The Property of Dichotomy for a Class of Interconnected Pendulum-like Systems[1)]

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Abstract The property of dichotomy of interconnected second-order pendulum-like systems with multiple equilibria is investigated. This interconnection can be viewed as harmonic control of independent sub-systems. Linear interconnections and a class of input and output interconnections are considered in this paper. The effects of input and output interconnections are shown through a permutation matrix. Frequency domain and linear matrix inequality (LMI) conditions of dichotomy of interconnected pendulum-like systems are derived. It is shown that global properties of two coupled systems can be changed significantly through interconnections. Examples are given to illustrate the results.

Key words Interconnected systems, pendulum-like systems, dichotomy

1 Introduction

In the area of large scale systems theory, decentralized control plays an important role in stability synthesis problems[1]. Its main characteristics are that there are no information interchanges among control stations, the control structure is simple, the closed-loop systems are always connectively stable[1] (systems are still stable under structural perturbations). For stability and reliability of systems, this kind of result is obviously satisfactong. However, conservativeness is resulted in since every control station takes decision only by itself, and interconnections are always viewed as disadvantages. In real life, we can always see the examples of collaborations among different individuals. How to realize the similar collaborative actions among independent subsystems becomes more and more important in the area of systems theory[2]. Recently, some interesting results have appeared to this line. For example, applications of small gain theorem to strengthen stability degree of interconnected systems were studied in [3]; coupled nonlinear systems were discussed in [4∼6]. It was pointed out that in order to stabilize an interconnected system under a special structure, some subsystems must be unstable. These results have shown the positive effects of interconnections.

This paper is devoted to studying the property of coupled second-order pendulum-like systems and discussing the effects of interconnections. The pendulum-like system which has important applications in phase-lock loops[7,8] is a special nonlinear system with infinite equilibria. For this kind of systems, the characteristics of solutions are more complicated than those systems with single equilibrium. The simplest pendulum-like system is the second-order mathematical pendulum equation which was first studied by F.Tricomi (1933)[9]. Then it was generalized to higher-order systems with periodic functions and new methods based on Lyapunov functions were established[10,11]. In this paper, we establish frequency and time domain criteria of dichotomy of interconnected second-order pendulum-like systems and discuss the effects of linear interconnections and a class of input and output coupling.

2 Interconnected pendulum-like systems

Let us consider two second-order pendulum-like systems represented by

$$\begin{cases} \dot{\eta}_1 = -a_1\eta_1 - \varphi_1(\sigma_1) \\ \dot{\sigma}_1 = \eta_1 \end{cases} \tag{1}$$

$$\begin{cases} \dot{\eta}_2 = -a_2\eta_2 - \varphi_2(\sigma_2) \\ \dot{\sigma}_2 = \eta_2 \end{cases} \tag{2}$$

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where a_i are positive numbers, $\varphi_i : \mathbb{R} \to \mathbb{R}$ are continuously differentiable on $\mathbb{R}$. We suppose also that for all $\sigma \in \mathbb{R}$ the following equations are true:

$$(\exists \Delta_i \in \mathbb{R})(\forall \sigma_i \in \mathbb{R}) : \varphi_i(\sigma_i + \Delta_i) = \varphi_i(\sigma_i) \quad i = 1, 2$$

where Δ_i are called the period in σ of nonlinearities $\varphi_i(\sigma_i)$.

The interconnected system composed of (1) and (2) is described as follows:

$$\begin{cases} \dot{\boldsymbol{\eta}} = A\boldsymbol{\eta} + B\boldsymbol{\varphi}(\boldsymbol{\sigma}) \\ \dot{\boldsymbol{\sigma}} = C\boldsymbol{\eta} \end{cases} \tag{3}$$

where $\boldsymbol{\eta} = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$, $\boldsymbol{\sigma} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix}$, $\boldsymbol{\varphi}(\boldsymbol{\sigma}) = \begin{bmatrix} \varphi_1(\sigma_1) \\ \varphi_2(\sigma_2) \end{bmatrix}$, $A = \begin{bmatrix} -a_1 & a_{12} \\ a_{21} & -a_2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Here, the coupled relation between the two sub-systems is expressed by two certain numbers a_{12} and a_{21}. The corresponding transfer function from $\boldsymbol{\varphi}$ to $\dot{\boldsymbol{\sigma}}$ is defined as follows

$$K(s) = C(sI - A)^{-1}B \tag{4}$$

Definition[11]**.** System (3) is said to be dichotomous if every its bounded solution is convergent.

