# Dynamics and Stable Control for a Class of Underactuated Mechanical Systems[1)]

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**Abstract** The control of underactuated mechanical systems is very complex for the loss of its control inputs. The model of underactuated mechanical systems in a potential field is built with Lagrangian method and its structural properties are analyzed in detail. A stable control approach is proposed for the class of underactuated mechanical systems. This approach is applied to an underactuated double-pendulum-type overhead crane and the simulation results illustrate the correctness of dynamics analysis and validity of the proposed control algorithm.

**Key words** Underactuated systems, system dynamics, stable control, pendulum, overhead crane

## 1 Introduction

Underactuated mechanical systems (UMSs) are a class of mechanical systems that have fewer control inputs than generalized coordinates variables. UMS has advantage over fully-actuated systems in energy saving, cost reducing, manufacturing and installing. The restriction of control inputs of UMS brings a challenging control problem. Moreover, a fully-actuated system may become an UMS because of actuator failure and so the control algorithm for UMS can be used as a kind of fault-tolerate control algorithm.

A remarkable research effort has been devoted to the UMS dynamics and control properties. The partial feedback linerization was put forward[1]. Three classes of control problems were analyzed[2]. Nonholonomic constraint and controllability were researched[3]. System dynamics, controllability and stabilizability results were derived[4]. However, there are few results that are applicable to entire class of UMSs[5]. In this paper, a class of UMSs in a potential field is considered: its dynamic model is built, its structural properties are analyzed, and a stable control approach is designed for the entire class of UMSs. Finally simulation is performed with an underactuated double-pendulum-type overhead crane (DPTOC).

## 2 Dynamics and properties of UMSs

According to the Lagrange mechanics, *i.e.*, Euler-Lagrange equation, the dynamics of the class of UMSs in a potential field can be built as follows.

$$M(\boldsymbol{q})\ddot{\boldsymbol{q}} + C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + G(\boldsymbol{q}) = \bar{\boldsymbol{\tau}} \tag{1}$$

where $\boldsymbol{q} = (q_1, \cdots, q_n)^{\mathrm{T}}$ is the generalized coordinates vector, $\bar{\boldsymbol{\tau}} = (\tau_1, \cdots, \tau_m, 0, \cdots, 0)^{\mathrm{T}} = (\boldsymbol{\tau}, \mathbf{0})^{\mathrm{T}}$ is the control vector. When $m < n$, the system is said to be underactuated. $M(\boldsymbol{q}) \in R^{n\times n}$ is the inertia matrix and $m_{ij}(\boldsymbol{q})$ is its matrix element. $C(\boldsymbol{q},\dot{\boldsymbol{q}})$ is the Centrifugal terms ($i = j$) and Coriolis terms ($i \neq j$), its matrix element is

$$c_{ij}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \sum_{k=1}^{n} \Gamma^{i}_{k,j}(\boldsymbol{q})\dot{\boldsymbol{q}}_k \tag{2}$$

$\Gamma^{k}_{i,j}(\boldsymbol{q})$ are called Christoffel symbols and defined as

$$\Gamma^{k}_{i,j}(\boldsymbol{q}) = \frac{1}{2}\left(\frac{\partial m_{kj}(\boldsymbol{q})}{\partial q_i} + \frac{\partial m_{ki}(\boldsymbol{q})}{\partial q_j} - \frac{\partial m_{ij}(\boldsymbol{q})}{\partial q_k}\right) \tag{3}$$

$G(\boldsymbol{q})$ contains the gravity terms

$$G(\boldsymbol{q}) = \partial P(\boldsymbol{q})/\partial \boldsymbol{q} \tag{4}$$

The UMSs have the following structural properties:

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**Property 1.** $M(\boldsymbol{q})$ is a positive definite symmetric matrix[6].

**Property 2.** $N(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{M}(\boldsymbol{q}) - 2C(\boldsymbol{q},\dot{\boldsymbol{q}})$ is a skew symmetric matrix.

