Analysis of Chaotic Synchronization Stability and Its Application to a Two-dimensional Advertising Competing Model1)

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Abstract A class of map in which chaotic synchronization can occur is defined. The transverse Lyapunov exponents are used to determine the stability of synchronized trajectories. Some complex phenomena closely related to chaotic synchronization, namely riddled basin, riddling bifurcation and blowout bifurcation are theoretically analyzed. Riddling bifurcation and blowout bifurcation may change the synchronization stability of the system. And two types of riddled basins, *i.e.*, global riddled basin and local riddled basin, may come into being after riddling bifurcation. An advertising competing model based on Vidale-Wolfe model is proposed and analyzed by the above theories at the end of the paper.

Key words Chaotic synchronization, riddling bifurcation, blowout bifurcation, riddled attracting basin, advertising competing model

1 Introduction

Synchronization of chaotic system was first introduced by Afraimovich *et al*[1], Fujisaka and Yamada[2]. The chaotic synchronization was first applied to secure communication in 1991. In the subsequent years the phenomenon was observed in many physical systems, such as circuit systems[3], electric power systems and laser systems[4]. At the same time chaotic synchronization has been applied to control fields[5,6], biomedicine engineering including brain and heart systems[7], parameter estimation from time series[8], and the stabilization of unstable periodic orbits in chaotic attractors[9]. The chaotic synchronization also can be found in economic systems. [10, 11] showed such examples of a competitive model for market attraction. Hence, it is necessary to study the stability of chaotic synchronization and some phenomenon closely related to chaotic synchronization, namely riddled basin[12∼15], riddling bifurcation and blowout bifurcation[16,17]. Riddling bifurcation and blowout bifurcation can change the synchronization stability of the system.

This paper presents a class of map in which chaotic synchronization can occur. We use the transverse Lyapunov exponent to determine the stability of synchronization. Then we analyze the changes of such stability across the riddling and blowout bifurcations and define the riddled basins and their classification. Finally, an advertising competing model based on Vidale-Wolfe (V-W) model is proposed and analyzed by the above theories at the end of the paper.

2 A class of map of chaotic synchronization

First, we define a class of map that can attain a state of chaotic synchronization under some parameters′ ranges

$$T : r\begin{cases} \boldsymbol{x}(t+1) = \boldsymbol{F}(\boldsymbol{x}(t), \boldsymbol{y}) \\ \boldsymbol{y}(t+1) = \boldsymbol{F}(\boldsymbol{y}(t), \boldsymbol{x}(t)) \end{cases} \tag{1}$$

where $\boldsymbol{x}, \boldsymbol{y} \in R^n, \boldsymbol{F} : R^{2n} \to R^n$. The symmetric property of system (1) implies that the set $\triangle = \{(\boldsymbol{x}(t), \boldsymbol{y}(t)) | \boldsymbol{x}(t) = \boldsymbol{y}(t)\}$ is an invariant subspace for map (1), *i.e.*, $\boldsymbol{F}(\triangle) \subseteq \triangle$. The trajectories of $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ are said to be in synchronization if $|\boldsymbol{x}(t) - \boldsymbol{y}(t)| \to 0$ as $t \to \infty$. In this case map (1)′s trajectories are embedded into the invariant subspace and are governed by the n-dimensional map

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$\boldsymbol{F}(\cdot) = T|_{\triangle} : \triangle \rightarrow \triangle$. And the chaotic synchronization occurs as trajectories of $\boldsymbol{F}(\cdot)$ follow the chaotic dynamics.

3 Transverse Lyapunov exponents

The synchronization stability usually can be determined by the transverse Lyapunov exponent[12~14] (abbreviated as TLE) $L_{\perp}$. If chaos occurs in the invariant subspace, a spectrum of TLEs $L_{\perp}^{\min}, \ldots, L_{\perp}^{nat}$, $\ldots, L_{\perp}^{\max}$ can be defined, which measure the transverse stability of deferent periodic orbits embedded in the chaotic attractor. $L_{\perp}^{nat}$ is said to be the natural TLE expressing the local transversely stable weight of average trajectories in the chaotic set.