From the definition of dichotomy, we know that the existence of chaotic attractors and limit cycles is impossible in dichotomous systems. It follows that every solution of such systems is either convergent to a certain equilibrium point or goes to infinity. Then, the corresponding frequency-domain criterion guaranteeing dichotomy of interconnected system (3) is given by [4,11].

Let us replace φ in system (3) with $\boldsymbol{\psi} = \begin{bmatrix} \varphi_1(\sigma_2) \\ \varphi_2(\sigma_1) \end{bmatrix}$, then we can get a class of interconnected system with input and output coupling as follows:

$$\begin{cases} \dot{\boldsymbol{\eta}} = A\boldsymbol{\eta} + B\boldsymbol{\psi}(\boldsymbol{\sigma}) \\ \dot{\boldsymbol{\sigma}} = C\boldsymbol{\eta} \end{cases} \tag{5}$$

3 Main results

Now, we consider the property of dichotomy of interconnected system (5). Let $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $z = P\boldsymbol{\sigma}$. System (5) can be converted to

$$\begin{cases} \dot{\boldsymbol{\eta}} = A\boldsymbol{\eta} + B\boldsymbol{\varphi}(\boldsymbol{z}) \\ \dot{z} = PC\boldsymbol{\eta} \end{cases} \tag{6}$$

where $\boldsymbol{\varphi}(\boldsymbol{z}) = \begin{bmatrix} \varphi_1(z_1) \\ \varphi_2(z_2) \end{bmatrix} = \boldsymbol{\psi}(\boldsymbol{\sigma})$. It is obvious that systems (6) and (5) have the same dichotomy and by (6) the transfer function from $\boldsymbol{\varphi}(\boldsymbol{z})$ to $\dot{\boldsymbol{z}}$ is $PK(s)$. Thus we can get the following theorem.

Theorem 1. Let the matrix A have no pure imaginary eigenvalues. Suppose the pair (A, B) is controllable and the pair (A, C) is observable. Suppose also φ_i has a finite number of isolated zeros in one period. If there exist diagonal matrices $\varepsilon = diag\{\varepsilon_1, \varepsilon_2\} > 0$ and $\kappa = diag\{\kappa_1, \kappa_2\} \neq 0$ such that the following frequency-domain inequality is true.

$$Re\{\kappa PK(i\omega) + K^*(i\omega)P^*\varepsilon PK(i\omega)\} \leqslant 0, \quad \forall \omega \in \mathbb{R} \tag{7}$$

Then system (5) is dichotomous.

In virtue of KYP Lemma given by [12], the frequency-domain condition (7) can be converted to LMI.

Corollary. Let the matrix A has no pure imaginary eigenvalues. Suppose the pair (A, B) is controllable, the pair (A, C) is observable. Suppose also φ_i has a finite number of isolated zeros in one period. If there exist diagonal matrices $\varepsilon = diag\{\varepsilon_1, \varepsilon_2\} > 0$, $\kappa = diag\{\kappa_1, \kappa_2\} \neq 0$ and a symmetric matrix X such that the following LMI is true:

$$\begin{bmatrix} C^*P^*\varepsilon PC & \frac{1}{2}C^*P\kappa \\ \frac{1}{2}\kappa PC & 0 \end{bmatrix} + \begin{bmatrix} XA + A^*X & XB \\ B^*X & 0 \end{bmatrix} \leqslant 0 \tag{8}$$

then system (5) is dichotomous.