**Proof.** Each element of the derivative of the inertia matrix is given by

$$\dot{m}_{ij}(\boldsymbol{q}) = \sum_{k=1}^{n} \frac{\partial m_{ij}(\boldsymbol{q})}{\partial q_k}\dot{q}_k$$

Each element of $N(\boldsymbol{q},\dot{\boldsymbol{q}})$ can be calculated from (2) and (3), *i.e.*,

$$n_{ij}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{m}_{ij}(\boldsymbol{q}) - 2c_{ij}(\boldsymbol{q}) = \sum_{k=1}^{n}\left(\frac{\partial m_{kj}(\boldsymbol{q})}{\partial q_i} - \frac{\partial m_{ik}(\boldsymbol{q})}{\partial q_j}\right)\dot{q}_k$$

Recalling Property 1, it is straightforward to deduce that

$$n_{ji}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \sum_{k=1}^{n}\left(\frac{\partial m_{ki}(\boldsymbol{q})}{\partial q_j} - \frac{\partial m_{jk}(\boldsymbol{q})}{\partial q_i}\right)\dot{q}_k = -\sum_{k=1}^{n}\left(\frac{\partial m_{kj}(\boldsymbol{q})}{\partial q_i} - \frac{\partial m_{ik}(\boldsymbol{q})}{\partial q_j}\right)\dot{q}_k = -n_{ij}(\boldsymbol{q},\dot{\boldsymbol{q}}) \tag{5}$$

*i.e.*, $\dot{M}(\boldsymbol{q}) - 2C(\boldsymbol{q},\dot{\boldsymbol{q}})$ is a skew symmetric matrix. □

**Property 3.** The UMSs are passive systems

**Proof.** The total energy function of the UMSs can be written as

$$E(\boldsymbol{q},\dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^{\mathrm{T}}M(\boldsymbol{q})\dot{\boldsymbol{q}} + P(\boldsymbol{q}) \tag{6}$$

where $\frac{1}{2}\dot{\boldsymbol{q}}^{\mathrm{T}}M(\boldsymbol{q})\dot{\boldsymbol{q}}$ is the system kinetic energy, and $P(\boldsymbol{q})$ is the system potential energy. A reference point of the potential energy can be chosen to make $P(\boldsymbol{q}) \geqslant 0$, and so $E(\boldsymbol{q},\dot{\boldsymbol{q}}) \geqslant 0$. The differential of $E(\boldsymbol{q},\dot{\boldsymbol{q}})$ can be computed using (1), (4) and Properties 1-2:

$$\dot{E}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^{\mathrm{T}}M(\boldsymbol{q})\ddot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^{\mathrm{T}}\dot{M}(\boldsymbol{q})\dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}^{\mathrm{T}}\frac{\partial P(\boldsymbol{q})}{\partial q} = \dot{\boldsymbol{q}}^{\mathrm{T}}\bar{\boldsymbol{\tau}} + \frac{1}{2}\dot{\boldsymbol{q}}^{\mathrm{T}}(\dot{M}(\boldsymbol{q}) - 2C(\boldsymbol{q},\dot{\boldsymbol{q}}))\dot{\boldsymbol{q}} = \dot{\boldsymbol{\Theta}}^{\mathrm{T}}\boldsymbol{\tau} \tag{7}$$

where $\boldsymbol{\Theta} = (q_1, q_2, \cdots, q_m)^{\mathrm{T}} = Z\boldsymbol{q}$, $\boldsymbol{\tau} = (\tau_1, \tau_2, \cdots, \tau_m)^{\mathrm{T}} = Z\bar{\boldsymbol{\tau}}$, and $Z = [I_m \quad 0]$. Therefore,

$$\int_0^t \dot{\boldsymbol{\Theta}}^{\mathrm{T}}\boldsymbol{\tau}\mathrm{d}t = E(\boldsymbol{q}(t),\dot{\boldsymbol{q}}(t)) - E(\boldsymbol{q}(0),\dot{\boldsymbol{q}}(0)) \geqslant -E(\boldsymbol{q}(0),\dot{\boldsymbol{q}}(0)) \ \textit{i.e.}\ \langle\boldsymbol{\tau}|\dot{\boldsymbol{\Theta}}\rangle_t \geqslant -E(\boldsymbol{q}(0),\dot{\boldsymbol{q}}(0)) \tag{8}$$

