On the Optimal Control Problem of a Nonlinear Elliptic Population System[1)]

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Abstract The optimal control problem for a nonlinear elliptic population system is considered. First, under certain hypotheses, the existence and uniqueness of coexistence state solutions are shown. Then the existence of the optimal control is given and the optimality system is established.

Key words Optimal control, nonlinear elliptic population system, sub-supper solution, optimality system

1 Introduction

We consider the optimal control problem for the following nonlinear elliptic system with Dirichlet boundary conditions

$$\begin{cases} -\triangle y_i = -u_i(x)y_i + \sum\limits_{j=1,j\neq i}^{N} a_{ij}(x)y_j - c_i y_i \sum\limits_{j=1}^{N} y_j, & \text{in } \Omega \\ y_i = 0, & \text{on } \partial\Omega \end{cases}, \quad i = 1, 2, \cdots, N \tag{1}$$

where $\Omega \subset \mathbb{R}^m$ is an open and connected bounded domain with smooth boundary $\partial\Omega$. $u_i(x)$ and $a_{ij}(x) \in L^\infty_+(\Omega)$ for $i, j \in \mathbb{Z}_N \stackrel{\triangle}{=} \{1, 2, \cdots, N\}$ and $j \neq i$, where $L^\infty_+(\Omega) = \{g \in L^\infty(\Omega) | g(x) \geqslant 0 \ a.e. \text{ in } \Omega\}$, and $c_1, c_2, \cdots, c_N$ are positive constants.

System (1) arises from population dynamics where it models the steady-state of a cooperative system of N populations. $y_1(x), \cdots, y_N(x)$ represent the concentrations of the populations, and the Laplacian operator shows the diffusive characterization of each y_i within Ω. $u_1(x), \cdots, u_N(x)$ reflect the results of harvesting a portion of each population due to fishing. $c_1, \cdots, c_N$ measure the strength of the crowding effect and the competition among $y_1, \cdots, y_N$. The weight function a_{ij} describes certain relation between y_i and y_j, such as the rate of y_j produced by y_i, *etc.* Moreover, the Dirichlet boundary condition may be interpreted as the condition that no population stays on $\partial\Omega$. We refer to [1∼4] and the references therein for more details on the applied background.

For $i \in \mathbb{Z}_N$ and $\delta_i > 0$, we define $C_{\delta_i} = \{g \in L^\infty_+(\Omega) | 0 \leqslant g \leqslant \delta_i \ a.e. \text{ in } \Omega \}$, and the admissible control set $\mathcal{U}_{ad} = \Pi_{i=1}^N C_{\delta_i}$. Let $U = [u_1, u_2, \cdots, u_N] \in \mathcal{U}_{ad}$ be an admissible control. Then it is natural for us to be interested in finding non-trivial and non-negative solutions to system (1). In fact, for each $U \in \mathcal{U}_{ad}$, system (1) has a unique positive solution $Y = [y_1, y_2, \cdots, y_N]$ (see Theorem 1 and Theorem 2). The cost-benefit functional $J : \mathcal{U}_{ad} \to \mathbb{R}$ is defined as

$$J(U) = \sum_{i=1}^{N} \int_\Omega \left(\lambda_i y_i(x) u_i(x) - (u_i(x))^2 \right) \mathrm{d}x \tag{2}$$

with $\lambda_i > 0, i = 1, \cdots, N$. The optimal control problem we will study reads

$$(P) \quad \max\{J(U) \,\Big|\, U = [u_1, u_2, \cdots, u_N] \in \mathcal{U}_{ad}\}$$

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As special cases where $N=2$ and $N=1$, such a problem was studied in [1~4] with either Neumann or Dirichlet boundary conditions.

The article is organized as follows. The existence and uniqueness of the coexistence state solution to (1) are given in Section 2. In Section 3, we show the existence of the optimal control to optimal controls of problem (P) and give the optimality conditions satisfied by the optimal control.

