$\boldsymbol{H}_\infty$ Filter Design for Discrete-time Systems with Missing Measurements[1)]

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Abstract For packet-based transmission of data over a network, or temporary sensor failure, *etc.*, data samples may be missing in the measured signals. This paper deals with the problem of H_∞ filter design for linear discrete-time systems with missing measurements. The missing measurements will happen at any sample time, and the probability of the occurrence of missing data is assumed to be known. The main purpose is to obtain both full-and reduced-order filters such that the filter error systems are exponentially mean-square stable and guarantee a prescribed H_∞ performance in terms of linear matrix inequality (LMI). A numerical example is provided to demonstrate the validity of the proposed design approach.

Key words Missing data, discrete-time system, H_∞ filter, LMI

1 Introduction

In many industrial applications, various filter schemes have been proposed recently for systems which assume that the measurements always contain a true signal[1]. However, due to sensor temporal failure and networks data transmission delay or loss, a measurement sequence may contain noise only, *i.e.*, the measurements are not consecutive but contain missing observations, which brings on degradation of the control-system performance including instability. The problem of missing measurements has attracted some research′s interest, see [2∼6] and references therein. The filtering problem with missing measurements was first introduced in [2], where the missing data were modeled by a binary switch sequence specified by a conditional probability distribution. A similar model was employed in [3, 4] to study the filter design problem. In [5], a measurement model with missing data using incompleteness matrix function was introduced to study the problem of state estimation and model validation. In [6], the filtering problem with missing data was investigated using Markov chains to describe probabilistic losses. However, the study of the systems with missing measurements has not been fully investigated and remains to be challenging.

In this paper, along the lines of [2∼4], we model the missing measurements as a Bernoulli distributed white sequence with a known conditional probability distribution. An H_∞ filtering problem is considered for the discrete-time systems with missing measurement. Some sufficient conditions for the existence of the full- and reduced-order filters are derived *via* LMI, which guarantee the filtering error system to be exponentially mean-square asymptotically stable with an H_∞-norm constraint for all possible missing observations.

2 Problem formulation

In this paper, we consider the class of stochastic discrete-time linear systems with missing measurements

$$\begin{aligned} \boldsymbol{x}(k+1) &= A\boldsymbol{x}(k) + B\boldsymbol{w}(k) \\ \boldsymbol{z}(k) &= L\boldsymbol{x}(k) + T\boldsymbol{w}(k) \\ \boldsymbol{y}(k) &= r(k)C\boldsymbol{x}(k) + D\boldsymbol{w}(k) \end{aligned} \tag{1}$$

where $\boldsymbol{x}(k) \in \Re^n$ is the state vector, $\boldsymbol{w}(k) \in \Re^m$ is the noise input, which belong to $l_2[0, \infty)$, $\boldsymbol{y} \in \Re^r$ is the measurement output, and $\boldsymbol{z}(k) \in \Re^p$ is the signal to be estimated. A, B, C, D, L, T are known real

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matrices. The stochastic variable $r(k) \in \Re$ is a Bernoulli distributed white sequence taking the values of 0 and 1 with

$$\text{Prob}\{r(k)=1\} = \text{E}\{r(k)\} := \bar{r} \tag{2}$$

$$\text{Prob}\{r(k)=0\} = 1-\text{E}\{r(k)\} := 1-\bar{r} \tag{3}$$

where $\bar{r}$ is a known positive constant.

Assumption 1. The system matrix A is stable, that is, all eigenvalues are located in the unit circle in the complex plane.

We consider the following filter of order k ($k = n$ for the full-order filter and $1 \leqslant k < n$ for reduced-order filters) described by

$$\hat{\boldsymbol{x}}(k+1) = A_f\hat{\boldsymbol{x}}(k) + B_f\boldsymbol{y}(k), \quad \hat{\boldsymbol{z}}(k) = C_f\hat{\boldsymbol{x}}(k) + D_f\boldsymbol{y}(k) \tag{4}$$

where $\hat{\boldsymbol{x}}(k) \in \Re^k$ is the estimated state, $\hat{\boldsymbol{z}}(k)$ is an estimate for $\boldsymbol{z}(k)$, and A_f, B_f, C_f and D_f are filter parameters to be determined.

