Suboptimal Strategies of Linear Quadratic Closed-loop Differential Games: An BMI Approach[1)]

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Abstract The suboptimal control program *via* memoryless state feedback strategies for LQ differential games with multiple players is studied in this paper. Sufficient conditions for the existence of the suboptimal strategies for LQ differential games are presented. It is shown that the suboptimal strategies of LQ differential games are associated with a coupled algebraic Riccati inequality. Furthermore, the problem of designing suboptimal strategies is considered. A non-convex optimization problem with BMI constrains is formulated to design the suboptimal strategies which minimizes the performance indices of the closed-loop LQ differential games and can be solved by using LMI Toolbox of MATLAB. An example is given to illustrate the proposed results.

Key words Differential game, suboptimal control, Nash equilibrium, bilinear matrix inequalities

1 Introduction

Linear quadratic differential games have many applications in the economy, military and intelligent robots. The concept of differential games was first introduced by Issac within the framework of two-person zero-sum games[1]. The non-zero games were introduced by Starr and Ho[2,3]. In the last decades, the nonzero-sum closed-loop games were studied by many researchers[4∼10]. In [5], the existence of linear Nash strategies for the LQ games was obtained by using Brower′s fixed-point theorem. The global existence of solutions of the related couple of Riccati equations was investigated in [8]. Recently, the asymptotic behavior of state feedback Nash equilibrium over infinite time period was discussed by Ween *et al.*[9]. The feedback equilibrium in the scalar infinite horizon LQ game was studied by Engwereda[10]. It is well known that the Nash equilibrium solution of LQ differential game over infinite time period can be expressed in terms of the solution of a set of algebraic Riccati equations (ARE). However, there is no general method to develop the solution of coupled algebraic Riccati equations, and the numerical determination of the solution of the nonlinear ARE may be difficult especially for high dimensional systems. Therefore, the problem of studying calculation methods to design suboptimal strategies of LQ differential games by using optimization tools such as Matlab Toolbox is an attractive subject in the game theory.

In this paper, we study the problem of designing suboptimal strategies LQ differential games with multiple players. The main objects of this paper are to obtain the existence conditions for designing suboptimal strategies and give an optimization algorithm for designing suboptimal strategies of LQ differential games based on the BMI approach. First, the problem of suboptimal strategies is formulated in Section 2. Then, sufficient conditions for the existence of state feedback suboptimal strategies are derived in Section 3. Furthermore, the problem of designing feedback suboptimal strategies is considered, optimization methods for calculation solution of a set of algebraic Riccati inequalities associated with suboptimal strategies of LQ games are proposed in Section 4. In Section 5, a numerical example is given to illustrate the main results.

2 Problem formulation

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Consider a class of linear quadratic (LQ) differential games with multiple players described by the following state-space equation:

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + \sum_{i=1}^{N} B_i \boldsymbol{u}_i(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{1}$$

where $\boldsymbol{x}(t)$ is the state vector, $\boldsymbol{u}_i(t)$ for $i = 1, 2, \dots, N$ are the control input vectors, A, B_i for $i = 1, 2, \dots, N$ are known constant real matrices.

Associated with system (1) are the performance indices

$$J_i(\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_N) = \int_0^{\infty} [\boldsymbol{x}^{\mathrm{T}} Q_i \boldsymbol{x} + \sum_{j=1}^{N} \boldsymbol{u}_i^{\mathrm{T}} R_{ij} \boldsymbol{u}_i] \mathrm{d}t, \quad i = 1, 2, \dots, N \tag{2}$$

where Q_i for $i = 1, 2, \dots, N$ and R_{ij} for $i, j = 1, 2, \dots, N$ are given positive-definite symmetric matrices.

In the game, all the players aim to select state feedback control laws

$$\boldsymbol{u}_i(t) = K_i \boldsymbol{x}(t), \quad i = 1, 2, \dots, N \tag{3}$$

and minimize their performance indices function $J_i(\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_N)$ such that the resulting closed-loop system

$$\dot{\boldsymbol{x}}(t) = [A + \sum_{i=1}^{N} B_i K_i] \boldsymbol{x}(t) \tag{4}$$

is asymptotically stable.