2 Existence and uniqueness of coexistence state solution

For each $q(\cdot)\in L^\infty(\Omega)$, let $\rho_1(q)$ represent the principal eigenvalue of the eigenvalue problem

$$\begin{cases} -\triangle y + qy = \rho y, & \text{in } \Omega \\ y = 0, & \text{on } \partial\Omega \end{cases} \tag{3}$$

It is well known that $\rho_1(q)$ is simple and verifies the variational characterization

$$\rho_1(q) = \inf\left\{ \int_\Omega |\nabla y|^2 \mathrm{d}x + \int_\Omega q|y|^2 \mathrm{d}x \Big| y \in H_0^1(\Omega),\ \int_\Omega |y|^2 \mathrm{d}x = 1 \right\}$$

and that it is possible to choose an associated eigenfunction $\varphi_1(q)\in C^{1,\alpha}(\bar{\Omega}), \forall\alpha\in(0,1)$, such that $\varphi_1(q)>0$ in Ω, and $\|\varphi_1(q)\|_{L^\infty(\Omega)}=1$. The following properties are direct consequences of the previous variational characterization: 1) $q_1\leqslant q_2 \Longrightarrow \rho_1(q_1)\leqslant\rho_1(q_2)$, and $\rho_1(q_1)<\rho_1(q_2)$ if $q_1<q_2$ on some subset of Ω with positive measure; 2) $\forall\delta\in\mathbb{R}$, $\rho_1(q+\delta)=\rho_1(q)+\delta$, and $\varphi_1(\delta)=\varphi_1(0)$. Throughout this paper, we use the following notations: for $\boldsymbol{X}=[x_1,\cdots,x_N], \boldsymbol{Y}=[y_1,\cdots,y_N]$, $\boldsymbol{X}\leqslant\boldsymbol{Y}$ if $x_i\leqslant y_i$ for $i\in\mathbb{Z}_N$, and

$$\underline{g} = \text{ess-}\inf_{x\in\Omega} |g(x)|, \quad \bar{g} = \text{ess-}\sup_{x\in\Omega} |g(x)| \quad \text{for} \quad g(\cdot)\in L^\infty(\Omega),$$
$$\underline{a}_i = \min\left\{\underline{a}_{ij} \Big| j\in\mathbb{Z}_N, j\neq i\right\}, \quad \bar{a}_i = \max\left\{\bar{a}_{ij}\Big| j\in\mathbb{Z}_N, j\neq i\right\}, \quad \forall i\in\mathbb{Z}_N,$$
$$\alpha = \max\left\{\underline{a}_i \Big| i\in\mathbb{Z}_N\right\}, \quad \gamma = \min\left\{c_i \Big| i\in\mathbb{Z}_N\right\}$$

Theorem 1. For system (1), assume that

$$\bar{a}_i - \underline{a}_i \leqslant \frac{1}{N-1}\frac{\gamma}{\alpha}\frac{\underline{a}_i^2}{c_i}, \quad \forall i\in\mathbb{Z}_N \tag{H$_1$}$$

$$\underline{a}_i\underline{a}_j > \rho_1(\delta_i)\rho_1(\delta_j), \quad \forall i,j\in\mathbb{Z}_N \tag{H$_2$}$$

where $\delta_1,\cdots,\delta_N$ are some fixed constants, Then $\forall\boldsymbol{U}=[u_1,\cdots,u_N]\in\mathcal{U}_{ad}$, there exists at least one coexistence state solution $\boldsymbol{Y}$ to (1) with each component $y_i>0$ strictly in Ω.

Proof. Let $\boldsymbol{W}=[w_1,\cdots,w_N]\triangleq[\underline{a}_1/c_1,\underline{a}_2/c_2,\cdots,\underline{a}_N/c_N]$. By hypothesis (H$_1$), we get

$$(\bar{a}_i-\underline{a}_i)\frac{\underline{a}_j}{c_j} - \frac{1}{N-1}\frac{\underline{a}_i^2}{c_i} \leqslant 0, \quad \forall j\in\mathbb{Z}_N\setminus\{i\}$$