For computing the transverse Lyapunov exponent, we need to consider the Jacobian matrix of system (1) in the invariant set

$$DF|_{\triangle} = \begin{bmatrix} F_1'(\boldsymbol{x}, \boldsymbol{x}) & F_2'(\boldsymbol{x}, \boldsymbol{x}) \\ F_2'(\boldsymbol{x}, \boldsymbol{x}) & F_1'(\boldsymbol{x}, \boldsymbol{x}) \end{bmatrix} \tag{2}$$

where $F_1'(\boldsymbol{x}, \boldsymbol{x})$ denotes the first order partial derivative matrix of $\boldsymbol{F}(\cdot)$ with respect to the first vector and is an n-dimensional matrix. And $F_2'(\boldsymbol{x}, \boldsymbol{x})$ denotes the first order partial derivative matrix of $\boldsymbol{F}(\cdot)$ with respect to the second vector. We can obtain $2n$ eigenvalues and eigenvectors from the Jacobain matrix. The transverse eigenvalues $\lambda_{\perp i}$ correspond to the eigenvectors in which the ith element and the $(i+n)$th one have the same absolute value but have the opposite signs, and the eigenvalues $\lambda_{\| i}$ along invariant subspace correspond to the eigenvectors which have the same value for the ith and $(i+n)$th elements $(i = 1, 2, \ldots, n)$. Particular values depend upon the special function. Hence, we define the transverse Lyapunov exponent as

$$L_{\perp i} = \lim_{n \to \infty} \frac{1}{n} \sum_{t=0}^{n} \ln|\lambda_{\perp i}(\boldsymbol{x}(t))| \tag{3}$$

The number of transverse Lyapunov exponents is n, equaling the dimensions of transverse eigenvalues. Each TLE can determine the synchronization stability between an element of vector $\boldsymbol{x}(t)$ and the coupled one of $\boldsymbol{y}(t)$. If there exists one exponent larger then zero, a pair coupled elements of vectors $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ will lose synchronization stability. This case can be called incomplete synchronization. Alternatively, we will talk about the complete synchronization if all exponents are less than zero. This paper mainly focuses on the complete synchronization, so only the maximum eigenvalue is used.

Without loss of generality we consider the following version of system (1)

$$T : \begin{cases} \boldsymbol{x}(t+1) = f(\boldsymbol{x}(t), \boldsymbol{y}(t)) \\ \boldsymbol{y}(t+1) = f(\boldsymbol{y}(t), \boldsymbol{x}(t)) \end{cases} \tag{4}$$

And the following transverse Lyapunov exponent can be obtained, according to the above analysis

$$L_{\perp} = \lim_{n \to \infty} \frac{1}{n} \sum_{t=0}^{n} \ln|f_1'(\boldsymbol{x}(t), \boldsymbol{x}(t) - f_2'(\boldsymbol{x}(t), \boldsymbol{x}(t))| \tag{5}$$

4 Riddled basin of attraction

If chaos occurs in the invariant set, $L_{\perp}^{nat}$ can be calculated by substituting an aperiodic point $(\in A_s)$ into (3). When all the periods embedded in chaotic attractor are transversely stable, *i.e.*, $L_{\perp}^{\max} < 0$, the chaotic attractor in the invariant set is asymptotically stable in the common Lyapunov sense. Once an unstable periodic orbit nested in chaotic attractor (periodic saddle) loses its transverse stability but the chaotic attractor still attracts its neighborhood points in the transverse direction, *i.e.*, $L_{\perp}^{\max} > 0$ and $L_{\perp}^{nat} < 0$, the chaotic attractor is weakly stable in the Milor sense[14]. Riddling bifurcation occurs where $L_{\perp}^{\max}$ changes its signs from negative to positive, and a riddled basin of the chaotic attractor comes into being.

$\beta(A_s)$ is referred to as riddled basin of attractor A_s, if for any point x in attractor A_s and $\delta > 0$, there exists $\mu(\beta(A_s) \cap U_\delta(x))\mu(\beta(A_s)^C \cap U_\delta(x)) > 0$, where $\mu(\cdot)$ denotes the Lebesgue measure and $U_\delta(x)$ means the δ-neighborhood of x. Moreover, we have $\lim_{\delta \to 0} \mu(\beta(A) \cap (U_\delta(A))/\mu(U_\delta(A)) = 1$[13].

