\in R^{r\times r}$ is a positive-definite matrix, $Q, Q_f\in R^{n\times n}$ are positive-semidefinite matrices. The original system′s optimal control problem is to find a control law $\boldsymbol{u}^*(k)$ such that the quadratic performance index (2) is minimized, while satisfying the dynamic equality constraint (1).

It is well known that the optimal control problem (1) and (2), by using the necessary optimality conditions, may lead to the following TPBV problem

$$\begin{aligned} &\boldsymbol{\lambda}(k)=\begin{cases} Q\boldsymbol{x}(k)+A^{\mathrm{T}}\boldsymbol{\lambda}(k+1)+A_1^{\mathrm{T}}\boldsymbol{\lambda}(k+h+1), & k=0,1,2,\cdots,N-h-1 \\ Q\boldsymbol{x}(k)+A^{\mathrm{T}}\boldsymbol{\lambda}(k+1), & k=N-h,N-h+1,\cdots,N-1 \end{cases} \\ &\boldsymbol{x}(k+1)=A\boldsymbol{x}(k)+A_1\boldsymbol{x}(k-h)-BR^{-1}B^{\mathrm{T}}\boldsymbol{\lambda}(k+1), \quad k=0,1,2,\cdots,N-1 \\ &\boldsymbol{x}(k)=\boldsymbol{\varphi}(k), \quad k=-h,-h+1,\cdots,0 \\ &\boldsymbol{\lambda}(N)=Q_f\boldsymbol{x}(N) \end{aligned} \tag{3}$$

The optimal control law can be described by

$$\boldsymbol{u}(k)=-R^{-1}B^{\mathrm{T}}\boldsymbol{\lambda}(k+1) \tag{4}$$

The TPBV problem in (3) involves both time-delay term $\boldsymbol{x}(k-h)$ and time-advance term $\boldsymbol{\lambda}(k+h+1)$, thus to find its exact solution is very difficult. In this paper, we will introduce a successive approximation approach to solve the optimal control problem of (1) and (2).

## 3 Preliminaries

We first propose two useful preliminary lemmas.

**Lemma 1.** The follow inequality holds

$$\sum_{i=1}^{k} i^j \leqslant \frac{(k+1)^{j+1}}{j+1}, \quad j=0,1,2,\cdots;\ k\in N \tag{5}$$

**Proof.** According to the Riemann integral definition, one gets

$$\frac{1}{k+1}\sum_{i=1}^{k}\left(\frac{i}{k+1}\right)^j \leqslant \int_0^1 x^j \mathrm{d}x=\frac{1}{j+1}, \quad j=0,1,2,\cdots;\ k\in N \tag{6}$$

The proof is complete. □

Consider autonomous discrete linear time-varying systems with time-delay described by

$$\begin{aligned} &\boldsymbol{x}(k+1)=G(k)\boldsymbol{x}(k)+G_1(k)\boldsymbol{x}(k-h), \quad k=0,1,2,\cdots,N-1 \\ &\boldsymbol{x}(k)=\boldsymbol{\varphi}(k), \quad k=-h,-h+1,\cdots,0 \end{aligned} \tag{7}$$

where $\boldsymbol{x}\in R^n$ is the state vector, $\boldsymbol{\varphi}(k)$ is the initial state vector, $G, G_1$ are time-varying matrices of appropriate dimensions, Define the vector function sequence $\{\boldsymbol{x}^{(j)}\}$ as

$$\boldsymbol{x}^{(0)}(k)=\prod_{m=1}^{k} G(k-m)\boldsymbol{\varphi}(0)$$

$$\boldsymbol{x}^{(j)}(k)=\prod_{m=1}^{k}G(k-m)\boldsymbol{\varphi}(0)+\sum_{i=0}^{k-1}\Big[\prod_{m=1}^{k-i-1}G(k-m)\Big]G_1(i)\boldsymbol{x}^{(j-1)}(i-h) \tag{8}$$
$$\boldsymbol{x}^{(j)}(k)=\boldsymbol{\varphi}(k),\quad k=-h,-h+1,\cdots,0;\quad j\in N$$

where $\prod_{m=1}^{0}G(k-m)=I$, $I$ is the unit matrix.