Then for any y_j with $0\leqslant y_j\leqslant \underline{a}_j/c_j$, we have

$$-\triangle w_i \geqslant -u_i w_i + \sum_{j\neq i} a_{ij}y_j - c_i w_i\left(\sum_{j\neq i} y_j + w_i\right) \tag{4}$$

which means that W is a supper solution to system (1). We now show that $\boldsymbol{Z}=[z_1,\cdots,z_N]\triangleq[\nu_1\tau\varphi_1(0),\nu_2\tau\varphi_1(0),\cdots,\nu_N\tau\varphi_1(0)]$ is a subsolution to (1) for $\tau>0$ small enough. In fact, first we choose positive constants $\nu_1,\cdots,\nu_N$, satisfying

$$\frac{\underline{a}_i}{\rho_1(\delta_i)} > \frac{\nu_i}{\nu_j} > \frac{\rho_1(\delta_j)}{\underline{a}_j}, \quad \forall\, i,j\in\mathbb{Z}_N, i\neq j \tag{5}$$

Then we select a positive τ which is independent of $i \in \mathbb{Z}_N$ such that $\nu_i \rho_1(\delta_i) \leqslant \nu_j \underline{a}_i - \tau c_i \nu_i((N-1)\nu_j + \nu_i)\varphi_1(0)$ for $i \in \mathbb{Z}_N$. Multiplying both sides of the above inequality by $\frac{1}{N-1}\tau\varphi_1(0)$, we obtain

$$\frac{1}{N-1}\nu_i\tau\varphi_1(0)\rho_1(0)+\frac{1}{N-1}\nu_i\tau\varphi_1(0)\delta_i \leqslant \frac{1}{N-1}\nu_j\tau\varphi_1(0)\Big(\underline{a}_i-(N-1)c_i\nu_i\tau\varphi_1(0)\Big)-\frac{1}{N-1}c_i(\tau\nu_i\varphi_1(0))^2$$

Now let $y_j \geqslant \nu_j \tau\varphi_1(0)$, then we have

$$z_i\rho_1(0) \leqslant -u_i z_i + \sum_{j\neq i} a_{ij} y_j - c_i z_i\bigg(\sum_{j\neq i} y_j + z_i\bigg)$$

Since $-\triangle z_i = \rho_1(0) z_i$, we obtain

$$-\triangle z_i \leqslant -u_i z_i + \sum_{j\neq i} a_{ij} y_j - c_i z_i\bigg(\sum_{j\neq i} y_j + z_i\bigg)$$

which implies that $\boldsymbol{Z}$ is a subsolution to (1). Thus we conclude that system (1) has at least one positive solution $\boldsymbol{Y} = [y_1, \cdots, y_N] \in (\boldsymbol{W}^{2,p}(\Omega))^N$ for any $p \in (1, \infty)$ with $\nu_i\tau\varphi_1(0) \leqslant y_i \leqslant \underline{a}_i/c_i, i \in \mathbb{Z}_N$. This completes the proof. □

By the regularity of elliptic equations, for the coexistence state solution $\boldsymbol{Y}$ to system (1), $y_i \in C^{1,\alpha}(\bar{\Omega}), \forall \alpha \in (0,1), i \in \mathbb{Z}_N$. We denote $[0, \underline{a}_i/c_i] = \{v \in C(\bar{\Omega}) | 0 \leqslant v(x) \leqslant \underline{a}_i/c_i \text{ in } \bar{\Omega}\}$.

Lemma 1. Let the hypotheses (H$_1$) and (H$_2$) are satisfied. Then for any fixed $y_j \in [0, \underline{a}_j/c_j], j \in \mathbb{Z}_N \setminus \{i\}$, the elliptic system

$$\begin{cases} -\triangle y = -u_i(x) y + \sum\limits_{j\neq i} a_{ij}(x) y_j - c_i y\bigg(\sum\limits_{j\neq i} y_j + y\bigg), & \text{in } \Omega \\ y = 0, & \text{on } \partial\Omega \end{cases} \tag{6}$$

has a unique non-negative solution $y_i \in [0, \underline{a}_i/c_i]$.