Furthermore, after riddling bifurcation, a dense set of repelling "tongues" opens from the transversely unstable repeller and its preimages which may form either local or global riddled basin of attraction. For local riddled basin, the trajectories starting from the repelling "tongues" region leave the attractor in a finite number of iterations but these trajectories are restricted by the nonlinear mechanism to move within an absorbing area[18] and finally reinject into the attractor. The absorbing area denoted by Abs is rigidly confined in the attraction basin and contains the attractor A_s with the properties: 1) there exists a neighborhood $Abs \subset U_\delta$ such that points in u_δ will enter Abs after a finite number of iterations; 2) Abs is trapping, which means trajectories entering Abs cannot leave it any longer, *i.e.*, $T(Abs) = Abs$.

In a global riddled basin, for the weak nonlinear mechanism, the repelling trajectories can not be restricted in the aforementioned absorbing area Abs, so the locally repelled trajectories will go to another coexistent attractor (or infinity) ultimately. In this case, many sets of repelling points will spread from the invariant set and the basin of attraction for the chaotic attractor is a Cantor set, signifying riddling.

Different types of riddled basins may come into being for different maps. For the map proposed in [12], a local riddled basin appears first, and then a global riddled basin emerges after a local-global riddled bifurcation. We will illustrate only emergence of global basin for the model of this paper in Section 5.

The parameter values where $L_\perp^{nat}$ changes its signs from negative to positive is referred to as blowout bifurcation points. At such critical values chaotic attractor loses its weak stability, *i.e.*, the whole chaotic attractor becomes a chaotic saddle.

Summarizing above analyses, the transformation process of transverse stability can be shown by Fig. 1. In Fig. 1, S denotes the invariant set and the broad-brush in the pane means chaotic attractor in the invariant set. The small arrowheads near the invariant set represent repelling direction of trajectories$'$ motion.

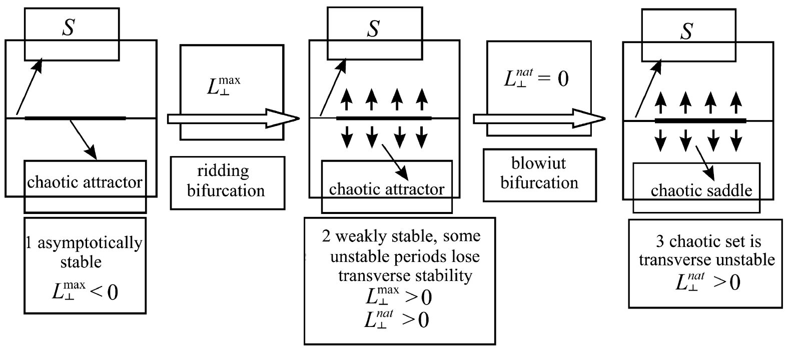

Fig. 1 The basic transformation process of transverse stability

5 An economic model

5.1 The description of model

This model is an extended competitive version of the classic V-W advertising model. The Lanchester combat model is used to describe the evolution of the market share of the two rivals. The model shows some chaotic synchronized dynamics after advertising efforts are used as linear feedback control variable. The model is defined as

$$\begin{aligned} x(t+1) &= a_1 x(t) + k_1 u_1(t)(1 - x(t)) - k_2 u_2(t) x(t) \\ y(t+1) &= a_2 y(t) + k_2 u_2(t)(1 - y(t)) - k_1 u_1(t) y(t) \end{aligned} \tag{6}$$

where $t = 0, 1, \ldots, T$, is the discrete time period; $x(t)$ and $y(t)$ denote the capture fractions of the market potential for two firms and satisfy $0 < x(t), y(t) < 1$; k_i is fixed positive values that measure the advertising effectiveness of firm i; a_i is the proportion of customers who will keep their purchasing habits with firm i in the next period. Furthermore, the parameters satisfy: $0 < k_i, a_i < 1 (i = 1, 2)$. Because a customer could buy both firms′ goods, or could buy no goods of the two firms, either $x(t)+y(t)>1$ or $x(t)+y(t)<1$ could have been appearing in the actual market.

