Delay-dependent Conditions for Absolute Stability of Lurie Control Systems with Time-varying Delay[1)]

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Abstract Some delay-dependent absolute stability criteria for Lurie control systems with time-varying delay are derived, in which some free-weighting matrices are used to express the relationships between the terms in the Leibniz-Newton formula. These criteria are based on linear matrix inequality(LMI) such that the upper bound of time-delay guaranteeing the absolute stability and the free-weighting matrices can be obtained through the solutions of the LMI. Moreover, the Lyapunov functional constructed by the solutions of these LMIs is adopted to guarantee the absolute stability of the systems. Finally, some examples are provided to demonstrate the effectiveness of the proposed methods.

Key words Delay-dependent, Lurie control system, absolute stability, linear matrix inequality

1 Introduction

The problem of the absolute stability of Lurie control systems with delay has been widely studied for several decades[1∼8]. Some delay-independent absolute stability criteria were derived in [1∼4], while the delay-dependent ones were obtained in [5∼8]. Since the delay-dependent criteria make use of information on the length of delays, they are less conservative than delay-independent ones. The delay-dependent absolute stability conditions in [5∼7] were formulated in terms of matrix measure and matrix norm. Although they are easy to check, they require the matrix measure to be negative such that they are only adapted to the systems with small sectors. On the other hand, they are only some existing conditions which, instead of being solvable, depend on the selection of some free parameters in the Lyapunov functional. Park′s inequality in [9] was extended to delay-dependent conditions for Lurie systems with delay in [8]. However, there is room for improvement in the handling of the delay term[10]. The free-weighting matrices approach presented in [10], which took the relationship between the terms in the Leibniz-Newton formula into account, is one of the most effective methods handling the delay-dependent stability problem.

In this paper, the free-weighting matrices approach[10] is employed to derive the delay-dependent absolute stability criteria for Lurie systems with time-varying delay. The relationship between the terms in the Leibniz-Newton formula is described by some free-weighting matrices such that the upper bound of delay guaranteeing that the system is absolutely stable can be derived through the solutions of linear matrix inequality(LMI). On the other hand, the delay-dependent conditions are adapted to both infinite sector and finite sector. Finally, some examples are employed to demonstrate the effectiveness and the improvement over some existing papers.

2 Main results

Consider a Lurie control system with time-delay

$$\begin{cases} \dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{x}(t-d(t)) + D\boldsymbol{f}(\boldsymbol{\sigma}) \\ \boldsymbol{\sigma} = C^{\mathrm{T}}\boldsymbol{x}(t) \\ \boldsymbol{x}(t) = \boldsymbol{\varphi}(t), \quad t \in [-\tau, 0] \end{cases} \tag{1}$$

where $\boldsymbol{x}(t) \in R^n$ is state vector, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \cdots, \sigma_m)^{\mathrm{T}}$, $\boldsymbol{f}(\boldsymbol{\sigma}) = (f_1(\sigma_1), f_2(\sigma_2), \cdots, f_m(\sigma_m))^{\mathrm{T}}$, $A, B \in$

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$R^{n\times n}$, $C, D \in R^{n\times m}$. The time delay, $\mathrm{d}(t)$ is a time-varying continuous function that satisfies

$$0 \leqslant \mathrm{d}(t) \leqslant \tau,\ \dot{\mathrm{d}} \leqslant \mu \leqslant 1 \tag{2}$$

Every non-linearity part satisfies the infinite sector condition

$$f_j(\cdot) \in K_j[0,\infty] = \{f_j(\sigma_j)|f_j(0)=0, \sigma_j f_j(\sigma_j) > 0(\sigma_j \neq 0)\}, \quad j=1,2,\cdots,m \tag{3}$$

or the finite sector condition

$$f_j(\cdot) \in K_j[0,k_j] = \{f_j(\sigma_j)|f_j(0)=0, 0<\sigma_j f_j(\sigma_j) \leqslant k_j\sigma_j^2(\sigma_j \neq 0)\}, \quad j=1,2,\cdots,m \tag{4}$$

Definition 1. System (1) is absolutely stable for any $f_j(\cdot)$ $(j=1,2,\cdots,m)$ satisfying (3)(or (4)), if system (1) is globally asymptotically stable for any functions in (3)(or (4)).

