Analysis for Robust Stability of Hopfield Neural Networks with Multiple Delays[1)]

ZHANG Hua-Guang JI Ce ZHANG Tie-Yan

(*Key Laboratory of Process Industry Automation, Ministry of Education, Northeastern University, Shenyang* 110004)

(E-mail: z_hg @tom.com, hanfj119@163.net, zty@syepi.edu.cn)

Abstract The robust stability of a class of Hopfield neural networks with multiple delays and parameter perturbations is analyzed. The sufficient conditions for the global robust stability of equilibrium point are given by way of constructing a suitable Lyapunov functional. The conditions take the form of linear matrix inequality (LMI), so they are computable and verifiable efficiently. Furthermore, all the results are obtained without assuming the differentiability and monotonicity of activation functions. From the viewpoint of system analysis, our results provide sufficient conditions for the global robust stability in a manner that they specify the size of perturbation that Hopfield neural networks can endure when the structure of the network is given. On the other hand, from the viewpoint of system synthesis, our results can answer how to choose the parameters of neural networks to endure a given perturbation.

Key words Hopfield neural networks, multiple delays, parameter perturbations, robust stability, Lyapunov functional, linear matrix inequality

1 Introduction

The investigation and application of Hopfield neural networks with symmetric interconnecting structure have extended to many fields and gained abundant fruits[1,2] in classification, parallel computing, associative memory, especially in solving some optimization problems[3]. However, it is impossible to realize the absolute symmetry of interconnecting structure due to the influences of parameter perturbations. On the other hand, time delay occurs inevitably and may bring oscillation during the implementation process of Hopfield neural networks[4]. Hence, it is important to consider the influences of time delay and interconnecting structure when we analyze the stability of Hopfield neural networks. [5,6] investigated respectively a class of Hopfield neural network models with special interconnecting structure, but they imposed strict restrictions on interconnecting structure and did not take into account the influences of time delays and parameter perturbations. [7] analyzed the parameter perturbations of interconnecting structure in detail, but the interconnecting matrix T was supposed to be symmetric after perturbation, which is very difficult to realize in practical application. To solve the above problems effectively, this paper will study the robust stability of a class of delayed Hopfield neural network models with parameter perturbations, which involve the perturbations of self-feedback terms and the perturbations of interconnecting structure. Some criteria for the global robust stability of Hopfield neural networks will be established by constructing a suitable Lyapunov functional. The conditions presented in the paper are in the form of linear matrix inequality, thus, have the advantage that they can be solved numerically and very effectively using the interior-point method. These sufficient conditions are very practical in the process of design and implementation of Hopfield neural networks.

2 Network model

The Hopfield neural network model with multiple delays can be described by

$$\dot{\boldsymbol{x}}(t) = -C\boldsymbol{x}(t) + T_0 S(\boldsymbol{x}(t)) + \sum_{k=1}^{K} T_k S(\boldsymbol{x}(t-\tau_k)) \tag{1}$$

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where $\boldsymbol{x} = (x_1, \cdots, x_n)^{\mathrm{T}} \in \Re^n$ denotes the state variables associated with the neurons, $C = \mathrm{diag}[c_1, \cdots, c_n]$ with $c_i > 0$, $i = 1, \cdots, n$, denotes the self-feedback matrix of the neurons. $T_0 \in \Re^{n\times n}$ denotes that part of the interconnecting structure which is not associated with delay, $T_k \in \Re^{n\times n}$ denotes that part of the interconnecting structure which is associated with delay τ_k, where τ_k denotes kth delay, $k = 1, \cdots, K$ and $0 < \tau_1 < \cdots < \tau_K < +\infty$. $S(\boldsymbol{x}) = [s_1(x_1), \cdots, s_n(x_n)]^{\mathrm{T}}$ denotes the activation functions, where $s_i(x_i)$, $i = 1, \cdots, n$ satisfies Assumption 1 in the following and $s_i(0) = 0$.

The initial condition is $\boldsymbol{x}(s) = \boldsymbol{\varphi}(s)$, for $s \in [-\tau_K, 0]$, where $\boldsymbol{\varphi} \in C([-\tau_K, 0], \Re^n)$. Here, $C([-\tau_K, 0], \Re^n)$ denotes the Banach space of continuous vector-valued functions mapping the interval $[-\tau_K, 0]$ into $\Re^n$ with a topology of uniform convergence.

