Anti-windup Compensator Gain Design for Time-delay Systems with Constraints[1)]

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Abstract Systems that are subject to both time-delay in state and input saturation are considered. We synthesize the anti-windup gain to enlarge the estimation of domain of attraction while guaranteeing the stability of the closed-loop system. An ellipsoid and a polyhedral set are used to bound the state of the system, which make a new sector condition valid. Other than an iterative algorithm, a direct designing algorithm is derived to compute the anti-windup compensator gain, which reduces the conservatism greatly. We analyze the delay-independent and delay-dependent cases, respectively. Finally, an optimization algorithm in the form of LMIs is constructed to compute the compensator gain which maximizes the estimation of domain of attraction. Numerical examples are presented to demonstrate the effectiveness of our approach.

Key words Actuator saturation, time-delay systems, compensation gain, Lyapunov-Krasovskii function

1 Introduction

Time-delay exists naturally in a large series of systems. More often than not, the presence of time-delays in the control loops usually degrades the performance of the system and complicates the analysis and design of the control systems. A lot of attentions has been paid in this area[1,2]. One of the main topics about the time-delay systems is to reduce the conservatism in the stability analysis. Recently, Moon *et al.* have constructed a new inequality[3], which has more free matrices and is widely used. To reduce the conservatism, model transformation technique is also widely used. A new model transformation has been used[4] , which leads to a comparable better result. In this paper, the above method will be extended.

The existence of input saturation may degrade the performance of the system, and even lead to loss of stability. To reduce the influence of input saturation, two main approaches are commonly adopted. One approach is to take control constraints into account at the beginning of the controller design[5,6]. Another approach is to first ignore the actuator saturation, and design a linear controller that satisfies the performance requirement, and then design an anti-windup compensator to weaken the influence of input saturation[7,8]. Kothare *et al.* have exploited a common framework for the study of anti-windup design[9,10].

We might observe actuator saturation and time-delays in the same system. During the past, few reports concerning this problem have been reported. The problem was reduced to an optimal design problem based on the Lyapunov-Krasovskii function method[11]. Then an iterative approach was proposed to obtain the compensation gain for systems subject to time-delay[12], which involved complex computation, similar to the approach proposed by Cao, *et al.*[13]. Later, a new descriptor method[4] was proposed to improve the regional stabilization results for time-delay systems with saturating actuators[11,13]. But this method imposes limitation on the derivative of the initial conditions.

In this paper, we will extend the idea of [13], and use compensator gain E_c as a free parameter to maximize the estimation of domain of attraction. By means of Lyapunov-Krasovskii method, a direct designing algorithm will be proposed to compute the anti-windup compensator gain, which can greatly reduce the computation burden compared with the earlier method[12,13] while enlarging the estimation of domain of attraction. Note that in [13] an iterative algorithm was introduced, which involved much computation. A new equality with several slack variables and a new sector condition[14] will be incorporated in the algorithm, and remarkably reduce the conservatism. Furthermore, an optimization algorithm in the form of LMIs was presented to estimate the domain of attraction.

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2 Problem statement and preliminaries

In this paper, we will revisit the system that is subject to both time-delays in state and input saturation. Consider the following original linear time-delay system

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}(t) + A_\tau \boldsymbol{x}(t-\tau) + B\boldsymbol{u}(t) \tag{1}$$

$$\boldsymbol{y}(t) = C\boldsymbol{x}(t) \tag{2}$$

with the initial conditions

$$\boldsymbol{x}(t) = \boldsymbol{\phi}(t), \quad \forall t \in [-\tau] \tag{3}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state, $\boldsymbol{u} \in \mathbb{R}^m$ the control input, $\boldsymbol{y} \in \mathbb{R}^p$ the measured output vectors. A, A_τ, B are real constant matrices of appropriate dimensions. The initial condition $\boldsymbol{\phi}$ is a continuous vector-valued function, *i.e.*, $\boldsymbol{\phi} \in C_{n,\tau}$. $C_{n,\pi} = C([-\tau, 0], \mathbb{R}^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence.