**Lemma 2.** The sequence $\{\boldsymbol{x}^{(j)}\}$ described by (8) uniformly converges to solution of the system (7).

**Proof.** From (8), we have

$$\boldsymbol{x}^{(1)}(k)-\boldsymbol{x}^{(0)}(k)=\sum_{i=0}^{k-1}\Big[\prod_{m=1}^{k-i-1}G(k-m)\Big]G_1(i)\boldsymbol{x}^{(0)}(i-h)=$$
$$\sum_{i=0}^{h}\Big[\prod_{m=1}^{k-i-1}G(k-m)\Big]G_1(i)\boldsymbol{\varphi}(i-h)+\sum_{i=h+1}^{k-1}\Big[\prod_{m=1}^{k-i-1}G(k-m)\Big]G_1(i)\Big[\prod_{m=1}^{i-h}G(i-h-m)\Big]\boldsymbol{\varphi}(0) \tag{9}$$

Let

$$\bar{a}=\sup_{k\in[0,N)}\left\{\left\|\prod_{m=1}^{k-i-1}G(k-m)\right\|,\ i=0,1,\cdots,k-j\right\}$$
$$a=\max\{1,\ \bar{a}\},\quad b=\sup_{k\in[0,N]}\|G_1(k)\|,\quad c=\sup_{k\in[-h,0]}\|\boldsymbol{\varphi}(k)\| \tag{10}$$

where $\bar{a},a,b$ and $c$ are some positive scalar constants. From (9) and (10), we have

$$\|\boldsymbol{x}^{(1)}(k)-\boldsymbol{x}^{(0)}(k)\|\leqslant abc[(h+1)+a(k-h-1)]\leqslant a^2bck \tag{11}$$

Similarly,

$$\|\boldsymbol{x}^{(2)}(k)-\boldsymbol{x}^{(1)}(k)\|\leqslant ab\sum_{i=1}^{k-1}\|\boldsymbol{x}^{(1)}(i)-\boldsymbol{x}^{(0)}(i)\|\leqslant a^3b^2c\sum_{i=1}^{k-1}i\leqslant a^3b^2c\frac{k^2}{2!} \tag{12}$$

By analogy and according to Lemma 1, one gets

$$\|\boldsymbol{x}^{(j)}(k)-\boldsymbol{x}^{(j-1)}(k)\|\leqslant\sum_{i=1}^{k-1}ab\|\boldsymbol{x}^{(j-1)}(i-h)-\boldsymbol{x}^{(j-2)}(i-h)\|\leqslant\frac{a^{j+1}b^jc}{(j-1)!}\sum_{i=1}^{k-1}i^{j-1}\leqslant a^{j+1}b^jc\frac{k^j}{j!},\ j\in N \tag{13}$$

From (13) and applying trigonometry inequality, it follows that

$$\|\boldsymbol{x}^{(j+l)}(k)-\boldsymbol{x}^{(j-1)}(k)\|\leqslant ac\sum_{i=j}^{j+l}\frac{(abk)^i}{i!},\quad\forall j,l\in N \tag{14}$$

For sequence $\{\boldsymbol{x}^{(j)}(k)\}$, once $k$ has been fixed in (14), $k$ is considered as a parametric variable. Inequality (14) implies

$$\lim_{k\to\infty}\|\boldsymbol{x}^{(j+l)}(k)-\boldsymbol{x}^{(j-1)}(k)\|=0,\quad\forall k,l\in N \tag{15}$$

Therefore, when $k$ is fixed, $\{\boldsymbol{x}^{(j)}(k)\}$ is a Cauchy sequence, *i.e.*, the sequence is uniformly convergent[9].