Assumption 1. For $i = 1, \cdots, n$, $s_i(x_i)$ is bounded and satisfies the following sector condition

$$0 \leqslant \frac{s_i(x_i)}{x_i} \leqslant \sigma_i^M \tag{2}$$

Considering the influences of perturbations, then (1) can be described as

$$\dot{\boldsymbol{x}}(t) = -(C + \Delta C)\boldsymbol{x}(t) + (T_0 + \Delta T_0)S(\boldsymbol{x}(t)) + \sum_{k=1}^{K}(T_k + \Delta T_k)S(\boldsymbol{x}(t - \tau_k)) \tag{3}$$

where $\Delta C = \mathrm{diag}[\Delta c_1, \cdots, \Delta c_n] \in \Re^{n\times n}$ and $\Delta T_k \in \Re^{n\times n}$, $k = 0, \cdots, K$ are time-invariant matrices representing the norm-bounded uncertainties.

Assumption 2. We assume that the norm of the perturbations ΔC and ΔT_k, $k = 0, \cdots, K$ are bounded and

$$[\Delta C \quad \Delta T_0 \cdots \Delta T_k] = HF[A \quad B_0 \cdots B_K] \tag{4}$$

where F is an unknown matrix representing parametric perturbations which satisfies

$$F^{\mathrm{T}}F \leqslant E \tag{5}$$

where E is an identical matrix, and $A, B_0, \cdots, B_K$ can be regarded as the known structural matrices of perturbations with appropriate dimensions.

Definition. The equilibrium point of system (1) is said to be globally robustly stable with respect to the uncertainties ΔC and ΔT_k, $k = 0, \cdots, K$, if the equilibrium point of system (3) is globally asymptotically stable.

Distinctly, the origin is an equilibrium point of (1) and (3). Thus, in order to study the global robust stability of the zero solution of system (1) with respect to parametric uncertainties ΔC and ΔT_k, $k = 0, \cdots, K$, it suffices to investigate the globally asymptotic stability of the zero solution of system (3). Now, the interconnecting matrix $T = T_0 + \Delta T_0 + \sum_{k=1}^{K}(T_k + \Delta T_k)$ is nonsymmetric due to the influences of uncertainties ΔT_k, $k = 0, \cdots, K$.

Lemma 1. ([8] Michel, *et al.*) For a functional differential equation with time delay $\dot{\boldsymbol{x}}(t) = f(t, \boldsymbol{x}_t)$, if there exists a continuous functional $V(t, \boldsymbol{\varphi})$ such that there exist non-decreasing continuous functions $u, v, w : \Re^+ \to \Re^+$, which satisfy $u(0) = v(0) = 0$, $u(\|\boldsymbol{\varphi}(0)\|) \leqslant V(t, \boldsymbol{\varphi}) \leqslant v(|\boldsymbol{\varphi}|)$ and $\dot{V}(t, \boldsymbol{\varphi}) \leqslant -w(\|\boldsymbol{\varphi}(0)\|)$, then the solution $\boldsymbol{x} = \mathbf{0}$ of the functional differential equation is asymptotically stable.

In the above Lemma, $\|\cdot\|$ denotes the Euclidean vector norm on $\Re^n$. $\boldsymbol{x}_t(\cdot)$ denotes the restriction of $\boldsymbol{x}(\cdot)$ to the interval $[t - \tau_K, t]$ translated to $[-\tau_K, 0]$. For $s \in [-\tau_K, 0]$, we have $\boldsymbol{x}_t(s) = \boldsymbol{x}(t + s)$, where $t > 0$. For any $\boldsymbol{\varphi} \in C([-\tau_K, 0], \Re^n)$, we define $|\boldsymbol{\varphi}| = \max\{\|\boldsymbol{\varphi}(t)\| : t \in [-\tau_K, 0]\}$.

Lemma 2[9]**.** If U, V and W are real matrices of appropriate dimensions with M satisfying $M = M^{\mathrm{T}}$, then

$$M + UVW + W^{\mathrm{T}}V^{\mathrm{T}}U^{\mathrm{T}} < 0 \tag{6}$$

for all $V^{\mathrm{T}}V \leqslant E$, if and only if there exists a positive constant ε such that

$$M + \varepsilon^{-1}UU^{\mathrm{T}} + \varepsilon W^{\mathrm{T}}W < 0 \tag{7}$$

In the following section, we will give the sufficient conditions for the globally asymptotic stability of equilibrium point $\boldsymbol{x}=\mathbf{0}$ of system (3).