Define the augmented state vector

$$\boldsymbol{x}_f(k) = \begin{bmatrix} \boldsymbol{x}(k) \\ \hat{\boldsymbol{x}}(k) \end{bmatrix} \tag{5}$$

the augmented system formed by system (1) and the filter (4) can be expressed by

$$\boldsymbol{x}_f(k+1) = A_{cl}\boldsymbol{x}_f(k) + B_{cl}\boldsymbol{w}(k), \quad \boldsymbol{z}_f(k) = C_{cl}\boldsymbol{x}_f(k) + D_{cl}\boldsymbol{w}(k) \tag{6}$$

where the filtering error output is denoted by $\boldsymbol{z}_f(k) = \boldsymbol{z}(k) - \hat{\boldsymbol{z}}(k)$, and

$$A_{cl} = A_{cl0} + (r(k)-\bar{r})A_{cl1},\ A_{cl0} = \begin{bmatrix} A & 0 \\ \bar{r}B_fC & A_f \end{bmatrix},\ A_{cl1} = \begin{bmatrix} 0 & 0 \\ B_fC & 0 \end{bmatrix},\ B_{cl} = \begin{bmatrix} B \\ B_fD \end{bmatrix}$$

$$C_{cl} = C_{cl0} + (r(k)-\bar{r})C_{cl1},\ C_{cl0} = \lfloor L-\bar{r}D_fC \quad -C_f \rfloor,\ C_{cl1} = \lfloor -D_fC \quad 0 \rfloor,\ D_{cl} = T - D_fD$$

Our objective is to design the full- and reduced-order filters of form (4) such that

1) The filtering error system (6) is exponentially mean-square asymptotically stable, *i.e.*, there exist constants $\alpha \geqslant 1$ and $0 < \tau < 1$, with $\boldsymbol{w}(k) = 0$, such that

$$\text{E}\{\|\boldsymbol{x}_f(k)\|\} \leqslant \alpha\tau^k\text{E}\{\|\boldsymbol{x}_f(0)\|^2\} \tag{7}$$

2) Under zero-initial condition, the filtering error output $\boldsymbol{z}_f(k)$ satisfies

$$\sum_{k=0}^{\infty}\text{E}\{\|\boldsymbol{z}_f(k)\|^2\} < \gamma^2\sum_{k=0}^{\infty}\text{E}\{\|\boldsymbol{w}(k)\|^2\} \tag{8}$$

for all nonzero $\boldsymbol{w}(k) \in l_2[0,\infty)$ and a given positive constant γ.

3 $\boldsymbol{H_\infty}$ filter analysis

Theorem 1. Given a scalar $\gamma > 0$ and the filter parameters A_f, B_f, C_f and D_f, the filtering error system (6) is exponentially mean-square asymptotically stable with an H_∞ noise attenuation level bound γ, if there exists a positive definite matrix $P = P^{\mathrm{T}} > 0$ satisfying

$$\begin{bmatrix} -P & 0 & A_{cl0}^{\mathrm{T}}P & C_{cl0}^{\mathrm{T}} & aA_{cl1}^{\mathrm{T}}P & aC_{cl1}^{\mathrm{T}} \\ 0 & -\gamma^2 I & B_{cl}^{\mathrm{T}}P & D_{cl}^{\mathrm{T}} & 0 & 0 \\ PA_{cl0} & PB_{cl} & -P & 0 & 0 & 0 \\ C_{cl0} & D_{cl} & 0 & -I & 0 & 0 \\ aPA_{cl1} & 0 & 0 & 0 & -aP & 0 \\ aC_{cl1} & 0 & 0 & 0 & 0 & -aI \end{bmatrix} < 0 \tag{9}$$

where $a = (1-\bar{r})\bar{r}$.

Proof. Define the Lyapunov functional as

$$V(k)=\boldsymbol{x}_f^{\mathrm T}(k)P\boldsymbol{x}_f(k) \tag{10}$$

where P is a positive definite matrix. The difference between the Lyapunov functional (10) from (6) is obtained as follows.