From (7) and (8) we obtain

$$\boldsymbol{x}(k)=\prod_{m=1}^{k}G(k-m)\boldsymbol{\varphi}(0)+\sum_{i=0}^{k-1}\Big[\prod_{m=1}^{k-i-1}G(k-m)\Big]G_1(i)\boldsymbol{x}(i-h)$$
$$\boldsymbol{x}(k)=\boldsymbol{\varphi}(k),\quad k=-h,-h+1,\cdots,0 \tag{16}$$

Since $l$ in (14) is arbitrary, the limit of sequence (8) uniformly converges to (16), and the limit of sequence $\{\boldsymbol{x}^{(j)}(k)\}$ is clearly the solution to system (7). □

## 4 Design of suboptimal control law

Construct the following TPBV problem sequence

$$\begin{aligned}
&\boldsymbol{x}^{(0)}(k)=\boldsymbol{\lambda}^{(0)}(k)=0\\
&\boldsymbol{\lambda}^{(j)}(k)=\begin{cases} Q\boldsymbol{x}^{(j)}(k)+A^{\mathrm{T}}\boldsymbol{\lambda}^{(j)}(k+1)+A_1^{\mathrm{T}}\boldsymbol{\lambda}^{(j-1)}(k+h+1), & k=0,1,2,\cdots,N-h-1\\ Q\boldsymbol{x}^{(j)}(k)+A^{\mathrm{T}}\boldsymbol{\lambda}^{(j)}(k+1), & k=N-h,N-h-1,\cdots,N-1\end{cases}\\
&\boldsymbol{x}^{(j)}(k+1)=A\boldsymbol{x}^{(j)}(k)+A_1\boldsymbol{x}^{(j-1)}(k-h)-BR^{-1}B^{\mathrm{T}}\boldsymbol{\lambda}^{(j)}(k+1)\\
&\boldsymbol{x}^{(j)}(k)=\boldsymbol{\varphi}(k), k=-h,-h+1,\cdots,0\\
&\boldsymbol{\lambda}^{(j)}(N)=Q_f\boldsymbol{x}(N),\quad j\in N
\end{aligned}\tag{17}$$

and the $j$-th iterative optimal control law is

$$\boldsymbol{u}^{(j)}(k)=-R^{-1}B^{\mathrm{T}}\boldsymbol{\lambda}^{(j)}(k+1),\quad j\in N\tag{18}$$

For the optimal control problem (1) with the quadratic performance index (2), we propose the following results based on a successive approximation approach.

**Theorem 1.** The optimal control law of (1) with the quadratic performance index (2) is

$$\boldsymbol{u}^*(k)=-S^{-1}(k+1)B^{\mathrm{T}}P(k+1)[A\boldsymbol{x}(k)+A_1\boldsymbol{x}(k-h)+\lim_{j\to\infty}\boldsymbol{g}^{(j)}(k+1)]\tag{19}$$

where $S(k+1)=R+B^{\mathrm{T}}P(k+1)B$, the matrix $P(k)$ is the unique positive-semidefinite solution of the following Riccati matrix difference equation

$$P(k)=A^{\mathrm{T}}[I-P(k+1)BS^{-1}(k+1)B^{\mathrm{T}}]P(k+1)A+Q,\quad P(N)=Q_f\tag{20}$$

and the $j$-th adjoint vector $\boldsymbol{g}^{(j)}(k)$ is the solution to the following adjoint vector difference equations