First, consider the absolute stability for system (1) in the infinite sector (3). Choose the Lyapunov functional candidate as

$$V(x_t) = \boldsymbol{x}^{\mathrm{T}}(t)P\boldsymbol{x}(t) + 2\sum_{j=1}^{m}\lambda_j\int_0^{\sigma_j} f_j(\sigma_j)\mathrm{d}\sigma_j + \int_{t-\mathrm{d}(t)}^{t}\boldsymbol{x}^{\mathrm{T}}(s)Q\boldsymbol{x}(s)\mathrm{d}s + \int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)Z\dot{\boldsymbol{x}}(s)\mathrm{d}s\mathrm{d}\theta \tag{5}$$

where $P>0,\ Q\geqslant 0,\ Z>0,\ \Lambda = \mathrm{diag}(\lambda_1,\lambda_2,\cdots,\lambda_m)\geqslant 0$ are to be determined. Calculating the derivative of $V(x_t)$ along the solutions of system (1) yields

$$\begin{aligned}\frac{\mathrm{d}V(x_t)}{\mathrm{d}t} \leqslant & 2[\boldsymbol{x}^{\mathrm{T}}(t)P + \boldsymbol{f}^{\mathrm{T}}(\boldsymbol{\sigma})\Lambda C^{\mathrm{T}}]\cdot[A\boldsymbol{x}(t) + B\boldsymbol{x}(t-\mathrm{d}(t)) + D\boldsymbol{f}(\boldsymbol{\sigma})] + \\ & \boldsymbol{x}^{\mathrm{T}}(t)Q\boldsymbol{x}(t) - (1-\mu)\boldsymbol{x}^{\mathrm{T}}(t-\mathrm{d}(t))Q\boldsymbol{x}(t-\mathrm{d}(t)) - \int_{t-\mathrm{d}(t)}^{t}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)Z\dot{\boldsymbol{x}}(s)\mathrm{d}s + \\ & \tau[A\boldsymbol{x}(t) + B\boldsymbol{x}(t-\mathrm{d}(t)) + D\boldsymbol{f}(\boldsymbol{\sigma})]^{\mathrm{T}}Z[A\boldsymbol{x}(t) + B\boldsymbol{x}(t-\mathrm{d}(t)) + D\boldsymbol{f}(\boldsymbol{\sigma})]\end{aligned} \tag{6}$$

Using the Leibniz-Newton formula, for any matrices $N_1, N_2 \in R^{n\times n}$, $N_3 \in R^{m\times n}$,

$$2[\boldsymbol{x}^{\mathrm{T}}(t)N_1 + \boldsymbol{x}^{\mathrm{T}}(t-\mathrm{d}(t))N_2 + \boldsymbol{f}^{\mathrm{T}}(\boldsymbol{\sigma})N_3]\cdot[\boldsymbol{x}(t) - \boldsymbol{x}(t-\mathrm{d}(t)) - \int_{t-\mathrm{d}(t)}^{t}\dot{\boldsymbol{x}}(s)\mathrm{d}s] = 0 \tag{7}$$

On the other hand, for any semi-positive definite matrix $X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^{\mathrm{T}} & X_{22} & X_{23} \\ X_{13}^{\mathrm{T}} & X_{23}^{\mathrm{T}} & X_{33}\end{bmatrix} \geqslant 0$, the following holds according to (2).

$$\boldsymbol{z}^{\mathrm{T}}(t)(\tau X)\boldsymbol{z}(t) - \int_{t-\mathrm{d}(t)}^{t}\boldsymbol{z}^{\mathrm{T}}(x)X\boldsymbol{z}(t)\mathrm{d}s \geqslant 0 \tag{8}$$

where $\boldsymbol{z}(t) = [\boldsymbol{x}^{\mathrm{T}}(t) \quad \boldsymbol{x}^{\mathrm{T}}(t-\mathrm{d}(t)) \quad \boldsymbol{f}^{\mathrm{T}}(\boldsymbol{\sigma})]^{\mathrm{T}}$. Adding (7) and (8) into (6) yields