Proof. From the proof of Theorem 1, we have seen that $z_i \equiv 0$ and $w_i \equiv \underline{a}_i/c_i$ are a pair of sub-supper solution to (6). Now let $\underline{y}_i, \bar{y}_i \in \boldsymbol{W}^{2,p}(\Omega)$ for $p > 1$ be the minimal and maximal coexistence state solutions to (6). Then $\underline{y}_i \leqslant \bar{y}_i$ and

$$-\triangle(\bar{y}_i - \underline{y}_i) = -u_i(\bar{y}_i - \underline{y}_i) - c_i(\bar{y}_i^2 - \underline{y}_i^2) - c_i(\bar{y}_i - \underline{y}_i)\sum_{j\neq i} y_j \leqslant 0$$

By the maximum principal, it follows that $\underline{y}_i = \bar{y}_i$, which implies that there exists a unique non-negative solution y_i satisfying $0 \leqslant y_i \leqslant \underline{a}_i/c_i$. Moreover, if $y_j = 0$ for all $j \in \mathbb{Z}_N \setminus \{i\}$, then $y_i = 0$, and if there exists some $j \neq i$ such that $y_j \geqslant 0$, $y_j \neq 0$ in Ω, then $y_i > 0$. □

Now we define the map $P_i : D^i \triangleq \prod_{j\neq i}[0, \underline{a}_j/c_j] \to [0, \underline{a}_i/c_i]$ as follows: for $Y^i \in D^i$ with components $y_j \in [0, \underline{a}_j/c_j], j \in \mathbb{Z}_N \setminus \{i\}$, $P_i(Y^i)$ is the non-negative solution to (6). Here and after we use the symbol X^i to represent the $(N-1)$-vector obtained by omitting the i-th component x_i from an N-vector $X = [x_1, \cdots, x_N]$ for the sake of convenience. By Lemma 1, P_i is well defined, and if there is a $y_j > 0$ strictly, then $P_i(Y^i) > 0$. Moreover, P_i has the following two important properties.

Lemma 2. Let the hypotheses (H$_1$) and (H$_2$) are satisfied. Then 1) P_i is strongly sublinear, i.e., for all $t \in (0,1)$, $P_i(tY^i) > tP_i(Y^i), \forall Y^i \in D^i$, $Y^i \neq 0$; 2) P_i is strongly increasing, i.e., for $Y^i, V^i \in D^i$, $Y^i \leqslant V^i \Longrightarrow P_i(Y^i) \leqslant P_i(V^i)$. Moreover, the above inequality is strict if $y_j < v_j$ strictly for some j.

Proof. We denote $\tilde{y}_i = P_i(y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_N)$. It is clear that $t\tilde{y}_i = 0$ on $\partial\Omega$ and

$$-\triangle(t\tilde{y}_i) < -u_i(t\tilde{y}_i) + \sum_{j\neq i} a_{ij}(ty_j) - c_i(t\tilde{y}_i)\bigg(\sum_{j\neq i} ty_j + t\tilde{y}_i\bigg)$$

which implies that $t\tilde{y}_i$ is a subsolution to system (6) for tY^i. So we obtain $P_i(tY^i) > tP_i(Y^i)$. If $Y^i \leqslant V^i$, then

$$-\triangle\tilde{y}_i \leqslant -u_i\tilde{y}_i + \sum_{j\neq i} a_{ij} v_j - c_i\tilde{y}_i\bigg(\sum_{j\neq i} v_j + \tilde{y}_i\bigg)$$

which implies that $\tilde{y}_i$ is a subsolution to system (6) for V^i. Consequently we get $P_i(Y^i) \leqslant P_i(V^i)$. If $y_j < v_j$ strictly for some j, then the above inequality is strict by Lemma 1. □

The uniqueness of the coexistence state solution is based on the following lemma which was developed in [5].

Lemma 3. Let $(E, \mathcal{C})$ be an ordered Banach space whose positive cone $\mathcal{C}$ has non-empty interior. Suppose that $\mathcal{D}$ is a star-shaped subset of $\mathcal{C}$ containing 0, and let $f: \mathcal{D} \to E$ be a strongly sublinear and strongly increasing map. Then f has at most one positive fixed point. Moreover, if $f(e) > e$, then f has no positive fixed point in the ordered interval $[0, e]$.