Thus, UMSs are passive systems with respect to input $\boldsymbol{\tau}$ and output $\dot{\boldsymbol{\Theta}}$[7]. □

## 3 Stable control for the UMSs

It is supposed that the control objective is to control the system to stay in $\boldsymbol{q}_d$, one of the equilibrium states, where $\boldsymbol{\Theta}_d$ is the control objective of the actuated part. The following Lyapunov function is defined

$$V(\boldsymbol{q},\dot{\boldsymbol{q}}) = \frac{1}{j}k_E(E(\boldsymbol{q},\dot{\boldsymbol{q}}) - P(\boldsymbol{q}_d))^j + \frac{1}{2}k_D\dot{\boldsymbol{\Theta}}^{\mathrm{T}}\dot{\boldsymbol{\Theta}} + \frac{1}{2}k_P(\boldsymbol{\Theta} - \boldsymbol{\Theta}_d)^{\mathrm{T}}(\boldsymbol{\Theta} - \boldsymbol{\Theta}_d) \tag{9}$$

where $k_E$ is a positive constant, $P(\boldsymbol{q}_d)$ is the potential energy at the desired position, $j$ is a constant. In order to make $V(\boldsymbol{q},\dot{\boldsymbol{q}}) \geqslant 0$, $j$ should be 1 when the desired position is the minimal potential energy point among all the accessible states, and $j$ should be 2 when the desired position is not the minimal potential energy point. Both $k_D$ and $k_P$ are positive constants.

When $j = 1$, the differential of Lyapunov function $V(\boldsymbol{q},\dot{\boldsymbol{q}})$ can be calculated as

$$\dot{V}(\boldsymbol{q},\dot{\boldsymbol{q}}) = k_E\dot{E}(\boldsymbol{q},\dot{\boldsymbol{q}}) + k_D\dot{\boldsymbol{\Theta}}^{\mathrm{T}}\ddot{\boldsymbol{\Theta}} + k_P(\dot{\boldsymbol{\Theta}} - \dot{\boldsymbol{\Theta}}_d)^{\mathrm{T}}(\boldsymbol{\Theta} - \boldsymbol{\Theta}_d) = \dot{\boldsymbol{\Theta}}^{\mathrm{T}}(k_E\tau + K_D\ddot{\boldsymbol{\Theta}} + k_P(\boldsymbol{\Theta} - \boldsymbol{\Theta}_d)) \tag{10}$$

In order to ensure the system stability, we take

$$\dot{V}(\boldsymbol{q},\dot{\boldsymbol{q}}) = -k\dot{\boldsymbol{\Theta}}^{\mathrm{T}}\dot{\boldsymbol{\Theta}} \tag{11}$$

where $k$ is a positive real number. Then following equation holds

$$k_E\boldsymbol{\tau} + k_D\ddot{\boldsymbol{\Theta}} + k_P(\boldsymbol{\Theta} - \boldsymbol{\Theta}_d) = -k\dot{\boldsymbol{\Theta}} \tag{12}$$

From (1) and (12), the control law can be obtained as follows.

$$\boldsymbol{\tau} = -(k_E \boldsymbol{I}_m + k_D Z M^{-1}(\boldsymbol{q}) Z^{\mathrm{T}})^{-1}(k_p(\boldsymbol{\Theta} - \boldsymbol{\Theta}_d) - k_D Z M^{-1}(\boldsymbol{q})(C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + G(\boldsymbol{q})) + k\dot{\boldsymbol{\Theta}}) \tag{13}$$

**Theorem 1.** For UMSs described by (1), when the equilibrium point $q_d$ is the minimal potential energy point in the system accessible space, controller (13) can make the closed loop system converge to the equilibrium point $q_d$ or the stable trajectory $\frac{1}{2}\dot{\boldsymbol{q}}^{\mathrm{T}} M(\boldsymbol{q})\dot{\boldsymbol{q}} + P(\boldsymbol{q}) = C$, where $C$ is a positive constant.