Definition 1. If there exist strategies $\boldsymbol{u}_i^*(t)$ for $i = 1, 2, \dots, N$ such that for any strategies $\boldsymbol{u}_i(t)$ for $i = 1, 2, \dots, N$ of LQ differential game (1), the closed-loop values of the performance indices (2) satisfy

$$J_i(\boldsymbol{u}_1^*, \boldsymbol{u}_2^*, \dots, \boldsymbol{u}_N^*) \leqslant J_i(\boldsymbol{u}_1^*, \dots, \boldsymbol{u}_{i-1}^*, \boldsymbol{u}_i, \boldsymbol{u}_{i+1}^*, \dots, \boldsymbol{u}_N^*), \quad i = 1, 2, \dots, N \tag{5}$$

then $\boldsymbol{u}_i^*(t)$ for $i = 1, 2, \dots, N$ are said to be optimal strategies of LQ differential game (1).

Definition 1 shows that the optimal strategies are just the Nash equilibrium of LQ differential game (1). The following well known result for the existence of the Nash equilibrium point was introduced by Starr and Ho (1965).

Proposition 1. If there exist optimal strategies of LQ differential game (1), the optimal strategies are given by

$$\boldsymbol{u}_i^*(t) = -R_{ii}^{-1} B_i^{\mathrm{T}} P_i \boldsymbol{x}(t), \quad i = 1, 2, \dots, N \tag{6}$$

and the symmetric matrices P_i for $i = 1, 2, \dots, N$ satisfy the following algebraic Riccati equations

$$Q_i + A^{\mathrm{T}} P_i + P_i A - P_i B_i R_{ii}^{-1} B_i^{\mathrm{T}} P_i + \sum_{j=1, j \neq i}^{N} P_j B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j^{\mathrm{T}} P_j -$$
$$\sum_{j=1, j \neq i}^{N} [P_j B_j R_{jj}^{-1} B_j^{\mathrm{T}} P_i + P_i B_j R_{jj}^{-1} B_j^{\mathrm{T}} P_j] = 0, \quad i = 1, 2, \dots, N \tag{7}$$

Remark 1. Although Proposition 1 has given the solutions of optimal strategies of LQ differential game (1) theoretically, the procedure of solving ARE (7) is very difficult since these equations are strongly coupled. There is no general method to solve ARE (7).

Definition 2. The strategies $\boldsymbol{u}_i(t)$ for $i = 1, 2, \dots, N$ are said to be the suboptimal strategies of LQ differential game (1) if there exists positive scalar δ, such that the closed-loop values of the performance indices (2) satisfy the following inequalities

$$J_i(\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_N) \leqslant J_i(\boldsymbol{u}_1^*, \boldsymbol{u}_2^*, \dots, \boldsymbol{u}_N^*) + \delta, \; i = 1, 2, \dots, N$$

The main objective of this paper is to obtain sufficient conditions for the existence of the suboptimal strategies and to develop a procedure of designing a set of stationary state feedback suboptimal strategies for LQ differential game (1) with multiple players. Furthermore, an algorithm method *via* bilinear matrix inequalities (BMI) is given to solve the related ARE and calculate the suboptimal strategies.

3 Suboptimal strategies of LQ differential games

At first, let us present some sufficient conditions for the existence of the state suboptimal strategies of LQ differential game (1).

Theorem 1. If there exist symmetric positive definite matrices P_i and matrices K_i for $i = 1, 2, \ldots, N$ such that the following matrix inequalities

$$Q_i + A^{\mathrm{T}}P_i + P_iA + \sum_{j=1}^{N} K_j^{\mathrm{T}}R_{ij}K_j + \sum_{j=1}^{N}[K_j^{\mathrm{T}}B_j^{\mathrm{T}}P_i + P_iB_jK_j] \leqslant 0,\ i = 1, 2, \ldots, N \tag{8}$$

hold, the suboptimal strategies of LQ differential game (1) are given by (3) and the corresponding values of performance indices (2) satisfy the upper bound

$$J_i(\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_N) \leqslant \boldsymbol{x}_0^{\mathrm{T}}P_i\boldsymbol{x}_0, \quad i = 1, 2, \ldots, N \tag{9}$$

Proof. Consider closed-loop system (4), and define Lyapunov functions

$$V_i(\boldsymbol{x}(t)) = \boldsymbol{x}^{\mathrm{T}}(t)P_i\boldsymbol{x}(t), \quad i = 1, 2, \ldots, N$$