To meet the performance specifications, a linear controller is defined as follows

$$\dot{\boldsymbol{\eta}}(t) = A_c\boldsymbol{\eta}(t) + B_c\boldsymbol{y}(t), \quad \boldsymbol{\nu}(t) = C_c\boldsymbol{\eta}(t) + D_c\boldsymbol{y}(t)$$

where $\boldsymbol{\eta} \in \mathbb{R}^{n_c}$ is the controller state, $\boldsymbol{y}(t)$ the controller input, and $\boldsymbol{\nu}(t)$ the controller output. This dynamic compensator has been designed to satisfy the stability and performance requirements of the closed-loop system in the absence of control saturation. In the presence of actuator saturation, the actual control signal of the system can be described as $\boldsymbol{u}(t) = \boldsymbol{\sigma}(\boldsymbol{\nu}(t)) = \boldsymbol{\sigma}(C_c\boldsymbol{\eta}(t) + D_cC\boldsymbol{x}(t))$. The function $\boldsymbol{\sigma} : \mathbb{R}^m \to \mathbb{R}^m$ is the standard saturation function defined by

$$\boldsymbol{\sigma}(\boldsymbol{u}) = [\boldsymbol{\sigma}(\boldsymbol{u}_1) \quad \boldsymbol{\sigma}(\boldsymbol{u}_2) \quad \cdots \quad \boldsymbol{\sigma}(\boldsymbol{u}_m)]^{\mathrm{T}} \text{ and } \boldsymbol{\sigma}(\boldsymbol{u}_i) = \mathrm{sign}(\boldsymbol{u}_i)\min\{\boldsymbol{u}_{0(i)}, |\boldsymbol{u}_i|\}$$

A typical anti-windup compensator involves adding a correction term of the form $E_c(\boldsymbol{\sigma}(\boldsymbol{\nu}(t)))$. The modified compensator has the form

$$\dot{\boldsymbol{\eta}} = A_c\boldsymbol{\eta}(t) + B_c\boldsymbol{y}(t) + E_c(\boldsymbol{\sigma}(\boldsymbol{\nu}(t)) - \boldsymbol{\nu}(t)), \quad \boldsymbol{\nu}(t) = C_c\boldsymbol{\eta}(t) + D_c\boldsymbol{y}(t)$$

In the stability analysis of system subject to input saturation, Popov and circle criteria were frequently used, although they are much conservative. Many attempts were paid to reduce the conservatism. Hu *et al*, has introduced a vertex criteria[15] which greatly reduced the conservatism. However, the vertex criteria can not be used to compute E_c directly. Here, a new sector condition[16] will be used and then an direct algorithm will be derived.

2.1 Closed-loop system structure

It is easy to see that $\boldsymbol{\psi}(\boldsymbol{\nu}) = \boldsymbol{\nu} - \boldsymbol{\sigma}(\boldsymbol{\nu})$ is a dead zone. If we use this dead zone representation under the compensated dynamic linear controller, the closed-loop system can be written as

$$\dot{\boldsymbol{\xi}}(t) = \bar{A}\boldsymbol{\xi}(t) + \bar{A}_\tau\boldsymbol{\xi}(t-\tau) - (\bar{B} + \bar{R}E_c)\boldsymbol{\psi}(F\boldsymbol{\xi}(t)) \tag{4}$$

where

$$\boldsymbol{\xi}(t) = \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{\eta}(t) \end{bmatrix}, \bar{A} = \begin{bmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{bmatrix}, \bar{A}_\tau = \begin{bmatrix} A_\tau & 0 \\ 0 & 0 \end{bmatrix}, \bar{B} = [\,B \quad 0\,], F = [D_cC \quad C_c], \bar{R} = [0 \quad I_{n_c}]^{\mathrm{T}}$$

It is easy to see in (4) that E_c is the only free and unknown parameter that can be used to enlarge the estimation of domain of attraction. In what follows, a new sector condition will be presented.