**Proof.** From (9) and (11), $V(\boldsymbol{q},\dot{\boldsymbol{q}}) \geqslant 0$ holds. There are two cases.

When $\boldsymbol{q} = \boldsymbol{q}_d$ and $\dot{\boldsymbol{q}} = \boldsymbol{0}$ hold at the same time, $V(\boldsymbol{q},\dot{\boldsymbol{q}}) = 0$ holds. From (13), $\boldsymbol{\tau} = 0$ holds. Therefore, the system is stabilized to the desired position.

When $\boldsymbol{q} \neq \boldsymbol{q}_d$ or $\dot{\boldsymbol{q}} \neq \boldsymbol{0}$, $V(\boldsymbol{q},\dot{\boldsymbol{q}})$ is a positive definite function. From (11), $\dot{V}(\boldsymbol{q},\dot{\boldsymbol{q}}) \leqslant 0$ holds. Therefore, $\dot{\boldsymbol{\Theta}} \in L_2 \cap L_\infty$, $E(\boldsymbol{q},\dot{\boldsymbol{q}})$ and $\boldsymbol{\Theta}$ are bounded. From (6), $q, \dot{\boldsymbol{q}}$ and $M(\boldsymbol{q})$ are bounded. $M(\boldsymbol{q})$ is a positive definite symmetric matrix indicating that $M^{-1}(\boldsymbol{q})$ is bounded. From (1), $\ddot{\boldsymbol{q}}$ is bounded, and then $\ddot{V}(\boldsymbol{q},\dot{\boldsymbol{q}}) = -2k\ddot{\boldsymbol{\Theta}}^{\mathrm{T}}\dot{\boldsymbol{\Theta}}$ is bounded. This indicates $\dot{V}(\boldsymbol{q},\dot{\boldsymbol{q}})$ is a uniformly continuous function. According to Barbalat′s lemma, $\lim_{t\to\infty}\dot{V}(\boldsymbol{q},\dot{\boldsymbol{q}}) = 0$ *i.e.*, $\lim_{t\to\infty}\boldsymbol{\Theta} = \boldsymbol{0}$. It can be seen that $\lim_{t\to\infty} V(\boldsymbol{q},\dot{\boldsymbol{q}})$ and $\lim_{t\to\infty}(\boldsymbol{\Theta} - \boldsymbol{\Theta}_d)$ are constant. From (7) and (13), $\lim_{t\to\infty} E(\boldsymbol{q},\dot{\boldsymbol{q}})$ and $\lim_{t\to\infty}\boldsymbol{\tau}$ are constant. If $\lim_{t\to\infty}\boldsymbol{\tau} \neq \boldsymbol{0}$ holds, then $\boldsymbol{\Theta}$ will change with $t \to \infty$. This is inconsistent with that $\lim_{t\to\infty}(\boldsymbol{\Theta} - \boldsymbol{\Theta}_d)$ is constant. Thus $\lim_{t\to\infty}\boldsymbol{\tau} = \boldsymbol{0}$ must be true. From (12), $\lim_{t\to\infty}\boldsymbol{\Theta} = \boldsymbol{\Theta}_d$ holds. The stability analyse can be divided into two cases.

When $\lim_{t\to\infty} E(\boldsymbol{q},\dot{\boldsymbol{q}}) = P(q_d)$, (6) can be written as $\frac{1}{2}\dot{\boldsymbol{q}}^{\mathrm{T}} M(\boldsymbol{q})\dot{\boldsymbol{q}} + (P(\boldsymbol{q}) - P(q_d)) = 0$ with $t \to \infty$, and both the two terms in the left side of the above equation are greater than or equal to zero. Therefore, the system kinetic energy is equal to zero, and potential energy is equal to the potential energy of the desired position, *i.e.*, $\boldsymbol{q} = \boldsymbol{q}_d$ and $\dot{\boldsymbol{q}} = \boldsymbol{0}$. That is to say, the system is stabilized to the desired position.