$$\frac{\mathrm{d}V(x_t)}{\mathrm{d}t} \leqslant \boldsymbol{z}^{\mathrm{T}}(t)\Phi\boldsymbol{z}(t) - \int_{t-\mathrm{d}(t)}^{t}\boldsymbol{z}_1^{\mathrm{T}}(t,s)\Psi\boldsymbol{z}_1(t,s)\mathrm{d}s \tag{9}$$

where $\boldsymbol{z}(t)$ is defined in (8) and $\boldsymbol{z}_1(t,s) = [\boldsymbol{z}^{\mathrm{T}}(t) \quad \dot{\boldsymbol{x}}^{\mathrm{T}}(s)]^{\mathrm{T}}$,

$$\Phi = \begin{bmatrix} \Phi_{11} + \tau A^{\mathrm{T}}ZA + \tau X_{11} & \Phi_{12} + \tau A^{\mathrm{T}}ZB + \tau X_{12} & \Phi_{13} + \tau A^{\mathrm{T}}ZD + \tau X_{13} \\ \Phi_{12}^{\mathrm{T}} + \tau B^{\mathrm{T}}ZA + \tau X_{12}^{\mathrm{T}} & \Phi_{22} + \tau B^{\mathrm{T}}ZB + \tau X_{22} & \Phi_{23} + \tau B^{\mathrm{T}}ZD + \tau X_{23} \\ \Phi_{13}^{\mathrm{T}} + \tau D^{\mathrm{T}}ZA + \tau X_{13}^{\mathrm{T}} & \Phi_{23}^{\mathrm{T}} + \tau D^{\mathrm{T}}ZB + \tau X_{23}^{\mathrm{T}} & \Phi_{33} + \tau D^{\mathrm{T}}ZD + \tau X_{33}\end{bmatrix}$$

$$\Psi = \begin{bmatrix} X_{11} & X_{12} & X_{13} & N_1 \\ X_{12}^{\mathrm{T}} & X_{22} & X_{23} & N_2 \\ X_{13}^{\mathrm{T}} & X_{23}^{\mathrm{T}} & X_{33} & N_3 \\ N_1^{\mathrm{T}} & N_2^{\mathrm{T}} & N_3^{\mathrm{T}} & Z\end{bmatrix}$$

$$\Phi_{11} = PA + A^{\mathrm{T}}P + Q + N_1 + N_1^{\mathrm{T}},\ \Phi_{12} = PB + N_2^{\mathrm{T}} - N_1,\ \Phi_{13} = PD + A^{\mathrm{T}}C\Lambda + N_3^{\mathrm{T}}$$

$$\Phi_{22} = -(1-\mu)Q - N_2 - N_2^{\mathrm{T}},\ \Phi_{23} = B^{\mathrm{T}}C\Lambda - N_3^{\mathrm{T}},\ \Phi_{33} = D^{\mathrm{T}}C\Lambda + \Lambda C^{\mathrm{T}}D$$

In addition, (3) can be expressed as

$$-f_j(\sigma_j)c_j\boldsymbol{x}(t) \leqslant 0, \quad j = 1, 2, \cdots, m \tag{10}$$

By applying S-procedure[11], $\dfrac{\mathrm{d}V(x_t)}{\mathrm{d}t} < 0$ for $(\boldsymbol{x}^{\mathrm{T}}, \boldsymbol{x}^{\mathrm{T}}(t-\mathrm{d}(t))) \neq 0$ and the restriction of the condition (10) if there exist $t_j \geqslant 0 (j = 1, 2, \cdots, m)$ such that

$$\boldsymbol{z}^{\mathrm{T}}(t)\varPhi\boldsymbol{z}(t) - \int_{t-\mathrm{d}(t)}^{t} \boldsymbol{z}_1^{\mathrm{T}}(t,s)\varPsi\boldsymbol{z}_1(t,s)\mathrm{d}s + 2\sum_{j=1}^{m} t_j f_j(\sigma_j)c_j\boldsymbol{x}(t) < 0 \tag{11}$$

for $\boldsymbol{z}(t) \neq 0$. If $\varPsi \geqslant 0$ and

$$\varPhi + \begin{bmatrix} 0 & 0 & CT \\ 0 & 0 & 0 \\ TC^{\mathrm{T}} & 0 & 0 \end{bmatrix} < 0 \tag{12}$$