2.2 Preliminaries

Consider a matrix $G \in \mathbb{R}^{m\times(n+n_c)}$ and define the following polyhedral set

$$\mathcal{S} = \{\boldsymbol{\xi} \in \mathbb{R}^{n+n}; -\boldsymbol{u}_{0(i)} \leqslant (F_i - G_i)\boldsymbol{\xi} \leqslant \boldsymbol{u}_{0(i)}\}, \quad i = 1, \cdots, m \tag{5}$$

Lemma 1[14]. Consider the function $\boldsymbol{\psi}(\boldsymbol{\nu})$ defined above. If $\boldsymbol{\xi} \in \mathcal{S}$, then the relation

$$\boldsymbol{\psi}^{\mathrm{T}}(F\boldsymbol{\xi})T[\boldsymbol{\psi}(F\boldsymbol{\xi}) - G\boldsymbol{\xi}] \leqslant 0 \tag{6}$$

holds for any diagonal matrix $0 < T \in \mathbb{R}^{m\times m}$.

2.3 Problem statement

Assume that system (4) admits an augmented initial condition

$$\phi_\xi(\theta) = \begin{bmatrix} \boldsymbol{x}(t_0+\theta) = \phi_x(\theta) \\ \boldsymbol{\eta}(t_0+\theta) = \phi_\eta(\theta) \end{bmatrix}, \quad \forall\theta \in [-\tau, 0]$$

Denote the state trajectory of (4) with the initial conditions $\boldsymbol{\xi}(t_0+\theta) = \phi_\xi(\theta)$ by $\boldsymbol{\xi}(t, \phi_\xi)$. The domain of attraction corresponding to all initial conditions ϕ_ξ of the closed-loop system (4) is then defined as the set $\mathcal{D} = \{\phi_\xi \in C_{n,\tau}[-\tau, 0] : \lim_{t\to\infty} \boldsymbol{\xi}(t, \phi_\xi) = 0\}$.

The trivial solution $\boldsymbol{\xi}(t, \phi_\xi)$ of system (4) is said to be asymptotically stable if for any initial condition satisfying $|\phi_\xi| \leqslant v$ with any finite $v > 0$, the trajectories of system (4) converge asymptotically to the origin[13]. Here, $|\bullet|$ denotes the vector norm. The determination of the exact domain of attraction is usually impossible and we also cannot find an invariant set to estimate the domain of attraction. However, note that if we can determine a set $\mathcal{X}_\delta$ defined by

$$\mathcal{X}_\delta = \{\phi_\xi \in C_{n,\tau}[-\tau, 0] : |\phi_\xi|^2 \leqslant v\} \tag{7}$$

such that $\mathcal{X}_\delta$ is contained in the domain of attraction, then $\mathcal{X}_\delta$ can be seen as an estimate of the domain of attraction[13]. The design objective of this paper is to design a compensator gain E_c to enlarge the estimation of the domain of attraction.

3 Anti-windup design for delay-independent case

We will give the approach to the design of the anti-windup compensator gain such that the closed-loop system is asymptotically stable at the origin with a domain of attraction as large as possible. The candidate Lyapunov-Krasovskii functional is defined as

$$V(\boldsymbol{\xi}(t)) = \boldsymbol{\xi}^{\mathrm{T}} P_1 \boldsymbol{\xi}(t) + \int_{t-\tau}^{t} \boldsymbol{\xi}^{\mathrm{T}}(s) Q \boldsymbol{\xi}(s) \mathrm{d}s \tag{8}$$

and $P_1 > 0, Q \geqslant 0$ are constant matrices to be determined. The second term of (8) corresponds to the delay-independent stability with respect to the discrete delays the in system state. Now we choose an ellipsoid to estimate the sector condition, which is defined as follows

$$\varOmega(P_1, \rho) = \{\boldsymbol{\xi} \in \mathbb{R}^{n+n_c} : \boldsymbol{\xi}^{\mathrm{T}} P_1 \boldsymbol{\xi} \leqslant \rho^{-1}\} \tag{9}$$

We use $\varOmega(P_1, \rho)$ to bound the states trajectories of system (4) for all the initial states contained in $\mathcal{X}_\delta$. With the above defined Lyapunov-Krasovskii functional, in what follows, we will give the conditions that satisfy the objective defined in Section 2.