The time derivative of $V_i(x)$ along any trajectory of the closed-loop system (4) is given by

$$\begin{aligned}\dot{V}_i(\boldsymbol{x}(t)) =& \boldsymbol{x}^{\mathrm{T}}(t)[(A + \sum_{j=1}^{N} B_iK_j)^{\mathrm{T}}P_i + P_i(A + \sum_{j=1}^{N} B_iK_j)]\boldsymbol{x}(t) = \boldsymbol{x}^{\mathrm{T}}(t)[Q_i + A^{\mathrm{T}}P_i + \\ & P_iA + \sum_{j=1}^{N} K_j^{\mathrm{T}}R_{ij}K_j + \sum_{j=1}^{N}(K_j^{\mathrm{T}}B_j^{\mathrm{T}}P_i + P_iB_jK_j)]\boldsymbol{x}(t) - \boldsymbol{x}^{\mathrm{T}}(t)[Q_i + \sum_{j=1}^{N} K_j^{\mathrm{T}}R_{ij}K_j]\boldsymbol{x}(t)\end{aligned}$$

Inequalities (8) imply

$$\dot{V}_i(\boldsymbol{x}(t)) \leqslant -\boldsymbol{x}^{\mathrm{T}}(t)[Q_i + \sum_{j=1}^{N} K_j^{\mathrm{T}}R_{ij}K_j]\boldsymbol{x}(t) \tag{10}$$

By integrating both sides of the inequality (10) from 0 to T and using the initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$, one obtains

$$-\int_0^{\mathrm{T}} \boldsymbol{x}^{\mathrm{T}}(s)[Q_i + \sum_{j=1}^{N} K_j^{\mathrm{T}}R_{ij}K_j]\boldsymbol{x}(s)\mathrm{d}s \geqslant \boldsymbol{x}^{\mathrm{T}}(T)P_i\boldsymbol{x}(T) - \boldsymbol{x}^{\mathrm{T}}(0)P_i\boldsymbol{x}(0)$$

Since P_i for $i = 1, 2, \cdots, N$ are positive definite matrices, the closed-loop system (4) is asymptotically stable. Thus

$$\lim_{T\to\infty} \boldsymbol{x}^{\mathrm{T}}(T)P_i\boldsymbol{x}(T) = 0, \quad i = 1, 2, \ldots, N$$

Hence, we get

$$\int_0^{\infty} \boldsymbol{x}^{\mathrm{T}}(s)[Q_i + \sum_{j=1}^{N} K_j^{\mathrm{T}}R_{ij}K_j]\boldsymbol{x}(s)\mathrm{d}s \leqslant \boldsymbol{x}^{\mathrm{T}}(0)P_i\boldsymbol{x}(0)$$

This completes the proof of Theorem 1.

By substituting $K_i = -R_{ii}^{-1}B_i^{\mathrm{T}}P_i$ for $i = 1, 2, \cdots, N$ into (8), we obtain the following main results.

Theorem 2. If there exist symmetric positive definite matrices P_i for $i = 1, 2, \ldots, N$ such that the following algebraic Riccati inequalities (ARI)

$$A^{\mathrm{T}}P_i + P_iA + Q_i - P_iB_iR_{ii}^{-1}B_i^{\mathrm{T}}P_i + \sum_{j=1, j\neq i}^{N} P_jB_jR_{jj}^{-1}R_{ij}R_{jj}^{-1}B_j^{\mathrm{T}}P_j -$$

$$\sum_{i=1,j\neq i}^{N}[P_jB_jR_{jj}^{-1}B_j^{\mathrm{T}}P_i+P_iB_jR_{jj}^{-1}B_j^{\mathrm{T}}P_j]\leqslant 0,\ i=1,2,\ldots,N \tag{11}$$

hold, the suboptimal strategies of LQ differential game (1) are given by

$$\boldsymbol{u}_i(t)=-R_{ii}^{-1}B_i^{\mathrm{T}}P_i\boldsymbol{x}(t),\ i=1,2,\ldots,N \tag{12}$$

and performance indices (2) satisfy the upper bound (9).

Remark 3. Inequalities (11) are coupled algebraic Riccati inequalities and play a key role in theory of LQ differential games. By Theorem 2, the feasible solution P_i for $i=1,2,\ldots N$ of (11) determines a suboptimal strategies of LQ game (1). In fact, the local suboptimal strategies can be obtained from the solution of inequalities of (11).

Due to the negative sign in the $-P_iB_iR_{ii}^{-1}B_i^{\mathrm{T}}P_i$ term, (11) cannot be simplified to LMIs. To accommodate the $-P_iB_iR_{ii}^{-1}B_i^{\mathrm{T}}P_i$ terms, we introduce communicating variables X_i for $i=1,2,\cdots,N$.