$$\begin{aligned}&\mathrm{E}\{V(k+1)|\boldsymbol{x}(k),\cdots,x(0),\hat{\boldsymbol{x}}(k),\cdots,\hat{\boldsymbol{x}}(0)\}-V(k)=(A_{cl0}\boldsymbol{x}_f(k)+B_{cl}\boldsymbol{w}(k))^{\mathrm T}P(A_{cl0}\boldsymbol{x}_f(k)+\\&B_{cl}\boldsymbol{w}(k))+\mathrm{E}(r(k)-\bar r)^2\boldsymbol{x}_f^{\mathrm T}(k)A_{cl1}^{\mathrm T}PA_{cl1}\boldsymbol{x}_f(k)-\boldsymbol{x}_f^{\mathrm T}(k)P\boldsymbol{x}_f(k)=(A_{cl0}\boldsymbol{x}_f(k)+\\&B_{cl}\boldsymbol{w}(k))^{\mathrm T}P(A_{cl0}\boldsymbol{x}_f(k)+B_{cl}\boldsymbol{w}(k))+a\boldsymbol{x}_f^{\mathrm T}(k)A_{cl1}^{\mathrm T}PA_{cl1}\boldsymbol{x}_f(k)-\boldsymbol{x}_f^{\mathrm T}(k)P\boldsymbol{x}_f(k)\end{aligned} \tag{11}$$

where $a=\mathrm{E}(r(k)-\bar r)^2=(1-\bar r)\bar r$. When $\boldsymbol{w}(k)=0$, we have

$$\begin{aligned}&\mathrm{E}\{V(k+1)|\boldsymbol{x}(k),\cdots,\boldsymbol{x}(0),\boldsymbol{x}_f(k),\cdots,\boldsymbol{x}_f(0)\}-V(k)=\\&\qquad\boldsymbol{x}_f^{\mathrm T}(k)(A_{cl0}^{\mathrm T}PA_{cl0}+aA_{cl1}^{\mathrm T}PA_{cl1}-P)\boldsymbol{x}_f(k)=\boldsymbol{x}_f^{\mathrm T}(k),\cdots,\boldsymbol{x}_f(0)\end{aligned} \tag{12}$$

By the Schur complement, LMI (9) implies $\Lambda<0$. Then we have

$$\mathrm{E}\{V(k+1)|\boldsymbol{x}(k),\cdots,\boldsymbol{x}(0),\hat{\boldsymbol{x}}(k),\cdots,\hat{\boldsymbol{x}}(0)\}-V(k)\leqslant-\lambda_{\min}(-\Lambda)\boldsymbol{x}_f^{\mathrm T}(k)\boldsymbol{x}_f(k)\leqslant-\alpha\boldsymbol{x}_f^{\mathrm T}(k)\boldsymbol{x}_f(k) \tag{13}$$

where $0<\alpha<\lambda_{\min}(-\Lambda)$.

We can find a scalar α such that $0<\alpha<\lambda_{\max}(P)$. Then

$$\mathrm{E}\{V(k+1)|\boldsymbol{x}(k),\cdots,\boldsymbol{x}(0),\hat{\boldsymbol{x}}(k),\cdots,\hat{\boldsymbol{x}}(0)\}-V(k)\leqslant-\alpha V(k)/\lambda_{\max}(P)=-\psi V(k) \tag{14}$$

where $0<\psi=\alpha/\lambda_{\max}(P)<1$.

From Lemma 1 of [7], we can conclude that the filtering error system (6) is exponentially mean-square asymptotically stable.

Next, for any nonzero $\boldsymbol{w}(k)$, it follows from (6) and (11) that

$$\begin{aligned}&\mathrm{E}\{V(k+1)\}-\mathrm{E}\{V(k)\}+\mathrm{E}\{\boldsymbol{z}_f^{\mathrm T}(k)\boldsymbol{z}_f(k)\}-\gamma^2\mathrm{E}\{w^{\mathrm T}(k)\boldsymbol{w}(k)\}=\\&\begin{bmatrix}\boldsymbol{x}_f(k)\\\boldsymbol{w}(k)\end{bmatrix}^{\mathrm T}\begin{bmatrix}A_{cl0}^{\mathrm T}PA_{cl0}+aA_{cl1}^{\mathrm T}PA_{cl1}-P+C_{cl0}^{\mathrm T}C_{cl0}+aC_{cl1}^{\mathrm T}C_{cl1} & A_{cl0}^{\mathrm T}PB_{cl}+C_{cl0}^{\mathrm T}D_{cl}\\ B_{cl}^{\mathrm T}PA_{cl0}+D_{cl}^{\mathrm T}C_{cl0} & D_{cl}^{\mathrm T}D_{cl}+B_{cl}^{\mathrm T}PB_{cl}-\gamma^2I\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_f(k)\\\boldsymbol{w}(k)\end{bmatrix}\end{aligned} \tag{15}$$