When $\lim_{t\to\infty} E(\boldsymbol{q},\dot{\boldsymbol{q}}) = C \neq P(q_d)$, the system converges to a stable trajectory $\frac{1}{2}\dot{\boldsymbol{q}}^{\mathrm{T}} M(\boldsymbol{q})\dot{\boldsymbol{q}} + P(\boldsymbol{q}) = C$, where $\boldsymbol{\Theta} = \boldsymbol{\Theta}_d$, $C$ is a constant greater than $P(q_d)$ and can be obtained with the above equation and (1) when $\bar{\boldsymbol{\tau}} = 0$. □

When $j = 2$, the following control law can be obtained

$$\boldsymbol{\tau} = -(k_E(E(\boldsymbol{q},\dot{\boldsymbol{q}}) - P(\boldsymbol{q}_d)) + k_D Z M^{-1}(\boldsymbol{q}) Z^{\mathrm{T}})^{-1}(k_P(\boldsymbol{\theta} - \boldsymbol{\theta}_d) - k_D Z M^{-1}(\boldsymbol{q})(C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + G(\boldsymbol{q})) + k\dot{\boldsymbol{\Theta}})$$

The system stability can be proved in the same way as Theorem 1.

## 4 Simulation studies

When the mass of crane hook cannot be ignored, the overhead crane performs as a double pendulum system[8], and the system is a typical UMS with one control input and three generalized coordinates variables[9].

### 4.1 Dynamics and properties of DPTOC

Fig. 1 shows the DPTOC, in which $m$ is the trolley mass, $m_1$ is the hook mass, $m_2$ is the load mass, $x$ is the trolley position, $\theta_1$ is the hook swing angle, $\theta_2$ is the load swing angle, $l_1$ is the cable length for the hook, $l_2$ is the cable length for load, $F$ is the trolley drive force. The friction and cable mass are ignored.

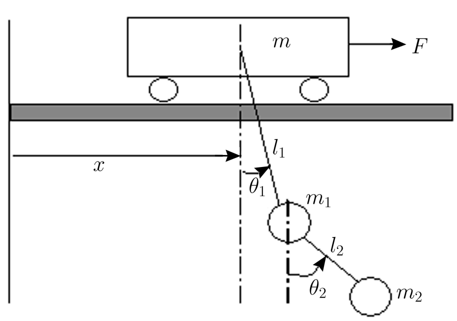



Fig. 1 DPTOC system scheme

The system dynamics can be described by (1), where

$$\bar{\boldsymbol{\tau}} = [F, 0, 0],\ \boldsymbol{q} = [x, \theta_1, \theta_2]^{\mathrm{T}},\ G(\boldsymbol{q}) = [0 \quad (m_1+m_2)gl_1\sin\theta_1 \quad m_2gl_2\sin\theta_2]^{\mathrm{T}}$$

$$M(\boldsymbol{q}) = \begin{bmatrix} m+m_1+m_2 & (m_1+m_2)l_1\cos\theta_1 & m_2l_2\cos\theta_2 \\ (m_1+m_2)l_1\cos\theta_1 & (m_1+m_2)l_1^2 & m_2l_1l_2\cos(\theta_1-\theta_2) \\ m_2l_2\cos\theta_2 & m_2l_1l_2\cos(\theta_1-\theta_2) & m_2l_2^2 \end{bmatrix}$$

$$C(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{bmatrix} 0 & -(m_1+m_2)l_1\dot{\theta}_1\sin\theta_1 & -m_2l_2\dot{\theta}_2\sin\theta_2 \\ 0 & 0 & m_2l_1l_2\dot{\theta}_1\sin(\theta_1-\theta_2) \\ 0 & -m_2l_1l_2\dot{\theta}_1\sin(\theta_1-\theta_2) & 0 \end{bmatrix}$$

In addition to Properties 1-3, the DPTOC system has two different natural frequencies that are calculated through the linearization of (1) around $\theta_1 = 0$ and $\theta_2 = 0$:

$$\bar{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + K\boldsymbol{q} = 0 \tag{14}$$

where $\bar{M}(\boldsymbol{q})$ is the linearization matrix of $M(\boldsymbol{q})$, and

$$K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & (m_1+m_2)gl_1 & 0 \\ 0 & 0 & m_2gl_2 \end{bmatrix}$$