Because $(X_i-P_i)B_iR_{ii}^{-1}B_i^{\mathrm{T}}(X_i-P_i)\geqslant 0$ for any symmetric matrices X_i and P_i of the same dimension, we obtain

$$X_iB_iR_{ii}^{-1}B_i^{\mathrm{T}}P_i+P_iB_iR_{ii}^{-1}B_i^{\mathrm{T}}X_i-X_iB_iR_{ii}^{-1}B_i^{\mathrm{T}}X_i\leqslant P_iB_iR_{ii}^{-1}B_i^{\mathrm{T}}P_i \tag{13}$$

where equalities hold when $X_i=P_i$ for $i=1,2,\cdots,N$. By combining (13) with ARI (11), we obtain

$$\begin{aligned}&A^{\mathrm{T}}P_i+P_iA+Q_i-X_iB_iR_{ii}^{-1}B_i^{\mathrm{T}}P_i-P_iB_iR_{ii}^{-1}B_i^{\mathrm{T}}X_i+X_iB_iR_{ii}^{-1}B_i^{\mathrm{T}}X_i+\\&\sum_{j=1,j\neq i}^{N}P_jB_jR_{jj}^{-1}R_{ij}R_{jj}^{-1}B_j^{\mathrm{T}}P_j-\sum_{i=1,j\neq i}^{N}[P_jB_jR_{jj}^{-1}B_j^{\mathrm{T}}P_i+P_iB_jR_{jj}^{-1}B_j^{\mathrm{T}}P_j]\leqslant 0,\ i=1,2,\ldots,N\end{aligned} \tag{14}$$

Thus, a sufficient condition for the existence of suboptimal strategies is given as follows.

Theorem 3. If there exist symmetric definite matrices P_i and X_i for i=1,2,...,N such that the following BMIs

$$\varGamma_i\triangleq\begin{bmatrix}\Sigma_i & P_1B_1 & \ldots & P_{i-1}B_{i-1} & X_iB_i & P_{i+1}B_{i+1} & \ldots & P_NB_N\\ B_1^{\mathrm{T}}P_1 & -S_{i1} & \ldots & O & O & O & \ldots & O\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ B_{i-1}^{\mathrm{T}}P_{i-1} & O & \ldots & -S_{i,i-1} & O & O & \ldots & O\\ B_i^{\mathrm{T}}X_i & O & \ldots & O & -S_{ii} & O & \ldots & O\\ B_{i+1}^{\mathrm{T}}P_{i+1} & O & \ldots & O & O & -S_{i,i+1} & \ldots & O\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ B_N^{\mathrm{T}}P_N & O & \ldots & O & O & O & \ldots & -S_{iN}\end{bmatrix} \tag{15}$$

where

$$\Sigma_i=A^{\mathrm{T}}P_i+P_iA+Q_i-X_iB_iR_{ii}^{-1}B_i^{\mathrm{T}}P_i-P_iB_iR_{ii}^{-1}B_i^{\mathrm{T}}X_i-\sum_{j=1,j\neq i}^{N}[P_jB_jR_{jj}^{-1}B_j^{\mathrm{T}}P_i+P_iB_jR_{jj}^{-1}B_j^{\mathrm{T}}P_j]$$

$$S_{ij}=\begin{cases}R_{jj}R_{ij}^{-1}R_{jj}, & i\neq j\\ R_{ii}, & i=j\end{cases},\quad i,j=1,2,\cdots,N$$

hold, the suboptimal strategies are given by (12) and the performance indices satisfy the upper bound (9).

Proof. By using the standard Schur complements, the strict inequalities (14) are equivalent to matrix inequalities (15). From Theorem 2 we can deduce this theorem. □

4 The optimization algorithm of ARI *via* BMI

Now, let us consider the problem of designing the suboptimal feedback strategies of LQ differential game (1).

According to the Theorem 3, the feasible solution of ARI (11) can be obtained by solving the following optimization problem

$$\min_{P_i, K_i,\ i=1,2,\cdots,N} \alpha \ \text{ subject to (15)} \tag{16}$$

where $\alpha = \max_{1\leqslant i\leqslant N}\{\alpha_i \mid \Gamma_i < -\alpha_i I\}$. This optimization problem is a standard BMI constraints optimization problem. Since matrix inequalities (15) are BMIs constraints, there are no direct methods to calculate the solution of problem (16). However, for any $1 \leqslant i \leqslant N$, if we fixed variables $P_1, \cdots, P_{i-1}, X_i, P_{i+1}, \cdots, P_N$, BMIs (15) can be formulated as LMIs in variables $X_1, \cdots, X_{i-1}, P_i, X_{i+1}, \cdots, X_N$, and if we fixed $X_1, \cdots, X_{i-1}, P_i, X_{i+1}, \cdots, X_N$, BMIs (15) can be formulated as LMIs in $P_1, \cdots, P_{i-1}, X_i, P_{i+1}, \cdots, P_N$. Thus, an obvious optimization approach for optimization problem (16) is to solve the following problems for $i = 1, 2, \cdots, N$