By the Schur complement, LMI (9) guarantees that

$$\mathrm{E}\{V(k+1)\}-\mathrm{E}\{V(k)\}+\mathrm{E}\{\boldsymbol{z}_f^{\mathrm T}(k)\boldsymbol{z}_f(k)\}-\gamma^2\mathrm{E}\{w^{\mathrm T}(k)\boldsymbol{w}(k)\}<0 \tag{16}$$

Now, summing (16) from 0 to ∞ with respect to k yields

$$\sum_{k=0}^{\infty}(E\{V(k+1)\}-\mathrm{E}\{V(k)\}+\mathrm{E}\{\boldsymbol{z}_f^{\mathrm T}(k)\boldsymbol{z}_f(k)\}-\gamma^2\mathrm{E}\{w^{\mathrm T}(k)\boldsymbol{w}(k)\})<0 \tag{17}$$

Since the filtering system is exponentially mean-square asymptotically stable, it is straightforward to see that

$$\sum_{k=0}^{\infty}\mathrm{E}\{\|\boldsymbol{z}_f(k)\|^2\}<\gamma^2\sum_{k=0}^{\infty}\mathrm{E}\{\|\boldsymbol{w}(k)\|^2\} \tag{18}$$

under the zero initial condition. This completes the proof. □

Theorem 2. Given a scalar $\gamma>0$ and the filter parameters A_f, B_f, C_f and D_f, the filtering error system (6) is exponentially mean-square asymptotically stable with an H_∞ noise attenuation level bound γ, if there exist matrices $P=P^{\mathrm T}>0$ and G satisfying

$$\begin{bmatrix}-P & 0 & A_{cl0}^{\mathrm T}G & C_{cl0}^{\mathrm T} & aA_{cl1}^{\mathrm T}G & aC_{cl1}^{\mathrm T}\\ 0 & -\gamma^2I & B_{cl}^{\mathrm T}G & D_{cl}^{\mathrm T} & 0 & 0\\ G^{\mathrm T}A_{cl0} & G^{\mathrm T}B_{cl} & P-G-G^{\mathrm T} & 0 & 0 & 0\\ C_{cl0} & D_{cl} & 0 & -I & 0 & 0\\ aG^{\mathrm T}A_{cl1} & 0 & 0 & 0 & a(P-G-G^{\mathrm T}) & 0\\ aC_{cl1} & 0 & 0 & 0 & 0 & -aI\end{bmatrix}<0 \tag{19}$$

where $a = (1-\bar{r})\bar{r}$.

Remark 1. Theorem 2 is equivalent to Theorem 1. The only difference between them lies in the fact that Theorem 2 eliminates the coupling between Lyapunov function and system matrices *via* replacing the Lyapunov matrix P by G, where the extra variable G does not present any structural constraint such as symmetry[8].

4 H_∞ filter design

In order to solve the filtering problem for the discrete-time system with missing measurements, some linearization procedures have to be adopted in this section. By using the standard linearization procedures proposed in [9], the full-order H_∞ filter in the following Theorem 3 can be obtained from (9). And we can get the reduced-order H_∞ filter in the following Theorem 4 from (19) *via* the linearization procedures proposed in [8].

4.1 Full-order filtering

Theorem 3. Given a scalar $\gamma > 0$, the filtering error system (6) is exponentially mean-square asymptotically stable with an H_∞-norm constraint (8) is achieved for all nonzero $\boldsymbol{w}(k)$, if there exist positive definite matrices $R = R^{\mathrm{T}} > 0$ and $Y = Y^{\mathrm{T}} > 0$, real matrices M, N, Z and D_f such that

$$\begin{bmatrix} -R & * & * & * & * & * & * & * & * \\ -R & -Y & * & * & * & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * & * & * & * \\ RA & RA & RB & -R & * & * & * & * & * \\ YA+\bar{r}ZC+M & YA+\bar{r}ZC & YB+ZD & -R & -Y & * & * & * & * \\ L-\bar{r}D_fC-N & L-\bar{r}D_fC & T-D_fD & 0 & -I & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & -aR & * & * \\ aZC & aZC & 0 & 0 & 0 & 0 & -aR & -aY & * \\ -aD_fC & -aD_fC & 0 & 0 & 0 & 0 & 0 & 0 & -aI \end{bmatrix} < 0 \tag{20}$$

where $a = (1-\bar{r})\bar{r}$.