3 Robust stability

Theorem. The equilibrium point $\boldsymbol{x}=\mathbf{0}$ of system (3) is globally asymptotically stable for arbitrarily bounded delay τ_k if there exists a positive definite matrix P, positive constant ε and positive diagonal matrixes $\Lambda=\mathrm{diag}[\lambda_{k1},\cdots,\lambda_{kn}]$, where $\lambda_{ki}>0$, $i=1,\cdots,n$, $k=0,\cdots,K$, such that the following linear matrix inequality (LMI) holds

$$\begin{bmatrix} -C^{\mathrm{T}}P-PC+\sum_{k=0}^{K}\Lambda_k+\varepsilon A^{\mathrm{T}}A & PT_KE^M-\varepsilon A^{\mathrm{T}}B_KE^M & \cdots & \cdots & TP_0E^M-\varepsilon A^{\mathrm{T}}B_0E^M & PH \\ E^MT_K^{\mathrm{T}}P-\varepsilon E^MB_K^{\mathrm{T}}A & -\Lambda_K+\varepsilon E^MB_K^{\mathrm{T}}B_KE^M & \cdots & \cdots & \varepsilon E^MB_K^{\mathrm{T}}B_0E^M & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \ddots & \vdots & \vdots \\ E^MT_0^{\mathrm{T}}P-\varepsilon E^MB_0^{\mathrm{T}}A & \varepsilon E^MB_0^{\mathrm{T}}B_KE^M & \cdots & \cdots & -\Lambda_0+\varepsilon E^MB_0^{\mathrm{T}}B_0E^M & 0 \\ H^{\mathrm{T}}P & 0 & \cdots & \cdots & 0 & -\varepsilon I \end{bmatrix}<0 \tag{8}$$

where $E^M=\mathrm{diag}[\sigma_1^M,\cdots,\sigma_n^M]$.

Proof. By (2), we can rewrite (3) as

$$\dot{\boldsymbol{x}}(t)=-(C+\Delta C)\boldsymbol{x}(t)+(T_0+\Delta T_0)E(\boldsymbol{x}(t))\boldsymbol{x}(t)+\sum_{k=1}^{K}(T_k+\Delta T_k)E(\boldsymbol{x}(t-\tau_k))\boldsymbol{x}(t-\tau_k) \tag{9}$$

where

$$E(\boldsymbol{x})=\mathrm{diag}[\sigma_1(x_1),\cdots,\sigma_n(x_n)],\quad \sigma_i(x_i)=s_i(x_i)/x_i,\quad i=1,\cdots,n$$

Then $\sigma_i(x_i)\in[0,\sigma_i^M]$.

Here, we introduce the following Lyapunov functional

$$V(\boldsymbol{x}_t)=\boldsymbol{x}^{\mathrm{T}}(t)P\boldsymbol{x}(t)+\sum_{k=1}^{K}\int_{-\tau_k}^{0}\boldsymbol{x}_t^{\mathrm{T}}(\theta)\Lambda_k\boldsymbol{x}_t(\theta)\mathrm{d}\theta \tag{10}$$

Clearly, we have $\lambda_{\min}(P)\|\boldsymbol{x}_t(\mathbf{0})\|^2\leqslant V(\boldsymbol{x}_t)\leqslant\left(\lambda_{\max}(P)+\sum_{k=1}^{K}\tau_k\lambda_{\max}(\Lambda_k)\right)|\boldsymbol{x}_t|^2$.