$$\min_{P_1,\cdots,P_{i-1},X_i,P_{i+1},\cdots,P_N} \alpha \ \text{ subject to (15)} \tag{17}$$

iteratively. Based on this idea, a local optimization algorithm, so called alternating minimization algorithm[11], for designing the suboptimal strategies is presented as follows.

Algorithm 1.

Initialize: $k = 0$, $(X_1, P_2, \cdots, P_N) = (X_1^{(0)}, \cdots, P_2^{(0)}, \cdots, P_N^{(0)})$.

Repeat. set $k = k + 1$ and $i = 1$

Do while $i \leqslant N$

Solve optimization problem (17) with

$(P_1, \cdots, P_{i-1}, X_i, P_{i+1}, \cdots, P_N) = (P_1^{(k)}, \cdots, P_{i-1}^{(k)}, X_i^{(k-1)}, P_{i+1}^{(k-1)}, \cdots, P_N^{(k-1)})$,

to obtain $X_1, \cdots, X_{i-1}, P_i, X_{i+1}, \cdots, X_N$; set

$(X_1^{(k)}, \cdots, X_{i-1}^{(k)}, P_i^{(k)}, X_{i+1}^{(k)}, \cdots, X_N^{(k)}) = (X_1, \cdots, X_{i-1}, P_i, X_{i+1}, \cdots, X_N)$.

Set $i = i + 1$.

Enddo

Until. For $i = 1, 2, \cdots, N$, $\|P_i^{(k)} - P_i^{(k-1)}\| < \varepsilon$, $\|K_i^{(k)} - K_i^{(k-1)}\| < \varepsilon$ (a prescribed tolerance), and (11) hold with $P_i = P_i^{(k)}, K_i = K_i^{(k)}, i = 1, 2, \cdots, N$

Remark 4. Using Algorithm 1 and MATLAB Toolbox[12~13], we can obtain a feasible solution of ARI (11). Since the minimization in Algorithm 1 is a local minimization, so the calculation cost depends largely on the initial point $(X_1^{(0)}, \cdots, P_2^{(0)}, \cdots, P_N^{(0)})$. If we select initial point $(X_1^{(0)}, \cdots, P_2^{(0)}, \cdots, P_N^{(0)})$ properly, the calculation cost can be decreased and more accurate solution can be obtained.

In the rest of this section, we present an method to calculate and choose the initial point. In terms of the standard Schur complements, the strict inequalities (8) are equivalent to

$$\begin{bmatrix} \Pi_i & K_1^{\mathrm{T}} & K_2^{\mathrm{T}} & \dots & K_N^{\mathrm{T}} & I \\ K_1 & -R_{i1}^{-1} & O & \dots & O & O \\ K_2 & O & -R_{i2}^{-1} & \dots & O & O \\ \dots & \dots & \dots & \dots & \dots & \dots \\ K_N & O & O & \dots & -R_{iN}^{-1} & O \\ I & O & O & \dots & O & -Q_i^{-1} \end{bmatrix} < 0, \quad i = 1, 2, \cdots, N \tag{18}$$

where $\Pi_i = A^{\mathrm{T}} P_i + P_i A + \sum_{i=1}^{N} [K_i^{\mathrm{T}} B_i^{\mathrm{T}} P_i + P_i B_i K_i]$, $i = 1, 2, \cdots, N$.