$$\begin{aligned}
&\boldsymbol{g}^{(0)}(k)=\boldsymbol{g}^{(1)}(k)=0\\
&\boldsymbol{g}^{(j)}(k)=\begin{cases} A^{\mathrm{T}}[I-P(k+1)BS^{-1}(k+1)B^{\mathrm{T}}][\boldsymbol{g}^{(j)}(k+1)+P(k+1)A_1\boldsymbol{x}^{(j-1)}(k-h)]+\\ \qquad A_1^{\mathrm{T}}P(k+h+1)\boldsymbol{x}^{(j-1)}(k+h+1)+A_1^{\mathrm{T}}\boldsymbol{g}^{(j-1)}(k+h+1),\quad k=0,1,2,\cdots,N-h-1\\ A^{\mathrm{T}}[I-P(k+1)BS^{-1}(k+1)B^{\mathrm{T}}][\boldsymbol{g}^{(j)}(k+1)+P(k+1)A_1\boldsymbol{x}^{(j-1)}(k-h)],\\ \qquad k=N-h,N-h+1,\cdots,N-1\end{cases}\\
&\boldsymbol{g}^{(j)}(N)=0,\quad j=2,3,\cdots
\end{aligned}\tag{21}$$

where $\boldsymbol{x}^{(j-1)}$ in (21) can be solved by the following equations

$$\begin{aligned}
&\boldsymbol{x}^{(0)}(k)=\boldsymbol{0},\quad k=-h,-h+1,\cdots,0,1,\cdots\\
&\boldsymbol{x}^{(j-1)}(k+1)=[I+BR^{-1}B^{\mathrm{T}}P(k+1)]^{-1}[A\boldsymbol{x}^{(j-1)}(k)+A_1\boldsymbol{x}^{(j-2)}(k-h)-BR^{-1}B^{\mathrm{T}}\boldsymbol{g}^{(j-1)}(k+1)]\\
&\boldsymbol{x}^{(j-2)}(k)=\boldsymbol{\varphi}(k),\quad k=-h,-h+1,\cdots,0;\quad j=2,3,\cdots
\end{aligned}\tag{22}$$

**Proof.** By Lemma 2, the solution sequence of the TPBV problems in (17) uniformly converges to the solution of the optimal solutions of (1) and (2). To decouple TPBV problems in (17), let

$$\boldsymbol{\lambda}^{(j)}(k)=P(k)\boldsymbol{x}^{(j)}(k)+\boldsymbol{g}^{(j)}(k),\quad j=0,1,2,\cdots\tag{23}$$

Using the final equation in (17) and (23), it follows that

$$\boldsymbol{g}^{(j)}(N)=\boldsymbol{0}\tag{24}$$

From (17), (18) and (23) we can obtain (20), (21), (22) and the $j$th optimal control law

$$\boldsymbol{u}^{(j)}(k)=-S^{-1}(k+1)B^{\mathrm{T}}P(k+1)[A_1\boldsymbol{x}^{(j)}(k)+A_1\boldsymbol{x}^{(j)}(k-h)]-S^{-1}(k+1)B^{\mathrm{T}}\boldsymbol{g}^{(j)}(k+1)\tag{25}$$

By Lemma 2, the optimal control law of (1) and (2) is as follows

$$\boldsymbol{u}^*(k)=\lim_{j\to\infty}\boldsymbol{u}^{(j)}(k)=-S^{-1}(k+1)B^{\mathrm{T}}P(k+1)[A\boldsymbol{x}(k)+A_1\boldsymbol{x}(k-h)+\lim_{j\to\infty}\boldsymbol{g}^{(j)}(k+1)]\tag{26}$$

The proof is complete. □

**Remark 1.** From (21), we have

$$\boldsymbol{g}^{(0)}(k)=\boldsymbol{g}^{(1)}(k)=\boldsymbol{0},\quad \boldsymbol{g}^{(j)}(k)=\sum_{i=0}^{N-k-1}\bar{A}^{i-1}\boldsymbol{f}^{(j-1)}(k+i),\quad j=2,3,\cdots \tag{27}$$

where

$$\bar{A}=A^{\mathrm{T}}-A^{\mathrm{T}}P(k+1)BS^{-1}(k+1)B^{\mathrm{T}}$$

$$\boldsymbol{f}^{(j)}(k)=\begin{cases}A_1^{\mathrm{T}}P(k+h+1)\boldsymbol{x}^{(j)}(k+h+1)+A_1^{\mathrm{T}}\boldsymbol{g}^{(j)}(k+h+1)+\\ \quad A^{\mathrm{T}}P(k+1)[A_1-BS^{-1}(k+1)B^{\mathrm{T}}P(k+1)A_1],\quad k=0,1,2,\cdots,N-h-1\\ A^{\mathrm{T}}P(k+1)[A_1-BS^{-1}(k+1)B^{\mathrm{T}}P(k+1)A_1],\quad k=N-h,N-h+1,\cdots,N-1\end{cases} \tag{28}$$

Therefore, the adjoint equations in (21) can be replaced with those in (27).