The derivative of $V(\boldsymbol{x}_t)$ with respect to t along any trajectory of system (9) is given by

$$\begin{aligned}
\dot{V}(\boldsymbol{x}_t)=&\dot{\boldsymbol{x}}^{\mathrm{T}}(t)P\boldsymbol{x}(t)+\boldsymbol{x}^{\mathrm{T}}(t)P\dot{\boldsymbol{x}}(t)+\sum_{k=1}^{K}\boldsymbol{x}^{\mathrm{T}}(t)\Lambda_k\boldsymbol{x}(t)-\sum_{k=1}^{K}\boldsymbol{x}^{\mathrm{T}}(t-\tau_k)\Lambda_k\boldsymbol{x}(t-\tau_k)=\\
&-\boldsymbol{x}^{\mathrm{T}}(t)[(C+\Delta C)^{\mathrm{T}}P+P(C+\Delta C)]\boldsymbol{x}(t)+\sum_{k=1}^{K}\boldsymbol{x}^{\mathrm{T}}(t)\Lambda_k\boldsymbol{x}(t)+\boldsymbol{x}^{\mathrm{T}}(t)\Lambda_0\boldsymbol{x}(t)+\\
&\boldsymbol{x}^{\mathrm{T}}(t)P(T_0+\Delta T_0)E(\boldsymbol{x}(t))\Lambda_0^{-1}E^{\mathrm{T}}(\boldsymbol{x}(t))(T_0+\Delta T_0)^{\mathrm{T}}P\boldsymbol{x}(t)-\\
&[\Lambda_0^{1/2}\boldsymbol{x}(t)-\Lambda_0^{-1/2}E^{\mathrm{T}}(\boldsymbol{x}(t))(T_0+\Delta T_0)^{\mathrm{T}}P\boldsymbol{x}(t)]^{\mathrm{T}}[\Lambda_0^{1/2}\boldsymbol{x}(t)-\Lambda_0^{-1/2}E^{\mathrm{T}}(\boldsymbol{x}(t))(T_0+\Delta T_0)^{\mathrm{T}}P\boldsymbol{x}(t)]+\\
&\sum_{k=1}^{K}\boldsymbol{x}^{\mathrm{T}}(t)P(T_k+\Delta T_k)E(\boldsymbol{x}(t-\tau_k))\Lambda_k^{-1}E^{\mathrm{T}}(\boldsymbol{x}(t-\tau_k))(T_k+\Delta T_k)^{\mathrm{T}}P\boldsymbol{x}(t)-\\
&\sum_{k=1}^{K}[\Lambda_k^{1/2}\boldsymbol{x}(t-\tau_k)-\Lambda_k^{-1/2}E^{\mathrm{T}}(\boldsymbol{x}(t-\tau_k))(T_k+\Delta T_k)^{\mathrm{T}}P\boldsymbol{x}(t)]^{\mathrm{T}}\times\\
&[\Lambda_k^{1/2}\boldsymbol{x}(t-\tau_k)-\Lambda_k^{-1/2}E^{\mathrm{T}}(\boldsymbol{x}(t-\tau_k))(T_k+\Delta T_k)^{\mathrm{T}}P\boldsymbol{x}(t)]\leqslant\\
&-\boldsymbol{x}^{\mathrm{T}}(t)[(C+\Delta C)^{\mathrm{T}}P+P(C+\Delta C)]\boldsymbol{x}(t)+\sum_{k=0}^{K}\boldsymbol{x}^{\mathrm{T}}(t)\Lambda_k\boldsymbol{x}(t)+
\end{aligned}$$

$$\boldsymbol{x}^{\mathrm{T}}(t)P(T_0+\Delta T_0)E(\boldsymbol{x}(t))\Lambda_0^{-1}E^{\mathrm{T}}(\boldsymbol{x}(t))(T_0+\Delta T_0)^{\mathrm{T}}P\boldsymbol{x}(t)+$$
$$\sum_{k=1}^{K}\boldsymbol{x}^{\mathrm{T}}(t)P(T_k+\Delta T_k)E(\boldsymbol{x}(t-\tau_k))\Lambda_k^{-1}E^{\mathrm{T}}(\boldsymbol{x}(t-\tau_k))(T_k+\Delta T_k)^{\mathrm{T}}P\boldsymbol{x}(t) \tag{11}$$