Pre- and post-multiplying both sides of (18) by $(N + 2) \times (N + 2)$ block diagonal matrices $Diag(P_i, I, \cdots, I)$ and denoting $X_i = P_i^{-1}, W_{ij} = K_j X_i$ for $i, j = 1, 2, \cdots, N$, one can yield the following linear matrix inequalities:

$$\begin{bmatrix} \Sigma_i & W_{i1}^{\mathrm{T}} & W_{i2}^{\mathrm{T}} & \dots & W_{iN}^{\mathrm{T}} & X_i \\ W_{i1} & -R_{i1}^{-1} & O & \dots & O & O \\ W_{i2} & O & -R_{i2}^{-1} & \dots & O & O \\ \dots & \dots & \dots & \dots & \dots & \dots \\ W_{iN} & O & O & \cdots & -R_{iN}^{-1} & O \\ X_i & O & O & \dots & O & -Q_i^{-1} \end{bmatrix} < 0, \quad i = 1, 2, \cdots, N \tag{19}$$

where $\Sigma_i = X_iA^{\mathrm{T}} + AX_i + \sum_{j=1}^{N}[W_{ij}^{\mathrm{T}}B_i^{\mathrm{T}} + B_iW_{ij}],\ i = 1, 2, \ldots, N.$

In general, LMIs (19) are called relaxations of BMIs (18). Since inequalities (19) for $i = 1, 2, \ldots, N$ are linear matrix inequalities (LMIs) in X_i for $i = 1, 2, \ldots, N$ and W_{ij} for $i, j = 1, 2, \ldots, N$, we can use MATLAB Toolbox to solve LMIs (19) and obtain feasible solutions $X_i, W_{ij}, i, j = 1, 2, \ldots, N$. problem (19). If problem (19) has feasible solutions $X_i, W_{ij}, i, j = 1, 2, \ldots, N$, the initial point is given by $(X_1^{(0)}, P_2^{(0)}, \cdots, P_N^{(0)})$ where $X_1^{(0)} = X_1, P_i^{(0)} = X_i^{-1}$ for $i = 2, 3, \cdots, N$.

5 Numerical Example

Consider triple players LQ differential game (1) and performance indices (2) with the following data:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix},\ \boldsymbol{B}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ \boldsymbol{B}_2 = \begin{bmatrix} 0.5 \\ 1 \\ 0 \end{bmatrix},\ \boldsymbol{B}_3 = \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix},\ \boldsymbol{x}_0 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}, Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$Q_1 = Q_2 = Q_3,\ R_{11} = R_{22} = 2,\ R_{12} = R_{13} = R_{31} = R_{32} = 0.5,\ R_{21} = R_{23} = R_{33} = 1.$

Solving LMIs (19) and problem (17) by matlab Toolbox, we obtain feasible solutions of ARI (11) as:

$$P_1 = \begin{bmatrix} 2.6200 & 1.0827 & -0.6509 \\ 1.0827 & 1.5957 & -0.4450 \\ -0.6509 & -0.4450 & 1.1509 \end{bmatrix},\quad P_2 = \begin{bmatrix} 2.7354 & 0.9833 & -0.3043 \\ 0.9833 & 1.5936 & -0.1153 \\ -0.3043 & -0.1153 & 0.9889 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 2.0080 & 0.4685 & -0.7343 \\ 0.4685 & 1.4790 & -0.5469 \\ -0.7343 & -0.5469 & 1.3746 \end{bmatrix}$$

The control gains as: $\boldsymbol{K}_1 = [-1.6886\ \ -1.2279\ 0.2603]$, $\boldsymbol{K}_2 = [-1.0233\ \ -0.9850\ \ -0.3607]$, $\boldsymbol{K}_3 = [-1.5079\ \ -0.6612\ \ -0.3669]$ and the guaranteed cost as: $J_1(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3) = 1.3350$, $J_2(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3) = 1.6113$, $J_3(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3) = 0.8091$. The following figures show the time behavior of variables and control inputs in this example.

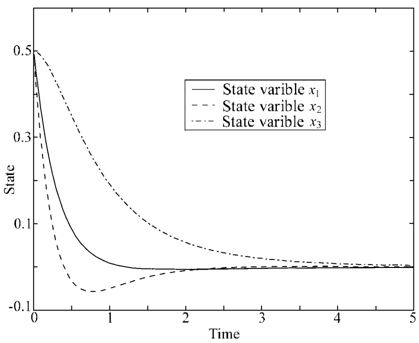

Fig. 1 Time behavior for state variables

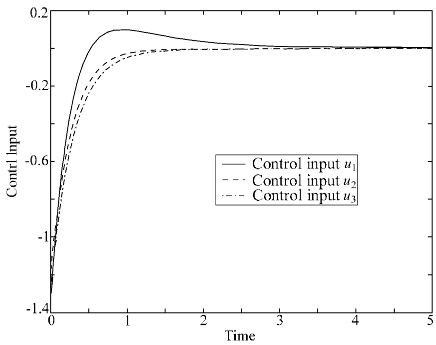

Fig. 2 Time behavior for control ioputs

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