Theorem 2. Parameters a_{12} and a_{21} can always be chosen such that the interconnected system (3) or system (5) is dichotomous.

4 Numerical example

Consider the second-order pendulum-like systems defined as (1) and (2), where $\varphi_i(\boldsymbol{\sigma}_i) = -(a_i + d_i \cos \boldsymbol{\sigma}_i)(c_i + d_i \sin \boldsymbol{\sigma}_i)$, $i = 1, 2$. In what follows, we consider the dichotomy of interconnected systems. Firstly, we consider system (5). Let $c_1 = 0.2$, $d_1 = 1$, $c_2 = 0.1$, $d_2 = 0.9$, $a_1 = 2$, $a_2 = 1$, $A = \begin{bmatrix} -a_1 & a_{12} \\ a_{21} & -a_2 \end{bmatrix}$. It follows from the qualitative analysis method given by [13] that solutions of subsystems (1) and (2) are all convergent (subsystems (1) and (2) are dichotomous). If we choose $a_{12} = 3$ and $a_{21} = 0$, then it is easy to show that the conditions of Corollary 3.1 are not fulfilled. The curves and phase portraits of solution $\boldsymbol{\sigma}$ and state variable $\boldsymbol{\eta}$ for system (5) with the initial value $\boldsymbol{\eta}_0 = [2 \quad -1]^*$ and $\boldsymbol{\sigma}_0 = [4 \quad -2]^*$ are shown in Fig. 1, Fig. 2, Fig. 3, and Fig. 4, respectively. It follows that the interconnected system (5) has a certain periodic orbit. Thus system is not dichotomous.

Let $a_{21} = 0.5$ and P be a unit matrix. It means that input and output coupling in system (5) is disappeared. Then there exist numbers $\kappa_1 = -1$, $\kappa_2 = -3$, $\varepsilon_1 = 0.2$, and $\varepsilon_2 = 0.1$ such that inequality (8) is satisfied. It follows that the interconnected system (3) is dichotomous. Referring to Fig. 5 and Fig. 6, the curves of solution $\boldsymbol{\sigma}$ and state variable $\boldsymbol{\eta}$ for system (3) with the initial value $\boldsymbol{\eta}_0 = [2 \quad -1]^*$ and $\boldsymbol{\sigma}_0 = [4 \quad -2]^*$ are convergent respectively. If the input and output coupling works in system (3), the phase portrait of solution $\boldsymbol{\sigma}$ and state variable $\boldsymbol{\eta}$ for interconnected system (5) are shown in Fig. 7 and Fig. 8. The solution $\boldsymbol{\sigma}$ is unbounded. However, as shown in Fig. 7 and Fig. 8, the state variable $\boldsymbol{\eta}$ appears to be a kind of chaotic phenomenon as $t \to +\infty$. It is shown that global properties of coupled systems can be changed significantly through input and output interconnection.

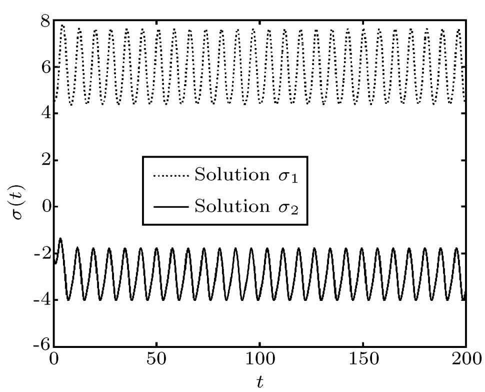

Fig. 1 The curve of solution $\boldsymbol{\sigma}$ for system (5)

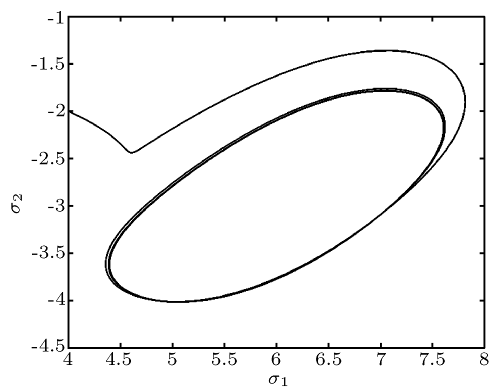

Fig. 2 The phase portrait of solution $\boldsymbol{\sigma}$ for (5)