Theorem 2. Under the hypotheses (H_1) and (H_2), the coexistence state solution to system (1) is unique.

Proof. Let $E = C(\bar{\Omega}), \mathcal{C} = \{u \in C(\bar{\Omega}) | u \geqslant 0\}$, and $\mathcal{D} = \prod_{i=1}^{N}[0, \underline{a}_i/c_i]$. It is easy to verify that $(E^N, \mathcal{C}^N)$ is an ordered Banach space, and that $\mathcal{D}$ is a star-shaped subset of $\mathcal{C}^N$ containing 0. The map $\Gamma: \mathcal{D} \to E^N$ is defined as follows:

$$\Gamma(Y) = [P_1(Y^1), P_2(Y^2), \cdots, P_N(Y^N)], \quad \forall Y \in \mathcal{D}$$

By Lemma 1, Γ is well defined. It follows from Lemma 2 that Γ is strongly sublinear and strongly increasing. Moveover, if Y is a coexistence state solution to system (1), then it is a non-negative fixed point for Γ. Conversely, any positive fixed point for Γ is a coexistence state solution for syetem (1). As shown in Theorem 1, Γ has at least one positive fixed point. Then by Lemma 3, we conclude that the coexistence state solution to system (1) is unique. □

3 Existence and characterization of optimal control

In this section, we first show the existence of optimal controls of problem (P), and then characterize them by the state system coupled with the adjoint one.

Theorem 3. Under hypotheses (H_1) and (H_2), the optimal control problem (P) has at least one solution, i.e., $\exists \boldsymbol{U}^* \triangleq [u_1^*, u_2^*, \cdots, u_N^*] \in \mathcal{U}_{ad}$, such that

$$J(\boldsymbol{U}^*) = \sup \left\{ J(\boldsymbol{U}) \middle| \boldsymbol{U} = [u_1, u_2, \cdots, u_N] \in \mathcal{U}_{ad} \right\}$$

Proof. Let $d \triangleq \sup \left\{ J(\boldsymbol{U}) | \boldsymbol{U} \in \mathcal{U}_{ad} \right\}$. Obviously, d is finite. Let $\{\boldsymbol{U}_n\}_{n=1}^{\infty} \subset \mathcal{U}_{ad}$ be a maximizing sequence such that $\lim_{n\to\infty} J(\boldsymbol{U}_n) = d$. Then there exists a subsequence of $\{\boldsymbol{U}_n\}$, still denoted by itself, such that

$$w - \lim_{n\to\infty} \boldsymbol{U}_n = \boldsymbol{U}^* \text{ in } (L^2(\Omega))^N \tag{7}$$

for some $\boldsymbol{U}^* = [u_1^*, u_2^*, \cdots, u_N^*] \in (L^2(\Omega))^N$. Denote $\boldsymbol{U}_n = [u_1^n, u_2^n, \cdots, u_N^n]$. Since $\{u_i^n\}_{n=1}^{\infty} \subset C_{\delta_i}$, we have $\boldsymbol{U}^* \in \mathcal{U}_{ad}$. Let $Y_n \in (W^{2,2}(\Omega))^N$ be the corresponding coexistence state solution to system (1) for $\boldsymbol{U}_n$. Noting that $(W^{2,2}(\Omega))^N$ is compactly imbedded in $(W^{1,2}(\Omega))^N$ and $\{Y_n\}_{n=1}^{\infty} \subset (W^{2,2}(\Omega))^N$ is bounded, we can extract a subsequence of $\{Y_n\}_{n=1}^{\infty}$, still denoted by itself, such that $\lim_{n\to\infty} Y_n = Y^*$ in $(W^{1,2}(\Omega))^N$ with some $Y^* = [y_1^*, y_2^*, \cdots, y_N^*] \in (W^{1,2}(\Omega))^N$. Letting $n \to \infty$ in (1) leads to that Y^* is a weak solution to (1) corresponding to U^*. By (7) we have