then (11) holds. Specifically, X can be chosen as $X = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} Z^{-1} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}^{\mathrm{T}}$. This ensures $X \geqslant 0$ and $\varPsi \geqslant 0$. In this case, according to Schur complements, (12) is equivalent to

$$\begin{bmatrix} \varPhi_{11} & \varPhi_{12} & \varPhi_{13} + CT & \tau N_1 & \tau A^{\mathrm{T}}Z \\ \varPhi_{12}^{\mathrm{T}} & \varPhi_{22} & \varPhi_{23} & \tau N_2 & \tau B^{\mathrm{T}}Z \\ \varPhi_{13}^{\mathrm{T}} + TC^{\mathrm{T}} & \varPhi_{23}^{\mathrm{T}} & \varPhi_{33} & \tau N_3 & \tau D^{\mathrm{T}}Z \\ \tau N_1^{\mathrm{T}} & \tau N_2^{\mathrm{T}} & \tau N_3^{\mathrm{T}} & -\tau Z & 0 \\ \tau ZA & \tau ZB & \tau ZD & 0 & -\tau Z \end{bmatrix} < 0 \tag{13}$$

Then, the following Theorems are given.

Theorem 1. For given $\tau > 0$ and μ, system (1) is absolutely stable with the restriction of (3) if there exist $P > 0$, $Q \geqslant 0$, $Z > 0$, $T \geqslant 0$, $\varLambda \geqslant 0$ and any matrices $N_i (i = 1, 2, 3)$ with appropriate dimension such that LMI (13) is feasible.

On the other hand, (4) can be expressed as

$$f_j(\sigma_j)(f_j(\sigma_j) - k_j c_j\boldsymbol{x}(t)) \leqslant 0, \; j = 1, 2, \cdots, m$$

Similarly, we also have

Theorem 2. For given $\tau > 0$ and μ, system (1) is absolutely stable with the restriction of (4) if there exist $P > 0$, $Q \geqslant 0$, $Z > 0$, $Z > 0$, $T \geqslant 0$, $\varLambda \geqslant 0$ and any matrices $N_i (i = 1, 2, 3)$ with appropriate dimension such that following LMI (15) is feasible.

$$\begin{bmatrix} \varPhi_{11} & \varPhi_{12} & \varPhi_{13} + CKT & \tau N_1 & \tau A^{\mathrm{T}}Z \\ \varPhi_{12}^{\mathrm{T}} & \varPhi_{22} & \varPhi_{23} & \tau N_2 & \tau B^{\mathrm{T}}Z \\ \varPhi_{13}^{\mathrm{T}} + TKC^{\mathrm{T}} & \varPhi_{23}^{\mathrm{T}} & \varPhi_{33} - 2T & \tau N_3 & \tau D^{\mathrm{T}}Z \\ \tau N_1^{\mathrm{T}} & \tau N_2^{\mathrm{T}} & \tau N_3^{\mathrm{T}} & -\tau Z & 0 \\ \tau ZA & \tau ZB & \tau ZD & 0 & -\tau Z \end{bmatrix} < 0$$

where $K = diag(k_1, k_2, \cdots, k_m)$, and $\varPhi_{ij} (i = 1, 2, 3; i \leqslant j \leqslant 3)$ are defined in (9).

4 Examples

Example 1. Consider system(1) with

$$A = \begin{bmatrix} -2 & 0 \\ -1 & -2 \end{bmatrix}, \; B = \begin{bmatrix} -0.2 & -0.5 \\ 0.5 & -0.2 \end{bmatrix}, \; \boldsymbol{D} = \begin{bmatrix} -0.2 \\ -0.3 \end{bmatrix}, \; \boldsymbol{C} = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}, \; f(\cdot) \in K[0, \infty)$$