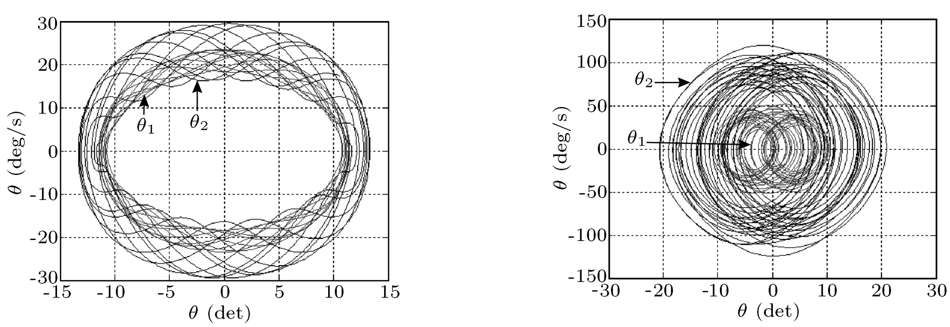
The natural frequencies can be obtained with nonzero eigenvalue of matrix $-\bar{M}^{-1}(\boldsymbol{q})K$[10]:

$$\omega_{1,2} = \sqrt{\frac{g}{2}(\alpha \pm \sqrt{\beta})} \tag{15}$$

where $\alpha = \dfrac{m_1+m_2}{m_1}\left(\dfrac{1}{l_1}+\dfrac{1}{l_2}\right)$, $\beta = \left(\dfrac{m_1+m_2}{m_1}\right)^2\left(\dfrac{1}{l_1}+\dfrac{1}{l_2}\right)^2 - 4\left(\dfrac{m_1+m_2}{m_1}\right)\dfrac{1}{l_1l_2}$.

### 4.2 Dynamics simulation for DPTOC

In DPTOC system, the parameters that always change in different transport tasks are the payload mass $m_2$ and the length of cable $l_1$. The effects of these parameters′ changes and different initial conditions are considered. In the simulation, the basic parameters are: $m = 5$Kg, $m_1 = 2$Kg, $m_2 = 5$Kg, $l_1 = 2$m, $l_2 = 1$m, and basic initial state is: $\theta_1 = 11.465°$, $\theta_2 = 11.465°$, $x = 0$m, $\dot{\theta}_1 = 0°$/s, $\dot{\theta}_2 = 0°$/s, $x = 0$m/s. The phase plane under the basic parameters and basic initial state are shown in Fig. 2 (a). Only one parameter or one initial state is changed in the following simulations. When parameter $\theta_2$ of the initial state is changed to $-11.465°$, the phase plane is shown in Fig. 2 (b). When parameter $m_2$ is changed to 50Kg, the phase plane is shown in Fig. 2 (c). When parameter $l_1$ is changed to 5m, the phase plane is shown in Fig. 2 (d). It can be seen that the dynamics are more complex than that of the single-pendulum-type overhead crane, and are affected by the system parameters′ change. Moreover, the dynamics are greatly affected by the initial conditions for its nonlinearity.




(a) (b)