$$d = \lim_{n\to\infty} \sum_{i=1}^{N} \int_{\Omega} (\lambda_i y_i^n u_i^n - |u_i^n|^2) \mathrm{d}x \leqslant \sum_{i=1}^{N} \int_{\Omega} (\lambda_i y_i^* u_i^* - |u_i^*|^2) \mathrm{d}x = J(\boldsymbol{U}^*)$$

which implies that $J(U^*) = d$. This completes the proof. □

Lemma 4. For any fixed $\boldsymbol{U} \in \mathcal{U}_{ad}$, let $\boldsymbol{Y}$ be the corresponding coexistence state solution to system (1). Then for any $\boldsymbol{F} = [f_1, f_2, \cdots, f_N] \in (L^{\infty}(\Omega))^N$, the following system

$$\begin{cases} -\triangle \xi_i + \left(u_i + c_i\left(y_i + \sum\limits_{j=1}^{N} y_j\right)\right)\xi_i - \sum\limits_{j\neq i}(a_{ij} - c_i y_i)\xi_j = -f_i y_i, & \text{in } \Omega \\ \xi_i = 0, & \text{on } \partial\Omega \end{cases} \tag{8}$$

for $i = 1, 2, \cdots, N$, has a unique solution $\boldsymbol{\xi} = [\xi_1, \xi_2, \cdots, \xi_N] \in (H_0^1(\Omega))^N$.

Proof. Let L be the following diagonal matrix operator of second order elliptic operators:

$$L = \begin{pmatrix} L_1 & & 0 \\ & \ddots & \\ 0 & & L_N \end{pmatrix} \quad \text{with} \quad L_i = -\triangle + \Big(u_i + c_i\Big(y_i + \sum_{j=1}^N y_j\Big)\Big)I, \quad i \in \mathbb{Z}_N$$

and B be the following $N \times N$ matrix of functions:

$$B = \begin{pmatrix} 0 & a_{12} - c_1 y_1 & \cdots & a_{1N} - c_1 y_1 \\ a_{21} - c_2 y_2 & 0 & \cdots & a_{2N} - c_2 y_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} - c_N y_N & a_{N2} - c_N y_N & \cdots & 0 \end{pmatrix}$$

Then (8) can be rewritten as

$$\begin{cases} L\boldsymbol{\xi} = B\boldsymbol{\xi} + Q, & \text{in } \Omega \\ \boldsymbol{\xi} = \mathbf{0}, & \text{on } \partial\Omega \end{cases} \tag{9}$$

with $\boldsymbol{Q} = [-f_1 y_1, -f_2 y_2, \cdots, -f_N y_N]$. One can easily check that B is essentially positive and fully coupled (see [6]). On the other hand, system (1) can be written in the form similar to (9):

$$\begin{cases} L\boldsymbol{Y} = B\boldsymbol{Y} + W, & \text{in } \Omega \\ \boldsymbol{Y} = \mathbf{0}, & \text{on } \partial\Omega \end{cases} \tag{1$'$}$$

with $\boldsymbol{W} = [c_1 y_1 \sum_{j=1}^N y_j, \cdots, c_N y_N \sum_{j=1}^N y_j]$. Since Y is the coexistence state solution to (1), we have $\boldsymbol{Y} > 0$ and $\boldsymbol{W} > 0$, which implies $(L - B)\boldsymbol{Y} \geqslant 0$ and $(L - B)\boldsymbol{Y} \neq 0$ in Ω. Then by Theorem 1.1 in [6] we conclude that system (8) has a unique solution. □

Lemma 5. Let the hypotheses (H$_1$) and (H$_2$) are satisfied. Let $U \in \mathcal{U}_{ad}$ and $F \in (L^\infty(\Omega))^N$. Thus $U + \varepsilon F \in \mathcal{U}_{ad}$ for $\varepsilon > 0$ and small enough. Let Y and Y^ε be the coexistence state solutions to system (1) corresponding to U and $U + \varepsilon F$ respectively. Then

$$\lim_{\varepsilon \to 0} \frac{Y^\varepsilon - Y}{\varepsilon} = \xi \ \text{ in } (H_0^1(\Omega))^N$$

where $\boldsymbol{\xi} = [\xi_1, \xi_2, \cdots, \xi_N]$ is the unique solution to system (8).