For any given t, if we let

$$\boldsymbol{y}_k^{\mathrm{T}}(t)=(y_{k1},\cdots,y_{kn})=\boldsymbol{x}^{\mathrm{T}}(t)P(T_k+\Delta T_k) \tag{12}$$

then the last term of (11) assumes the form

$$\sum_{k=1}^{K}\boldsymbol{y}_k^{\mathrm{T}}(t)E(\boldsymbol{x}(t-\tau_k))\Lambda_k^{-1}E^{\mathrm{T}}(\boldsymbol{x}(t-\tau_k))\boldsymbol{y}_k(t)=\sum_{k=1}^{K}\sum_{i=1}^{n}y_{ki}^2\lambda_{ki}^{-1}\sigma_i^2(\boldsymbol{x}_i(t-\tau_k))\leqslant\sum_{k=1}^{K}\sum_{i=1}^{n}y_{ki}^2\lambda_{ki}^{-1}(\sigma_i^M)^2=$$
$$\sum_{k=1}^{K}\boldsymbol{x}^{\mathrm{T}}(t)P(T_k+\Delta T_k)E^M\Lambda_k^{-1}E^M(T_k+\Delta T_k)^{\mathrm{T}}P\boldsymbol{x}(t) \tag{13}$$

Similarly, we have

$$\boldsymbol{x}^{\mathrm{T}}(t)P(T_0+\Delta T_0)E(\boldsymbol{x}(t))\Lambda_0^{-1}E^{\mathrm{T}}(\boldsymbol{x}(t))(T_0+\Delta T_0)^{\mathrm{T}}P\boldsymbol{x}(t)\leqslant$$
$$\boldsymbol{x}^{\mathrm{T}}(t)P(T_0+\Delta T_0)E^M\Lambda_0^{-1}E^M(T_0+\Delta T_0)^{\mathrm{T}}P\boldsymbol{x}(t) \tag{14}$$

From (13) and (14), we can express (11) as

$$\dot V(\boldsymbol{x}_t)\leqslant\boldsymbol{x}^{\mathrm{T}}(t)\Big\{-[(C+\Delta C)^{\mathrm{T}}P+P(C+\Delta C)]+\sum_{k=0}^{K}\Lambda_k+$$
$$\sum_{k=0}^{K}P(T_k+\Delta T_k)E^M\Lambda_k^{-1}E^M(T_k+\Delta T_k)^{\mathrm{T}}P\Big\}\boldsymbol{x}(t) \tag{15}$$

Here, we define

$$S^M=-[(C+\Delta C)^{\mathrm{T}}P+P(C+\Delta C)]+\sum_{k=0}^{K}\Lambda_k+\sum_{k=0}^{K}P(T_k+\Delta T_k)E^M\Lambda_k^{-1}E^M(T_k+\Delta T_k)^{\mathrm{T}}P \tag{16}$$

By (15), $\dot V(\boldsymbol{x}_t)\leqslant-\lambda_{\max}(-S^M)\|\boldsymbol{x}(t)\|^2=-\lambda_{\max}(-S^M)\|\boldsymbol{x}_t(\boldsymbol{0})\|^2$ can be derived. From Lemma 1, we know that the equilibrium point $\boldsymbol{x}=\boldsymbol{0}$ of system (3) is globally asymptotically stable when $S^M<0$. Then, according to Schur Complement[10], $S^M<0$ can be expressed by the following linear matrix inequality

$$\begin{bmatrix} -[(C+\Delta C)^{\mathrm{T}}P+P(C+\Delta C)]+\sum_{k=0}^{K}\Lambda_k & P(T_K+\Delta T_K)E^M & \cdots & \cdots & P(T_0+\Delta T_0)E^M \\ E^M(T_K+\Delta T_K)^{\mathrm{T}}P & -\Lambda_K & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ E^M(T_0+\Delta T_0)^{\mathrm{T}}P & 0 & \cdots & 0 & -\Lambda_0 \end{bmatrix}<0 \tag{17}$$

In fact, (17) is exactly

$$\begin{bmatrix} -C^{\mathrm{T}}P-PC+\sum_{k=0}^{K}\Lambda_k & PT_KE^M & \cdots & \cdots & PT_0E^M \\ E^MT_K^{\mathrm{T}}P & -\Lambda_K & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ E^MT_0^{\mathrm{T}}P & 0 & \cdots & 0 & -\Lambda_0 \end{bmatrix}+$$

$$\begin{bmatrix} -\Delta C^{\mathrm{T}}P-P\Delta C & P\Delta T_K E^M & \cdots & \cdots & P\Delta T_0 E^M \\ E^M \Delta T_K^{\mathrm{T}} P & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ E^M \Delta T_0^{\mathrm{T}} P & 0 & \cdots & \cdots & 0 \end{bmatrix} < 0 \tag{18}$$