In fact, it is almost impossible to obtain $\boldsymbol{g}^{(\infty)}(k+1)$ in (19). We may, in practical applications, get a suboptimal control law by replacing $\infty$ with a finite positive integer $M$ in (19). Therefore the Mth order suboptimal control law can be obtained as follows.

$$\boldsymbol{u}_M(k)=-S^{-1}(k+1)B^{\mathrm{T}}P(k+1)[A\boldsymbol{x}(k)+A_1\boldsymbol{x}(k-h)]-S^{-1}(k+1)B^{\mathrm{T}}\boldsymbol{g}^{(M)}(k+1) \tag{29}$$

The computational procedure to determine the $j$-th order suboptimal control law can be summarized as follows.

**Algorithm 1.**

**Step 1.** Solve $P(k)$ from Riccati matrix difference equation (20). Let $J_0=\infty, j=1, \boldsymbol{x}^{(0)}(k)=\boldsymbol{g}^{(0)}(k)=\boldsymbol{g}^{(1)}(k)=\boldsymbol{0}$. Give some positive constant $\varepsilon>0$.

**Step 2.** Obtain the jth adjoint vector $\boldsymbol{g}^{(j)}(k)$ from (21) or (27).

**Step 3.** Letting $M=j$, Calculate $\boldsymbol{u}_M(k)$ from (29).

**Step 4.** Calculate $J_M$ from

$$J_M=\frac{1}{2}\boldsymbol{x}^{\mathrm{T}}(N)Q_f\boldsymbol{x}(N)+\frac{1}{2}\sum_{k=0}^{N-1}\{\boldsymbol{x}^{\mathrm{T}}(k)Q\boldsymbol{x}(k)+[\boldsymbol{u}_M(k)]^{\mathrm{T}}R\boldsymbol{u}_M(k)\} \tag{30}$$

**Step 5.** If $|(J_M-J_{M-1})/J_M|<\varepsilon$ then stop and output $\boldsymbol{u}_M(k)$, else calculate the $j$-th order state vector $\boldsymbol{x}^{(j)}(k)$ from (22).

**Step 6.** Letting $j=j+1$, go to Step 2.

**Remark 2.** $\boldsymbol{x}(k)$ and $\boldsymbol{x}(k-h)$ in the first term in (29) are the accurate solutions. Only the second term $\boldsymbol{g}^{(M)}(k+1)$ in (29) is the $M$-th iterative result in place of $\boldsymbol{g}^{(\infty)}(k+1)$, thus the suboptimal control law $\boldsymbol{u}_M(k)$ is closer to the optimal control law $\boldsymbol{u}^*(k)$ than the $M$-th iterative optimal control $\boldsymbol{u}^{(M)}(k)$.

**Remark 3.** From the proof of Lemma 2, the finite time $N$ in quadratic performance index (2) can be very large. In practical control system, if the finite time $N$ is sufficiently large, we can take $N\to\infty$ as true. Therefore, the algorithm mentioned above is compatible to the situation as $N\to\infty$. Thus the quadratic performance index (2) can be transformed to

$$J=\frac{1}{2}\sum_{k=0}^{\infty}[\boldsymbol{x}^{\mathrm{T}}(k)Q\boldsymbol{x}(k)+\boldsymbol{u}^{\mathrm{T}}(k)R\boldsymbol{u}(k)] \tag{31}$$

If $(A,B)$ and $(A,Q^{\frac{1}{2}})$ are controllable and observable, respectively, the Riccati matrix difference equation in (20) can be transformed into the following discrete Riccati matrix equation

$$A^{\mathrm{T}}PA-P-A^{\mathrm{T}}PBS^{-1}B^{\mathrm{T}}PA+Q=0 \tag{32}$$

where the unique positive-definite matrix $P$ is a constant matrix.