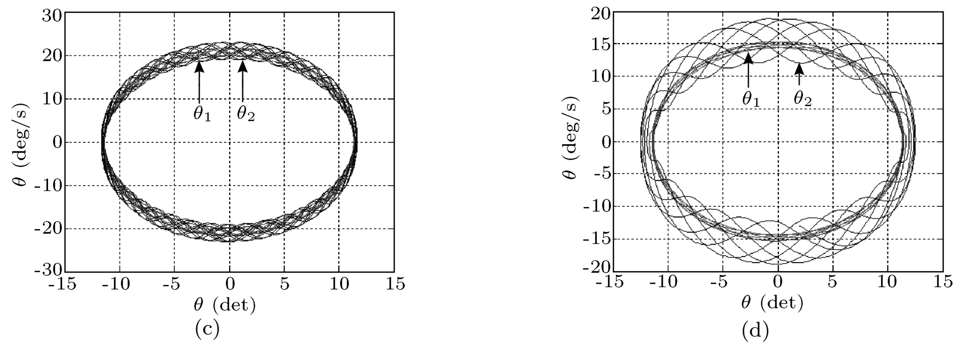



Fig. 2 Simulation results for system dynamics

### 4.3 Stable control simulation

The proposed stable control algorithm is added to the DPTOC system, and the desired position is chosen as the reference point of the potential energy. The following controller parameters are used: $k_E = 1$, $k_D = 0$, $k_P = 10$, $k = 20$. In the simulation, the initial position $x = 0$m of the basic initial state is changed to $x = -30$m. The system dynamics under the basic parameters and basic initial state is shown in Fig. 3 (a). When parameter $\theta_2$ of the initial state is changed to $-11.465^\circ$, the system dynamics is shown in Fig. 3 (b). When system parameter $m_2$ is changed to 50Kg, the system dynamics is shown in Fig. 3 (c). When system parameter $l_1$ is changed to 5m, the system dynamics is shown in Fig.3 (d). It is clear that the proposed control law can transport the payload to the desired position while damping swing angle, and has some robustness to the parameters changes. Moreover, the control performance is improved with a larger load mass and is degraded with a larger cable length $l_1$, and the system may be stabilized to a stable trajectory.

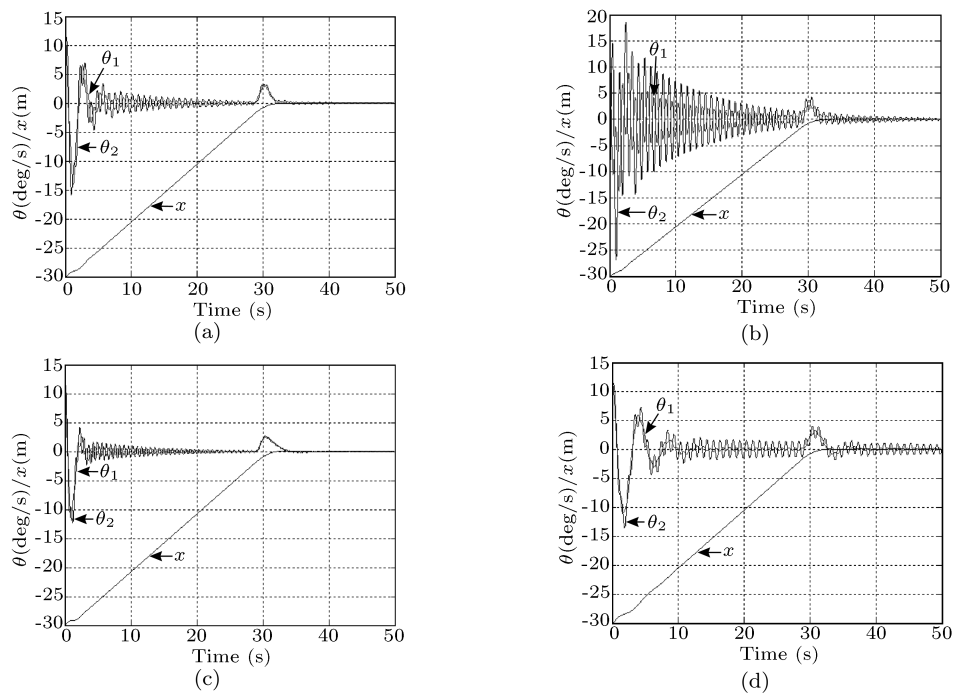



Fig. 3 Simulation results for stable control

## 5 Conclusions

Control of UMSs is currently an active field of research due to their broad applications while the restriction of control inputs of UMSs brings forward a challenging control problem. The dynamic model of the UMSs is built with Lagrangian method and their several structural properties such as the positive definite symmetric inertia matrix and the passivity are analyzed. A stable control method is proposed for a class of UMSs. The underactuated DPTOC system is used to validate the proposed control algorithm. Simulation results illustrate the complex dynamics of the DPTOC and the effectiveness of proposed control algorithm.

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