Because of $[\Delta C \quad \Delta T_0 \cdots \Delta T_K] = HF[A \quad B_0 \cdots B_K]$, (18) can be expressed as

$$\begin{bmatrix} -C^{\mathrm{T}}P-PC+\sum_{k=0}^{K}\Lambda_k & PT_K E^M & \cdots & \cdots & PT_0 E^M \\ E^M T_K^{\mathrm{T}} P & -\Lambda_K & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ E^M T_0^{\mathrm{T}} P & 0 & \cdots & 0 & -\Lambda_0 \end{bmatrix} + \begin{bmatrix} PH \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} F[-A \quad B_K E^M \quad \cdots \cdots \quad B_0 E^M] +$$

$$\begin{bmatrix} -A^{\mathrm{T}} \\ E^M B_K^{\mathrm{T}} \\ \vdots \\ \vdots \\ E^M B_0^{\mathrm{T}} \end{bmatrix} F^{\mathrm{T}}[H^{\mathrm{T}}P \quad 0 \quad \cdots \cdots \quad 0] < 0 \tag{19}$$

Using Lemma 2, we know that (19) holds for all $F^{\mathrm{T}}F \leqslant I$ if and only if there exists a constant $\varepsilon > 0$ such that

$$\begin{bmatrix} -C^{\mathrm{T}}P-PC+\sum_{k=0}^{K}\Lambda_k & PT_K E^M & \cdots & \cdots & PT_0 E^M \\ E^M T_K^{\mathrm{T}} P & -\Lambda_K & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ E^M T_0^{\mathrm{T}} P & 0 & \cdots & 0 & -\Lambda_0 \end{bmatrix} + \frac{1}{\varepsilon}\begin{bmatrix} PHH^{\mathrm{T}}P & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix} +$$

$$\varepsilon \begin{bmatrix} A^{\mathrm{T}}A & -A^{\mathrm{T}}B_K E^M & \cdots & \cdots & -A^{\mathrm{T}}B_0 E^M \\ -E^M B_K^{\mathrm{T}} A & E^M B_K^{\mathrm{T}} B_K E^M & \cdots & \cdots & E^M B_K^{\mathrm{T}} B_0 E^M \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ -E^M B_0^{\mathrm{T}} A & E^M B_0^{\mathrm{T}} B_K E^M & \cdots & \cdots & E^M B_0^{\mathrm{T}} B_0 E^M \end{bmatrix} < 0 \tag{20}$$

Rearranging (20), we get

$$\begin{bmatrix} -C^{\mathrm{T}}P-PC+\sum_{k=0}^{K}\Lambda_k+\frac{1}{\varepsilon}PHH^{\mathrm{T}}P+\varepsilon A^{\mathrm{T}}A & PT_K E^M-\varepsilon A^{\mathrm{T}}B_K E^M & \cdots\cdots & PT_0 E^M-\varepsilon A^{\mathrm{T}}B_0 E^M \\ E^M T_K^{\mathrm{T}} P-\varepsilon E^M B_K^{\mathrm{T}} A & -\Lambda_K+\varepsilon E^M B_K^{\mathrm{T}} B_K E^M & \cdots\cdots & \varepsilon E^M B_K^{\mathrm{T}} B_0 E^M \\ \vdots & \vdots & \ddots \cdots & \vdots \\ \vdots & \vdots & \cdots \ddots & \vdots \\ E^M T_0^{\mathrm{T}} P-\varepsilon E^M B_0^{\mathrm{T}} A & \varepsilon E^M B_0^{\mathrm{T}} B_K E^M & \cdots\cdots & -\Lambda_0+\varepsilon E^M B_0^{\mathrm{T}} B_0 E^M \end{bmatrix} < 0 \tag{21}$$

By use of Schur complement, (21) is equivalent to the condition (8). The proof is completed. □

Since the norm of the perturbations ΔC and ΔT_k, $k = 0, \cdots, K$ is bounded, the corollary below can be obtained easily by the definition and properties of matrix norm.