Theorem 1. If there exist matrices $X_1 > 0, H > 0$, a diagonal matrix $S > 0$, matrices X_2, Z, Y of appropriate dimensions and constants $\varepsilon, \varepsilon_\tau$, such that the following LMIs hold

$$\begin{bmatrix} \varXi_1 + \varLambda_1 & \bar{\varPhi} \\ * & -2S \end{bmatrix} < 0 \tag{10}$$

$$\begin{bmatrix} X_1 & X_2^{\mathrm{T}} F_{(i)}^{\mathrm{T}} - Y_{(i)}^{\mathrm{T}} \\ * & \rho \boldsymbol{u}_{0(i)}^2 \end{bmatrix} \geqslant 0, \quad \forall i = 1, \cdots, m \tag{11}$$

where

$$\varXi_1 = \begin{bmatrix} X_2^{\mathrm{T}} \bar{A}^{\mathrm{T}} + \bar{A} X_2 & X_1 - X_2 + \varepsilon X_2^{\mathrm{T}} \bar{A}^{\mathrm{T}} & \varepsilon_\tau X_2^{\mathrm{T}} \bar{A}^{\mathrm{T}} + \bar{A}_\tau X_2 \\ * & -\varepsilon X_2^{\mathrm{T}} - \varepsilon X_2 & \varepsilon \bar{A}_\tau X_2 - \varepsilon_\tau X_2^{\mathrm{T}} \\ * & * & \varepsilon_\tau X_2^{\mathrm{T}} \bar{A}_\tau^{\mathrm{T}} + \varepsilon_\tau \bar{A}_\tau X_2 \end{bmatrix} \tag{12}$$

$$\bar{\varLambda}_1 = \begin{bmatrix} H & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -H \end{bmatrix} \tag{13}$$

$$\bar{\varPhi} = [-(-Y^{\mathrm{T}} + \bar{R}Z + \bar{B}S)^{\mathrm{T}} \quad -\varepsilon(\bar{R}Z + \bar{B}S)^{\mathrm{T}} \quad -\varepsilon_\tau(\bar{R}Z + \bar{B}S)^{\mathrm{T}}]^{\mathrm{T}} \tag{14}$$

then the closed-loop system under compensator gain $E_c = ZS^{-1}$ is asymptotically stable at the origin for any initial condition $\phi_\xi(\theta) \in \mathcal{X}_\delta$ with $\delta = 1/\rho\delta_{T_1}$ and $\delta_{T_1} = \bar{\sigma}(X_2^{-\mathrm{T}} X_1 X_2^{-1}) + \tau\bar{\sigma}(X_2^{-\mathrm{T}} H X_2^{-1})$, where $G = YX_2^{-1}$.

Proof. It is easy to see that inequality (1) ensures $W(P_1,\rho) \subset s$. Considering the Lyapunov-Krasovskii functional (8). First, we will introduce several slack variables by a new equality condition[17].

Define $\boldsymbol{z}(t) = \dot{\boldsymbol{\xi}}$. System (4) can be represented equally as

$$\bar{A}\boldsymbol{\xi}(t) - \boldsymbol{z}(t) + \bar{A}_\tau\boldsymbol{\xi}(t-\tau) - (\bar{B} + \bar{R}E_c)\boldsymbol{\psi} = 0 \tag{15}$$

For simplicity, in what follows, we will denote

$$\boldsymbol{\xi}_\tau = \boldsymbol{\xi}(t-\tau),\ \tilde{\boldsymbol{\xi}}(t) = [\boldsymbol{\xi}^{\mathrm{T}}(t) \quad \boldsymbol{z}^{\mathrm{T}}(t) \quad \boldsymbol{\xi}_\tau^{\mathrm{T}}]^{\mathrm{T}},\ \tilde{A} = [\bar{A} \quad -I \quad \bar{A}_\tau],\ \mathcal{P} = [P_2 \quad P_3 \quad P_\tau]$$

We then have

$$2(P_2\boldsymbol{\xi} + P_3\boldsymbol{z} + P\boldsymbol{\xi}_\tau)^{\mathrm{T}}(\bar{A}\boldsymbol{\xi}(t) - \boldsymbol{z}(t) + \bar{A}_\tau\boldsymbol{\xi}_\tau - (\bar{B} + \bar{R}E_c)\boldsymbol{\psi}) = 0 \tag{16}$$

for any weighting matrices P_2, P_3, and P_τ of compatible dimensions. These three matrices are slack variables which can provide freedom in the stability analysis, thus reduce the conservatism.