Correspondingly, the time-varying matrix $P(\cdot)$ in each expression can be substituted by constant positive-definite matrix $P$, the $j$-th adjoint vector $\boldsymbol{g}^{(j)}(k)$ can be found from the following equations

$$\boldsymbol{g}^{(0)}(k)=\boldsymbol{g}^{(1)}(k)=\boldsymbol{0}$$

$$\boldsymbol{g}^{(j)}(k) = A^{\mathrm{T}}[I - PBS^{-1}B^{\mathrm{T}}][\boldsymbol{g}^{(j)}(k+1) + PA_1\boldsymbol{x}^{(j-1)}(k-h)]+$$
$$A_1^{\mathrm{T}}P\boldsymbol{x}^{(j-1)}(k+h+1) + A_1^{\mathrm{T}}\boldsymbol{g}^{(j-1)}(k+h+1), \quad k = 1, 2, \cdots$$
$$\boldsymbol{g}^{(j)}(N) = 0, \quad j = 2, 3, \cdots \tag{33}$$

The state vector $\boldsymbol{x}^{(j-1)}$ in (33) is the solution to the followings

$$\boldsymbol{x}^{(0)}(k) = \boldsymbol{0}$$
$$\boldsymbol{x}^{(j-1)}(k+1) = [I + BR^{-1}B^{\mathrm{T}}P]^{-1}[A\boldsymbol{x}^{(j-1)}(k) + A_1\boldsymbol{x}^{(j-2)}(k-h) - BR^{-1}B^{\mathrm{T}}\boldsymbol{g}^{(j-1)}(k+1)]$$
$$\boldsymbol{x}^{(j-2)}(k) = \boldsymbol{\varphi}(k), \quad k = -h, -h+1, \cdots, 0, \ j = 2, 3, \cdots \tag{34}$$

In this case, the computational procedure to determine the $j$-th order suboptimal control law is similar to Algorithm 1.

## 5 Simulation examples

Consider a second-order discrete linear system with time-delay described by

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.57 \\ 0.9 & 2.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_x(k) \end{bmatrix} + \begin{bmatrix} 0.4 & 0.228 \\ 0.36 & 1 \end{bmatrix} \begin{bmatrix} x_1(k-h) \\ x_2(k-h) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \boldsymbol{u}(k)$$
$$\boldsymbol{x}(k) = [1 \quad 0]^{\mathrm{T}}, \quad k = -h, -h+1, \cdots, 0 \tag{35}$$

The quadratic performance index is selected as

$$J = \frac{1}{2}\sum_{k=0}^{\infty}(x_1^2(k) + x_2^2(k) + \boldsymbol{u}^2(k)) \tag{36}$$

For time-delay $h = 1$, the simulation results and the performance index values with respect to the iteration times $j = 1, 2, 3, 4$ are shown in Fig. 1 and listed in Table 1, respectively. It is obvious that $J_1 > J_2 > J_3 > J_4$. If $\varepsilon = 0.01$, we obtain $|(J_4 - J_3)/J_4| = 0.0068 < \varepsilon$. Thus $u_4(k)$ may be considered as a suboptimal control law. Fig. 1 and Table 1 show the algorithm is valid for the optimal control of discrete linear time-delay systems. It is clear that the more the iteration times, the better the control precision. With the increase of the iteration times, the errors of the curves are smaller and smaller, which indicates the suboptimal control law is close to the optimal control law sufficiently.