Moreover, the filter parameters are given by

$$A_f = (R-Y)^{-1}M,\ B_f = (R-Y)^{-1}Z,\ C_f = N,\ D_f = D_f$$

4.2 Reduced-order filtering

Theorem 4. Given a scalar $\gamma > 0$, the filtering error system (6) is exponentially mean-square asymptotically stable with an H_∞-norm constraint (8) is achieved for all nonzero $\boldsymbol{w}(k)$, if there exist positive definite matrices $P_1 = P_1^{\mathrm{T}} > 0$ and $P_3 = P_3^{\mathrm{T}} > 0$, real matrices P_2, V_1, V_2, V_3, $\hat{A}_f$, $\hat{B}_f$, $\hat{C}_f$, $\hat{D}_f$ such that

$$\begin{bmatrix} -P_1 & * & * & * & * & * & * & * & * \\ -P_2^{\mathrm{T}} & -P_3 & * & * & * & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * & * & * & * \\ V_1^{\mathrm{T}}A+\bar{r}E\hat{B}_fC & E\hat{A}_f & V_1^{\mathrm{T}}B+E\hat{B}_fD & P_1-V_1-V_1^{\mathrm{T}} & * & * & * & * & * \\ V_3^{\mathrm{T}}A+\bar{r}\hat{B}_fC & \hat{A}_f & V_3^{\mathrm{T}}B+\hat{B}_fD & P_2^{\mathrm{T}}-V_2^{\mathrm{T}}E^{\mathrm{T}}-V_3^{\mathrm{T}} & P_3-V_2-V_2^{\mathrm{T}} & * & * & * & * \\ L-\bar{r}\hat{D}_fC & -\hat{C}_f & T-\hat{D}_fD & 0 & 0 & -I & * & * & * \\ E\hat{B}_fC & 0 & 0 & 0 & 0 & 0 & a(P_1-V_1-V_1^{\mathrm{T}}) & * & * \\ a\hat{B}_fC & 0 & 0 & 0 & 0 & 0 & a(P_2^{\mathrm{T}}-V_2^{\mathrm{T}}E^{\mathrm{T}}-V_3^{\mathrm{T}}) & a(P_3-V_2-V_2^{\mathrm{T}}) & * \\ -a\hat{D}_fC & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -aI \end{bmatrix} < 0 \tag{21}$$

where $a = (1-\bar{r})\bar{r}$, $E = \begin{bmatrix} I_{k*k} \\ 0_{(n-k)*k} \end{bmatrix}$.

Moreover, the filter parameters are given by

$$A_f = V_2^{-1}\hat{A}_f,\ B_f = V_2^{-1}\hat{B}_f,\ C_f = \hat{C}_f,\ D_f = \hat{D}_f$$

5 An illustrative example

In this section, we shall present an example to demonstrate the effectiveness and applicability of the proposed algorithms. Consider the system described by (1) with parameters as follows

$$A = \begin{bmatrix} 0 & 0.3 \\ -0.2 & 0.4 \end{bmatrix},\ \boldsymbol{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \boldsymbol{C} = [1 \quad 0],\ \boldsymbol{D} = 1,\ \boldsymbol{L} = [1 \quad 2],\ T = 0$$

and $\bar{r} = 0.8$.

First, consider the full-order filter design problem. By solving the LMI (20) in Theorem 3 by Matlab LMI Toolbox, the minimum noise attenuation level bound is obtained with $\gamma^* = 0.4102$ and the corresponding filter parameters are given by

$$A_f = \begin{bmatrix} -0.0091 & 0.2847 \\ -0.9838 & 0.4309 \end{bmatrix}, \ \boldsymbol{B}_f = \begin{bmatrix} 0.0209 \\ 0.9581 \end{bmatrix}, \ \boldsymbol{C}_f = [0.9945 \quad 1.9829], \ D_f = 0.0221$$

Next, we consider the reduced-order filter design problem. By solving the LMI (21) in Theorem 4, we obtain the minimum noise attenuation level bound with $\gamma^* = 2.5187$ and the corresponding filter parameters are given by

$$A_f = -0.3158, \ B_f = -0.4147, \ C_f = -1.0751, \ D_f = 0.6305$$

6 Conclusion

The problem of H_∞ filter design has been considered in this paper for stochastic discrete-time systems with missing measurements. Both full- and reduced-order filters have been designed in terms of the Matlab LMI toolbox, which guarantee the filtering error system to be exponentially mean-square asymptotically stable and the filtering error output to satisfy H_∞ performance constraint for all possible missing observations.

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