Proof. Let $\xi_i^\varepsilon = \dfrac{y_i^\varepsilon - y_i}{\varepsilon}$ for $i \in \mathbb{Z}_N$. It is easily seen that $\boldsymbol{\xi}^\varepsilon = [\xi_1^\varepsilon, \xi_2^\varepsilon, \cdots, \xi_N^\varepsilon]$ solves the problem

$$\begin{cases} -\triangle \xi_i^\varepsilon + \Big(u_i + c_i\Big(y_i^\varepsilon + \sum_{j=1}^N y_j\Big)\Big)\xi_i^\varepsilon - \sum_{j\neq i}(a_{ij} - c_i y_i^\varepsilon)\xi_j^\varepsilon = -f_i y_i^\varepsilon, & \text{in } \Omega \\ \xi_i^\varepsilon = 0, & \text{on } \partial\Omega \end{cases}$$

By elliptic estimates, there exists a constant $M > 0$ independent of ε, such that $\|\xi^\varepsilon\|_{(H_0^2(\Omega))^N} \leqslant M$. Then for any positive number sequence $\{\varepsilon_n\}_{n=1}^\infty$ with $\varepsilon_n \to 0$ as $n \to \infty$, $\lim_{n\to\infty} \xi^{\varepsilon_n} = \xi \text{in} (H_0^2(\Omega))^N$. The uniqueness of ξ by Lemma 4 assures the convergence of the whole sequence. □

In the sequel, we will use the notation $f^+(x) = \max\{f(x), 0\}$ for $x \in \Omega$.

Theorem 4. Under the hypotheses (H$_1$) and (H$_2$), the optimal control $\boldsymbol{U} = [u_1, \cdots, u_N]$ of problem (P) can be expressed in the form

$$u_i = \min\left\{\frac{1}{2} y_i (\lambda_i - z_i)^+, \delta_i\right\} \text{ in } \Omega, \quad \forall i \in \mathbb{Z}_N$$

where $Y = [y_1, y_2, \cdots, y_N]$ is the corresponding coexistence state solution to (1), and $Z = [z_1, z_2, \cdots, z_N]$ is the unique solution to the following adjoint system: for $i = 1, 2, \cdots, N$

$$\begin{cases} -\triangle z_i + \Big(u_i + c_i\Big(y_i + \sum_{j=1}^N y_j\Big)\Big)z_i - \sum_{j\neq i}(a_{ji} - c_j y_j)z_j = \lambda_i u_i, & \text{in } \Omega \\ z_i = 0, & \text{on } \partial\Omega \end{cases} \tag{10}$$

Proof. For any $p > m$, we define a linear operator $\mathcal{A} : (L^p(\Omega))^N \to (W^{2,p}(\Omega))^N$ as follows: for any $G = [g_1, g_2, \cdots, g_N] \in (L^p(\Omega))^N$, $\mathcal{A}(G)$ is the unique solution to the equations

$$\begin{cases} -\triangle z_i + \Big(u_i + c_i\Big(y_i + \sum_{j=1}^{N} y_j\Big)\Big) z_i = g_i, & \text{in } \Omega \\ z_i = 0, & \text{on } \partial\Omega \end{cases}, \quad i = 1, \cdots, N \tag{11}$$

Then Z is a solution to (10) if and only if

$$(I - \mathcal{AB})(Z) = \mathcal{A}(\lambda_1 u_1, \lambda_2 u_2, \cdots, \lambda_N u_N) \tag{12}$$

with

$$\mathcal{B}(Z) = \Big[\sum_{j=2}^{N}(a_{j1} - c_j y_j)z_j, \sum_{j=1, j\neq 2}^{N}(a_{j2} - c_j y_j)z_j, \cdots, \sum_{j=1}^{N-1}(a_{jN} - c_j y_j)z_j\Big]$$