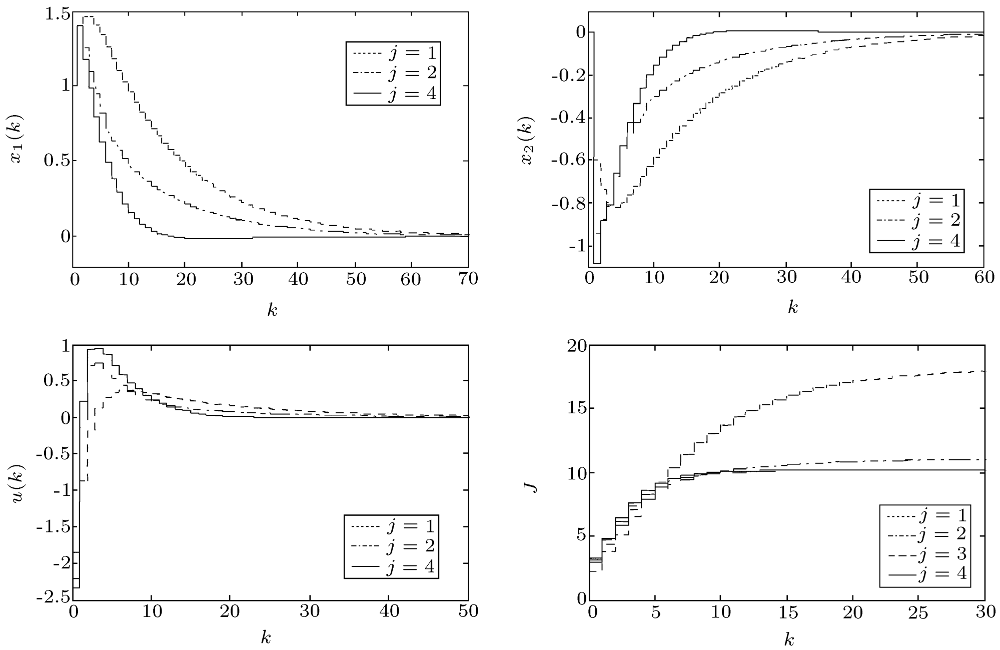


Fig. 1 Simulation curves of the system when $h = 1$

Table 1 Performance index values at the different iteration times when $h = 1$

| iteration time $j$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| performance index value $J_M$ | 18.1550 | 11.0616 | 10.1822 | 10.1138 |

In order to make a further study on the relations between time-delays and convergent speed, we consider the performances at the different time delays of $h$. When $M = 4$, the simulation results of different time delays are shown in Fig. 2, and the performance index values are listed in Table 2.