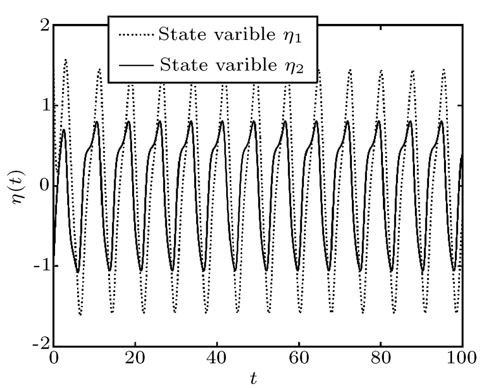

Fig. 3 The curve of state variable $\boldsymbol{\eta}$ for (5)

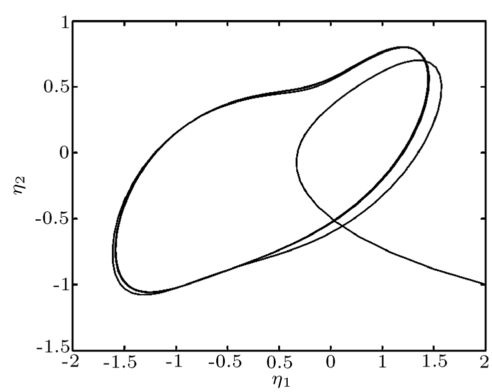

Fig. 4 The phase portrait of state variable $\boldsymbol{\eta}$ for (5)

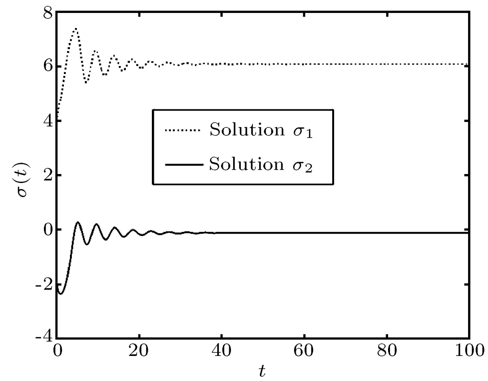

Fig. 5 The curve of $\boldsymbol{\sigma}$ for (3)

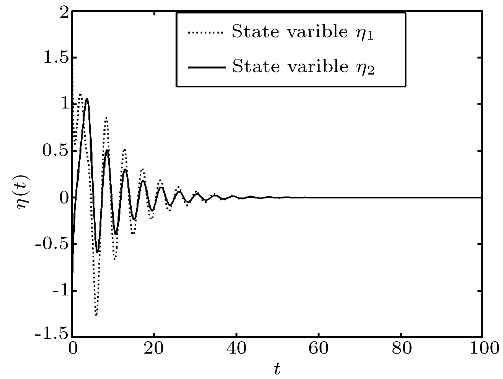

Fig. 6 The curve of $\boldsymbol{\eta}$ for (3)

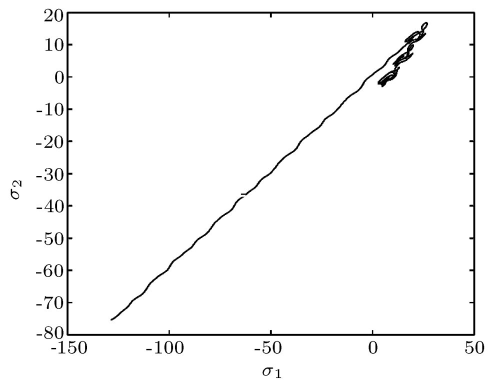

Fig. 7 The phase portrait of $\boldsymbol{\sigma}$ for (5)