By (16), derive $V(\boldsymbol{\xi}(t))$ along the solution of (4). Then we obtain

$$\dot{V}(\boldsymbol{\xi}) = 2\boldsymbol{\xi}^{\mathrm{T}}P_1\boldsymbol{z} + \boldsymbol{\xi}^{\mathrm{T}}Q\boldsymbol{\xi} - \boldsymbol{\xi}_\tau^{\mathrm{T}}Q\boldsymbol{\xi}_\tau + 2(P_2\boldsymbol{\xi} + P_3\boldsymbol{z} + P\boldsymbol{\xi}_\tau)^{\mathrm{T}}(\bar{A}\boldsymbol{\xi}(t) - \boldsymbol{z}(t) + \bar{A}_\tau\boldsymbol{\xi}_\tau - (\bar{B} + \bar{R}E_c)\boldsymbol{\psi})$$

By the sector condition (6), the derivative of $V(\boldsymbol{\xi}(t))$ can be relaxed as

$$\dot{V}(\boldsymbol{\xi}(t)) \leqslant \dot{V}(\boldsymbol{\xi}) + 2\boldsymbol{\psi}^{\mathrm{T}}TG\boldsymbol{\xi}(t) - 2\boldsymbol{\psi}^{\mathrm{T}}T\boldsymbol{\psi} = \boldsymbol{\zeta}^{\mathrm{T}}\Gamma\boldsymbol{\zeta}$$

where

$$\boldsymbol{\zeta} = [\boldsymbol{\xi}^{\mathrm{T}} \quad \boldsymbol{z}^{\mathrm{T}} \quad \boldsymbol{\xi}_\tau^{\mathrm{T}} \quad \boldsymbol{\psi}^{\mathrm{T}}]^{\mathrm{T}},\quad \Gamma = \begin{bmatrix} \Sigma_1 + \Lambda_1 & \Phi \\ * & -2T \end{bmatrix}$$

and

$$\Sigma_1 = \begin{bmatrix} P_2^{\mathrm{T}}\bar{A} + \bar{A}^{\mathrm{T}}P_2 & P_1 - P_2^{\mathrm{T}} + \bar{A}^{\mathrm{T}}P_3 & \bar{A}^{\mathrm{T}}P_\tau + P_2^{\mathrm{T}}\bar{A}_\tau \\ * & -P_3^{\mathrm{T}} - P_3 & P_3^{\mathrm{T}}\bar{A}_\tau - P_\tau \\ * & * & P_\tau^{\mathrm{T}}\bar{A}_\tau + \bar{A}_\tau^{\mathrm{T}}P_\tau \end{bmatrix}$$

$$\Lambda_1 = \begin{bmatrix} Q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -Q \end{bmatrix},\quad \Phi = \begin{bmatrix} (TG)^{\mathrm{T}} - P_2^{\mathrm{T}}(\bar{B} + \bar{R}E_c) \\ -P_3^{\mathrm{T}}(\bar{B} + \bar{R}E_c) \\ -P_\tau^{\mathrm{T}}(\bar{B} + \bar{R}E_c) \end{bmatrix}$$

Obviously inequality

$$\Gamma < 0 \tag{17}$$

ensures $\dot{V}(\boldsymbol{\xi}(t)) < 0$. Let $X_2 = P_2^{-1}$, $X_1 = X_2^{\mathrm{T}}P_1X_2$, $P_3 = \varepsilon P_2$, $P_\tau = \varepsilon_\tau P_2$, $S = T^{-1}$, $H = X_2^{\mathrm{T}}QX_2$, $Z = E_cS$, $Y = GX_2$, and $\Delta = \mathrm{diag}\{X_2, X_2, X_2, S\}$. Multiply (17) by Δ^{T} and Δ on the left and right sides, respectively and we get inequality (10).