$u_1(t)$ and $u_2(t)$ are the current advertising efforts invested by firms 1 and 2, respectively. We choose an advertising efforts policy named sales-percent policy that is applied by many corporations to actual market[19,20], *i.e.*, take the sum of one firm sales′ proportion and its rivals′ one as the advertising effort. Hence it is a linear function as follows.

$$u_1(t)=\eta_1\pi Bx(t)+\lambda_1 pBy(t),\quad u_2(t)=\eta_2 pBy(t)+\lambda_2 pBx(t) \tag{7}$$

where B denotes the population of potential customers, p is the price of goods, and the percents taken from the firm′s own sales are denoted by η_1 and η_2, respectively. λ_1 and λ_2 are the proportions of added advertising expenditures to the rival′s sales, which expresse the advertising will be enhanced as the rival′s sales increase. For convenience, let

$$h_1=\eta_1 pB,\quad n_2=\eta_2 pB,\quad j_1=\lambda_1 pB,\quad j_2=\lambda_2 pB \tag{8}$$

We can obtain the following map by substituting advertising efforts into state equation as linear feedback control variable.

$$\begin{aligned}x(t+1)&=a_1x(t)+k_1(h_1x(t)+j_1y(t))(1-x(t))-k_2(h_2y(t)+j_2x(t))x(t)\\ y(t+1)&=a_2y(t)+k_2(h_2y(t)+j_2x(t))(1-y(t))-k_1(h_1x(t)+j_1y(t))y(t)\end{aligned} \tag{9}$$

5.2 Transformation of the system

Assume that two competitive firms produce the same daily expendable goods and have the same parameters. Given that two firms can get the rival′s entire policy by way of business information, the firm who wants to win in the competition would increase its advertising expenditure beyond the rival′s. However, increasing expenditure boundlessly can damage firm′s profit, and finally the proportion η,λ chosen by the two firms will coincide. Hence, we get a symmetric two-dimensional dynamic system as the following

$$\begin{aligned}x(t+1)&=ax(t)+k(hx(t)+jy(t))(1-x(t))-k(hy(t)+jx(t))x(t)\\ y(t+1)&=ay(t)+k(hy(t)+jx(t))(1-y(t))-k(hx(t)+jy(t))y(t)\end{aligned} \tag{10}$$

Let $A=kh+kj, C=a+kh, D=kj$ to simplify the system. Restricting the system in the invariant set $\triangle$, we can get a one-dimensional map

$$f(x)=-2Ax^2+(C+D)x \tag{11}$$

Map (10) is a particular version of map (4), so we have

$$L_\perp=\lim_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n}\ln|-2Ax(t)+C-D| \tag{12}$$

We mainly study the dynamics in the invariant set and its neighborhood, so we do the homeomorphous transformation as the following.
Let

$$z_\parallel=(x+y)/2,\quad z_\perp=(y-x)/2 \tag{13}$$

We have

$$z_\parallel(t+1)=-2Az_\parallel^2(t)+(C+D)z_\parallel(t),\quad z_\perp(t+1)=-2Az_\parallel(t)z_\perp(t)+(C-D)z_\perp(t) \tag{14}$$

where $z_\parallel$ and $z_\perp$ express the dynamics along and perpendicular to the invariant set, respectively. The invariant set $z_\perp=0$ of system (14) is equivalent to the synchronized invariant set $\triangle$ of original system (10), and two systems have the same dynamic function in invariant set. Hence this transformation keeps the properties of system (10) in the neighborhood of invariant set unchanged.

Furthermore, using the linear transformation

$$z_{\|} = \frac{C+D}{2A}xx, \quad z_{\perp} = yy \tag{15}$$

we obtain the Logistic map respect to state variable xx

$$xx(t+1) = (C+D)xx(t)(1-xx(t)), \quad yy(t+1) = -(C+D)xx(t)yy(t) + (C-D)yy(t) \tag{16}$$

Letting parameter $r = C+D, C-D = r-2D$, we get the corresponding relation of these parameters: $r = a + kh + kj, D = kj$.

5.3 Synchronized stability and numerical results

We only need to discuss the transformed system (16) according to the above analysis. As an approach to determine the transverse stability we may evaluate transverse Lyapunov exponent first

$$L_{\perp} = \lim_{n\to\infty} \frac{1}{n}\sum_{t=0}^{n-1}[-rxx(t) + r - 2D] \tag{17}$$

From (17), we can find that $L_{\perp}$ is related to parameters r, D and the trajectories of $xx(t)$ in the invariant set. From the properties of Logistic map, we know that system (16) will show complex dynamics in the invariant set as $r > 3$[21].