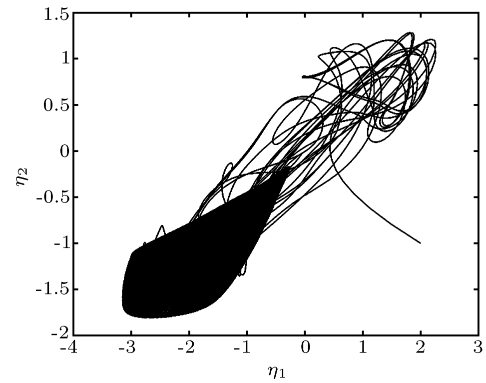

Fig. 8 The phase portrait of $\boldsymbol{\eta}$ for (5)

5 Conclusion

In this paper, the effects of interconnections between two independent second-order pendulum-like systems have been investigated. Linear interconnection and a class of input and output interconnections have been involved. Some frequency domain and LMI conditions of dichotomy for interconnected pendulum-like systems have been established. Examples show that the input and output interchange can result in great changes in some practical systems. For example, chaotic phenomenon in partial variables may appear by adding interconnections between two independent second-order pendulum-like systems which are dichotomous. Since the solution $\boldsymbol{\sigma}$ is unbounded, there is no chaotic phenomenon in plane phase space. However, chaotic phenomenon appears on the cylindrical surface of cylindrical phase space, called the chaos on cylindrical surface, which was never studied. It shows the complexity of physical property in concrete systems even if they are dichotomous. This also indicates that it is possible for the existence of chaotic attractors in pendulum-like systems.

References

1 Siljak D D. Decentralized Control of Complex Systems. New York: Academic, 1991

2 Huang L, Duan Z S. Complexity of control science. *Acta Automatica Sinica*, 2003, **29**(5): 748∼754

3 Duan Z S, Huang L, Wang L, Wang J Z. Some applications of small gain theorem to interconnected systems. *Systems and Control Letters*, 2004, **52**(3-4): 263∼273

4 Duan Z S, Wang J Z, Huang L. Multi-input and multi-output nonlinear systems: Interconnected Chua′s circuit. *International Journal of Bifurcation and Chaos*, 2004, **14**(9): 3065∼3081

5 Duan Z S, Wang J Z, Huang L. Frequency-domain method for the property of dichotomy of modified Chua′s equations. *International Journal of Bifurcation and Chaos*, 2005, **15**(8): to appear

6 Duan Z S, Wang J Z, Huang L. Input and output coupled nonlinear systems. *IEEE Transactions on Circuits and Systems-I Regular Papers*, 2005, **52**(3): 567∼575

7 Leonov G A, Smirnova V B. Analysis of frequency-of-oscillations-controlled systems. In: Proceedings of International Conference on Control of Oscillations and Chaos, St. Petersburg, Russia: 1997. **2**: 439∼441

8 Leonov G A, Tomayev A, Chshiyeva T. Stability of frequency-phase locked automatic frequency control systems. *Soviet Journal of Communications Technology and Electronics,* 1992, **37**(11): 1∼9

9 Tricomi F. Integrazione di unequazione differenziale presentatasi in electrotechnica. *Annali della Roma Scuola Normale Superiore de Pisa: Scienza Phys.e Mat.*, 1933, **2**: 1∼20

10 Hayes W D. On the equation for a damped pendulum under constant torque. *Z.A.M.Ph.*, 1953, **4**(5): 398∼401

11 Leonov G A, Ponomarenko D V, Smirnova V B. Frequency-Domain Methods for Nonlinear Analysis. Singapore: World Scientific, 1996

12 Rantzer A. On the Kalman-Yakubovich-Popov lemma. *Systems and Control Letters,* 1996, **28**: 7∼10

13 Li X B, Huang Y N, Yang Y. Huang L. Critical damping of the second-order pendulum-like systems. *Applied Mathematics and Mechanics*, 2005, **26**(1): 7∼16

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