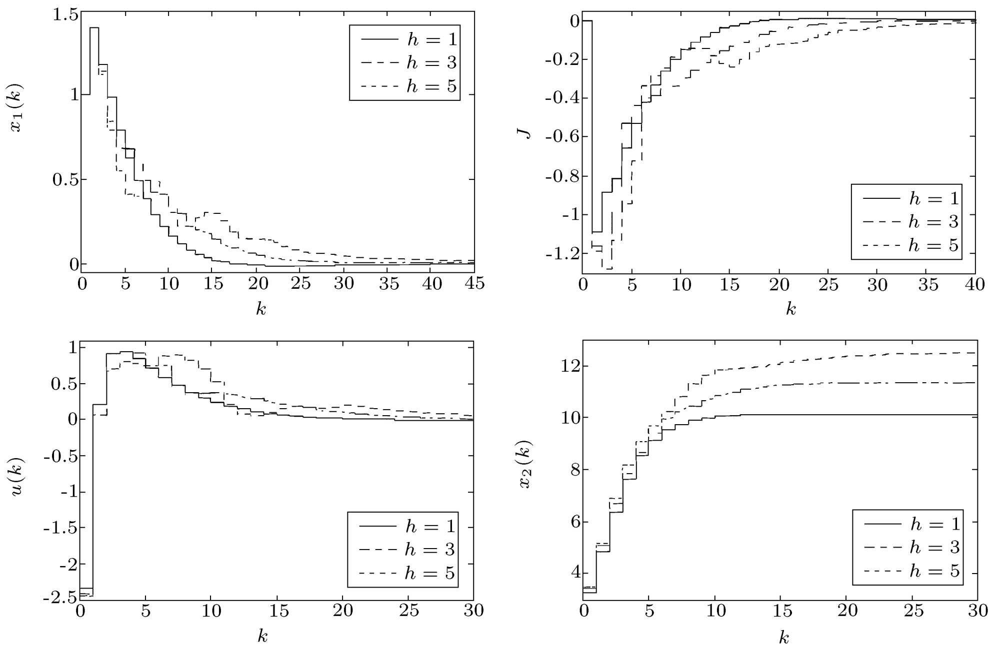


Fig. 2 Simulation curves of the system when $M = 4$

Table 2 Performance index values at different delays when $M = 4$

| time-delay $h$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| performance index value $J_M$ | 10.1138 | 10.8078 | 11.3501 | 11.8916 | 12.4897 |

Fig. 2 and Table 2 show that with the increase of the time delay, the settling time of the system increases correspondingly. Note that too more iterations would affect the computing speed of the systems, thus it is more practical to find a tradeoff between the iteration times and the precision.

## 6 Conclusion

In this paper, a successive approximation approach for the optimal control of a discrete linear system with state delay is presented. It is clear that the errors of the suboptimal control law mainly come from the results of the truncation of $\boldsymbol{g}^{(M)}(k)$. The smaller of the delay is, the smaller the errors would be, which make the suboptimal control trajectory very close to the theoretic optimal control law, and the iteration times $M$ can be smaller accordingly. Thus the computing work can be reduced. An example is given to test the validity of the algorithm. The results show that the algorithm is easy to implement, the convergence speed is high, and the computing work is small.

## References

1 Liu Yong-Qing, Tang Gong-You. Theory and Application of Large-Scale Dynamic Systems: Delays, Stability and Control. Guangzhou: The South China University of Technology Press, 1992

2 Malek-Zavarei M, Jamshidi M. Time-Delay Systems: Analysis, Optimization and Applications. New York: North-Holland Press, 1987

3 Chu J. Application of a discrete optimal tracking controller to an industrial electric heater with pure delays, *Journal of Process Control*, 1995, **5**(1): 3∼8

4 Liu Fei, Su Hong-Ye, Jiang Peng-Gang, Chu Jian. Guaranteed cost control with $H_\infty$ disturbance attenuation for polytopic uncertain discrete-time systems with delay, *Control and Decision,* 2002, **17**(1): 103∼106

5 Yu Li, Feng Hua. Guaranteed cost control of discrete-time uncertain time-delay systems. *Acta Automatic Sinica*, 2001, **27**(3): 392∼396

6 Mariani L, Nicoletti B. Optimal discrete systems with pure delays. *IEEE Transactions on Automatic Control*, 1973, **18**(3): 311∼313

7 Lu Peng-Fei, Tang Gong-You, Jia Xiao-Bo, Tao Ye. Successive approximation approach of suboptimal control for nonlinear time-delay systems. *Control and Decision*, 2004, **19**(2): 230∼234

8 Tang Gong-You, Wang Fang. Suboptimal control for linear large-scale systems with small time-delay. *Control Theory & Applications*, 2003, **20**(1): 281∼286

9 Taylor A E, Lay D C. Introduction to Functional Analysis (2nd ed.), New York: John Wiley and Sons, Inc., 1980



**TANG Gong-You** Professor of the College of Information Science and Engineering, Ocean University of China. He received his Ph. D. degree from the South China University of Technology in 1991. His main research interests include analysis and control for time-delay systems and nonlinear systems, theory and applications of large-scale systems.

**WANG Hai-Hong** Ph. D. candidate of College of Information Science and Engineering, Ocean University China. Her research interests include computer control, analysis and control for time-delay systems, and nonlinear systems.