By the regularity of elliptic equations, $\mathcal{AB}$ is a compact operator from $(C(\bar{\Omega}))^N$ to itself (see [6]). We now show that the kernel space $\mathcal{N}(I - \mathcal{AB}) = \{0\}$. In fact, if $(I - \mathcal{AB})(Z) = 0$, then

$$\begin{cases} -\triangle z_i + \Big(u_i + c_i\Big(y_i + \sum_{j=1}^{N} y_j\Big)\Big) z_i - \sum_{j\neq i}(a_{ji} - c_j y_j)z_j = 0, & \text{in } \Omega \\ z_i = 0, & \text{on } \partial\Omega \end{cases}, \quad i = 1, 2, \cdots, N$$

For any $[h_1, h_2, \cdots, h_N] \in (L^p(\Omega))^N$, let $[\eta_1, \eta_2, \cdots, \eta_N]$ be the unique solution to the system

$$\begin{cases} -\triangle \eta_i + \Big(u_i + c_i\Big(y_i + \sum_{j=1}^{N} y_j\Big)\Big) \eta_i - \sum_{j\neq i}(a_{ji} - c_j y_j)\eta_j = h_i, & \text{in } \Omega \\ \eta_i = 0, & \text{on } \partial\Omega \end{cases}, \quad i = 1, 2, \cdots, N$$

After some calculations we get $\sum_{i=1}^{N}\int_\Omega z_i h_i \mathrm{d}x = 0$, which implies $Z = 0$. Thus system (10) has a unique solution Z for $p > m$ (see [10]).

Let $F = [f_1, f_2, \cdots, f_n] \in (L^\infty(\Omega))^N$, Then $U + \varepsilon F \in \mathcal{U}_{ad}$ for $\varepsilon > 0$ small enough. Let $\boldsymbol{Y}^\varepsilon = (y_1^\varepsilon, y_2^\varepsilon, \cdots, y_N^\varepsilon)$ be the corresponding solution to (1) for $U + \varepsilon F$. Since $J(U) \geqslant J(U + \varepsilon F)$, we get $\sum_{i=1}^{N}\int_\Omega(\lambda_i u_i(y_i^\varepsilon - y_i) + \varepsilon\lambda_i f_i y_i^\varepsilon - \varepsilon(2u_i + \varepsilon f_i)f_i)\mathrm{d}x \leqslant 0$. Dividing by ε and letting $\varepsilon \to 0^+$, by Lemma 5 we obtain

$$\sum_{i=1}^{N}\int_\Omega \Big(\lambda_i u_i \xi_i + \lambda_i f_i y_i - 2u_i f_i\Big) \leqslant 0 \tag{13}$$

Multiplying both sides of (8) and (10) by z_i and $\lambda_i\xi_i$ respectively, and integrating over Ω, we get

$$\sum_{i=1}^{N}\int_\Omega \Big(\lambda_i u_i \xi_i + z_i f_i y_i\Big)\mathrm{d}x = 0 \tag{14}$$

Combing (13) and (14), we deduce that

$$\sum_{i=1}^{N}\int_\Omega \Big(\lambda_i f_i y_i - 2u_i f_i - z_i f_i y_i\Big)\mathrm{d}x \leqslant 0 \tag{15}$$

Taking $f_2 = f_3 = \cdots = f_N \equiv 0$ in (15), we obtain $\int_\Omega f_1[y_1(\lambda_1 - z_1) - 2u_1]\mathrm{d}x \leqslant 0$. By the standard argument concerning the sign of the variation f_1 depending on the size of u_1, we can obtain $u_1 = \min\Big\{\frac{1}{2}y_1(\lambda_1 - z_1)^+, \delta_1\Big\}$ in Ω. Similarly, we have $u_i = \min\Big\{\frac{1}{2}y_i(\lambda_i - z_i)^+, \delta_i\Big\}$ in Ω, $i = 2, 3, \cdots, N$. This completes the proof. □

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