Fig. 2 shows the blowout bifurcation points in parameter space of r and D. We perform 172400 scans of $L_{\perp}^{nat}$ for every different value of r and D, by ranging r from 3.569 to 4 with step of 0.001 and ranging D from 0 to 2 with step 0.005. Then we draw the blowout bifurcation points where $abs(L_{\perp}^{nat}) < 10^{-10}$ in Fig. 2.

From above numerical results, we find that the values of blowout bifurcation points are almost two regular straight lines: $D = 0$ and $D = (r-1)/2$, except in the stable period windows. Fig. 2 shows the several remarkable regions of period windows in which blowout bifurcation points are not regular. And in such period windows, the value of $L_{\perp}^{nat}$ alternates between negative and positive as parameter D rises. It is worth noting that the distinction between asymptotic and weak stability disappears in periodic windows for only one TLE exists in this case.

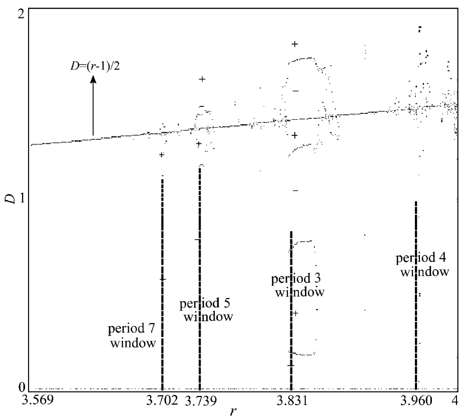

Fig. 2 Points in parameter plane at which bifurcation occurs

For the fixed point $1-1/r$ in the invariant set, according to equation (17), we can get the parameter region of D : $[0,1]$, in which the fixed point is transversely stable. Similarly, we get the region of D in which the period-2 points $(1+r-\sqrt{r^2-2r-3})/2r$ and $(1+r+\sqrt{r^2-2r-3})/2r$ are transversely stable

$$D \in \begin{cases} [0, (r-1)/4 - \sqrt{r^2-2r-7} \text{ or } [(r-1)/4 - \sqrt{r^2-2r-7}], (r-1)/2], & \text{for } r \geqslant 1+2\sqrt{2} \\ [0, (r-1)/2], & \text{for } r < 1+2\sqrt{2} \end{cases} \tag{18}$$

It is difficult to get the analyzed solutions of other periodic points, so the only way that we can use is numerical methods.

An interesting result is that the upper values of blowout points coincide with the upper values $(r-1)/2$ of transversely stable region for period 2, and the lower ones are the lower values 0 of transversely stable region for fixed point. This shows that the periods 1 and 2 orbits play an important role in dynamics of whole system.

The other finding is related to riddled basin. For system in this paper, only global riddled basin can come into being. As D rises and is slightly larger than 1, the riddling bifurcation occurs and the fixed point $(1-1/r)$ loses its transverse stability $(L_{\perp}^{fix} \geqslant 0)$. As a result, many repelling tongues appear in the neighborhood of fixed point and its preimages along the transverse direction. Because the map function perpendicular to the invariant set is linear, the local properties of the invariant set $yy=0$ can be extended to the whole state space. So these local repelling sets will be perpendicularly repelled from the neighborhood of invariant set and finally go to infinity. On the other hand, the TLE reflecting the result of the linear approximation of map (10) around invariant set not only can determine the local transverse stability, but also shows the global transverse stability for the attractor in invariant set.

Given the assumption of the model, we can get the following results in economy: If competition is not intense, *i.e.*, the competitive parameters h and j are small, two firms will converge to the same market share whatever their initial conditions are. If the competition begins intense, the final market share will be entirely dependent on the initial condition and market will fluctuate so sharply that it will lose stability. The later case should be avoided in actual marketing.

6 Conclusion

This paper presents a class of map in which chaotic synchronization could emerge under parameters′ some range and defines the calculating method of transverse Lyapunov exponents. A spectrum of TLEs is defined as chaotic trajectories appear in invariant set, where the natural TLE is calculated with the chaotic trajectories. The blowout bifurcation in which natural TLE changes signs may let the attractor in invariant set lose weak stability and the riddling bifurcation in which the maximum TLE changes signs may let the attractor lose asymptotic stability. Two types of riddled basin will come into being after riddling bifurcation for maps′ different nonlinear mechanisms. An economic instance-two dimensional competitive model of advertising is proposed. The blowout bifurcation points are numerically computed and a conclusion is obtained that only global riddled basin comes into being after riddling bifurcation for this model.

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