From Theorem 1, system (1) is absolutely stability in the infinite sector restriction (3) for any delay when $\mu = 0$. The best value for the upper bound of delay given in [5∼8] was 2.055 in [8]. When $\mu = 0.9$, for any given delay, system (1) with the restriction of (3) is still absolutely stable. In fact, solving LMI (13) for $\tau = 100$, we have

$$P = \begin{bmatrix} 4.7087 & -0.2022 \\ -0.2022 & 3.5361 \end{bmatrix}, \; Q = \begin{bmatrix} 8.6172 & 1.6617 \\ 1.6617 & 7.1791 \end{bmatrix}, \; Z = \begin{bmatrix} 0.0127 & -0.0022 \\ -0.0022 & 0.0088 \end{bmatrix}$$

$$\Lambda = [0.6649],\ T = [1.7421],\ N_1 = \begin{bmatrix} 0.1721 & -0.2381 \\ 0.3923 & -0.2081 \end{bmatrix} \times 10^{-4}$$

$$N_2 = \begin{bmatrix} 0.1407 & -0.0458 \\ 0.0126 & 0.0834 \end{bmatrix} \times 10^{-3},\ \boldsymbol{N}_3 = [0.2370 \quad -0.0231] \times 10^{-4}$$

and system (1) with the restriction of (3) is absolutely stable, while [5∼8] could not derive the corresponding results.

Example 2. Consider system (1) with

$$A = \begin{bmatrix} -2 & 0 \\ -1 & -2 \end{bmatrix},\ B = \begin{bmatrix} -0.2 & -0.5 \\ 0.5 & -0.2 \end{bmatrix},\ \boldsymbol{D} = \begin{bmatrix} -0.2 \\ -0.3 \end{bmatrix},\ \boldsymbol{C} = \begin{bmatrix} -0.6 \\ 0.8 \end{bmatrix}$$

the upper bounds of time-delay, which guantee that system (1) is stable for various μ and K, are listed in Table 1.

Table 1 Relationship among sector, the upper bound of the derivative of delay and the upper bound of delay

$f(\cdot)$	$K[0,\infty)$	$K[0,100]$	$K[0,10)$
$\mu=0$	1.1263	1.7541	any delay
$\mu=0.5$	0.8848	1.2282	Any delay
$\mu=0.9$	0.6781	0.8472	3.0656

5 Conclusion

In this paper, some delay-dependent conditions for Lurie systems with time-varying delay are derived, in which the free-weighting matrices are employed to express the relationship between the terms in the the Leibniz-Newton formula. The free-weighting matrices and the free parameters constructing the Lyapunov functional can be obtained by the solutions of LMI. Finally, numerical examples demonstrate the effectiveness of the methods presented in this paper.

References

1 Popov V M, Halanay A. About stability of nonlinear controlled systems with delay. *Automat Remote Control*, 1962, **23**(7): 849∼851

2 Somolines A. Stability of Lurie-type functional equations. *Journal of Differential Equations*, 1977, **26**(2): 191∼199

3 Gan Z X, Ge W G. Lyapunov functional for multiple nonlinear Lurie control systems with delay. *Journal of Mathematical Analysis and Applications*, 2001, **259**(1): 596∼608

4 Bliman P A. Extension of Popov absolute stability criterion to non-autonomous systems with delays. *International Journal of Control*, 2000, **73**(15): 1349∼1361

5 Nian Xiao-Hong. Delay dependent conditions for absolute stability of Lurie type control systems. *Acta Automatica Sinica*, 1999, **25**(4): 564∼566

6 Yang Bin, Chen Mian-Yun. Absolute stability of a class of control systems with discrete and distributed delays. *Control and Decision*, 2001, **16**(3): 362∼364

7 Xu Bing-Ji, Liao Xiao-Xin. Absolute stability criteria of delay-dependent for Lurie control systems. *Acta Automatica Sinica*, 2002, **28**(2): 317∼320

8 Yang Bin, Chen Mian-Yun. Delay-dependent criterion for absolute stability of Lurie type control systems with time delay. *Control Theory and Applications*, 2001, **18**(6): 929∼931

9 Park P. A delay-dependent stability criterion for systems with uncertain time-invariant delays. *IEEE Transactions on Automatic Control*, 1999, **44**(4): 876∼877

10 He Y, Wu M, She J H, Liu G P. Delay-dependent robust stability criteria for uncertain neutral systems with mixed delays. *Systems & Control Letters*, 2004, **51**(1): 57∼65

11 Boyd S, Ghaoui L El, Feron E, Balakrishnan V. Linear Matrix Inequality in System and Control Theory. Philadelphia: Society for Industrial and Applied Mathematics, 1994

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