Corollary 1. For arbitrarily bounded delay $\tau_k > 0$, $k = 1, \cdots, K$, the equilibrium point $\boldsymbol{x} = \boldsymbol{0}$ of system (3) is asymptotically stable if

$$2\|P_0\| \cdot \|\Delta C\| + \sum_{k=0}^{K} \|\Lambda_k\| + (\bar{\sigma}^M)^2 \cdot \|P_0\|^2 \cdot \sum_{k=0}^{K} \|\Lambda_k^{-1}\| \cdot (\|T_k\| + \|\Delta T_k\|)^2 < 2 \tag{22}$$

where $P_0 = \text{diag}[1/c_1, \cdots, 1/c_n]$, $\bar{\sigma}^M = \max\{\sigma_i^M : 1 \leqslant i \leqslant n\}$. Here, $\|\cdot\|$ denotes the matrix norm induced by the Euclidean vector norm, *i.e.*, $\|P_0\| = \sqrt{\lambda_{\max}(P_0^{\mathrm{T}} P_0)}$.

The corollary follows from $S^M < 0$ by choosing P as P_0.

Corollary 1 gives the condition under which small parameter perturbations cannot result in change of the asymptotically stable equilibrium point $\boldsymbol{x} = \boldsymbol{0}$. Meanwhile, this corollary is helpful to adjust the parameters of network to minimize the influences of perturbations when we design Hopfield neural networks.

Using Lemma 2 many times in the proof of the theory, we can have the following corollary.

Corollary 2. The equilibrium point $\boldsymbol{x} = \boldsymbol{0}$ of system (3) is globally asymptotically stable for arbitrarily bounded delay τ_k if there exists a positive definite matrix P, positive constants ε_k, $k = 0, \cdots, K+1$ and positive diagonal matrixes $\Lambda_k = \text{diag}[\lambda_{k1}, \cdots, \lambda_{kn}]$, where $\lambda_{ki} > 0$, $i = 1, \cdots, n$, $k = 0, \cdots, K$, such that the following linear matrix inequality (LMI) holds

$$\begin{bmatrix}
-C^{\mathrm{T}}P - PC + \sum_{k=0}^{K}\Lambda_k + \varepsilon_{K+1}A^{\mathrm{T}}A & PT_KE^M & \cdots & \cdots & PT_0E^M & PH & \cdots & \cdots & PH \\
E^MT_K^{\mathrm{T}}P & -\Lambda_K + \varepsilon_KE^MB_K^{\mathrm{T}}B_KE^M & 0\ \cdots & 0 & 0 & \cdots & \cdots & 0 & \\
\vdots & 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\
E^MT_0^{\mathrm{T}}P & 0 & \cdots & 0 & -\Lambda_0 + \varepsilon_0E^MB_0^{\mathrm{T}}B_0E^M & 0 & \cdots & \cdots & 0 \\
H^{\mathrm{T}}P & 0 & \cdots & \cdots & 0 & -\varepsilon_{K+1}I & 0\ \cdots & 0 & \\
\vdots & \vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\
H^{\mathrm{T}}P & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & -\varepsilon_0I
\end{bmatrix} < 0 \tag{23}$$

The proof of the corollary and the simulations are omitted due to space limitation.

4 Conclusion

During the implementation process of Hopfield neural networks by electronic circuits, time delays and parameter perturbations are inevitable. This paper studies the robust stability of a class of Hopfield neural network models with multiple delays and parameter perturbations, and gives the sufficient conditions for the asymptotic stability of equilibrium point $\boldsymbol{x} = \boldsymbol{0}$ for arbitrarily bounded delay τ_k, $k = 1, \cdots, K$, which take the form of linear matrix inequality. Since the perturbation norms are bounded in general, we give a useful corollary by means of the definition and properties of matrix norm.

In applications, the bound of delays is frequently not very large and is usually known. Therefore, the next research work is to discuss further whether we can obtain the sufficient conditions for the robust stability of equilibrium point, which depend on time delay τ_k, $k = 1, \cdots, K$.

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ZHANG Hua-Guang Received his Ph. D. degree from Southeastern University in 1991. He entered Automatic Control Department, Northeastern University in 1992, as a postdoctoral fellow. Since 1994, he has been a professor at Northeastern University. His research interests include fuzzy control, chaos control, neural networks based control, nonlinear control, signal processing, and their industrial application.

JI Ce Received her master and Ph. D. degrees form Northeastern University in 1997 and 2005, respectively. Now she works at Northeastern University, as an instructor. Her research interests include neural networks and nonlinear system.

ZHANG Tie-Yan Received his master degree from Harbin Institute of Technology in 1989. Now he is a professor. His research interests include fuzzy adaptive control, fuzzy neural networks, and their application.