From $\dot{V}(\boldsymbol{\xi}(t)) < 0$, it follows that $V(\boldsymbol{\xi}(t)) < V(\boldsymbol{\xi}(0))$. And Let $\bar{\sigma}(G)$ denote the largest singular value of G, therefore, we obtain

$$\boldsymbol{\xi}^{\mathrm{T}}P_1\boldsymbol{\xi}(t) \leqslant V(\boldsymbol{\xi}(t)) < V(\boldsymbol{\xi}(0)) \leqslant |\boldsymbol{\phi}_\xi|^2\delta_{T_1} \leqslant \rho^{-1}$$

Hence, with any initial condition $\boldsymbol{\phi}_\xi(\theta) \in \mathcal{X}_\delta$, the trajectories $\boldsymbol{\xi}(t, \boldsymbol{\phi}_\xi)$ of the closed-loop system remain within $\Omega(P_1,\rho)$ provided that (10) is satisfied. □

Remark 1. In the proof of Theorem 1, a model transformation is actually introduced when we set $\boldsymbol{z}(t) = \dot{\boldsymbol{\xi}}(t)$. This transformation is less conservative than other model transformation[18]. Comparing Theorem 1 with Theorem 1 of [16], there are two free variables $\varepsilon, \varepsilon_\tau$ in Theorem 1, which could provide more freedom for search of the optimal solution. Actually, in the proof of Theorem 1, three slack variables P_2, P_3, and P_τ are introduced.

4 Anti-windup design for delay-dependent case

In this section, we will do delay-dependent stability analysis for system (4) and present a compensation method to enlarge the estimation of domain of attraction by designing the anti-windup compensation gain E_c.

4.1 Delay-dependent result

Denote $z(t) = \dot{\xi}(t)$ and we choose the following Lyapunov-Krasovskii functional

$$V(\boldsymbol{\xi}(t)) = \boldsymbol{\xi}^{\mathrm{T}}P_1\boldsymbol{\xi}(t) + V_2 + V_3 \tag{18}$$

where

$$V_2 = \int_{t-\tau}^{t} \boldsymbol{\xi}^{\mathrm{T}}(s) Q_1 \boldsymbol{\xi}(s) \mathrm{d}s, \quad V_3 = \int_{-\tau}^{0}\int_{t+\theta}^{t} \boldsymbol{z}^{\mathrm{T}}(s) Q_2 \boldsymbol{z}(s) \mathrm{d}s\mathrm{d}\theta$$

With the above defined system transformation and Lyapunov-Krasovskii functional, the following theorem can be obtained.

Theorem 2. If there exist matrices $X_1 > 0, H_1 > 0, H_2 > 0, \tilde{M} > 0$, a diagonal matrix $S > 0$, matrices $\tilde{N}, X_2, Z, Y$ of appropriate dimensions and constants $\varepsilon, \varepsilon_\tau, \tau_0$ such that the following LMIs hold

$$\begin{bmatrix} \Xi_1 + \Xi + \bar{\Lambda}_2 & \bar{\Phi} \\ * & -2S \end{bmatrix} < 0 \tag{19}$$

$$\begin{bmatrix} X_1 & X_2^{\mathrm{T}} F_{(i)}^{\mathrm{T}} - Y_{(i)}^{\mathrm{T}} \\ * & \rho \boldsymbol{u}_{0(i)}^2 \end{bmatrix} \geqslant, \quad \forall i = 1, \cdots, m \tag{20}$$

$$\begin{bmatrix} \tilde{M} & \tilde{N}^{\mathrm{T}} \\ * & H_2 \end{bmatrix} \geqslant 0 \tag{21}$$

where $\Lambda_2 = \begin{bmatrix} H_1 & 0 & 0 \\ 0 & \tau_0 H_2 & 0 \\ 0 & 0 & -H_1 \end{bmatrix}$, $\Xi = \tau_0 \tilde{M} + \tilde{N}\Pi + \Pi^{\mathrm{T}}\tilde{N}$ with, $\Pi = [I \quad 0 \quad -I]$, Ξ, $\bar{\Phi}_1$, $\bar{\Phi}_2, S$ and G defined in Theorem 1, then the closed-loop system (4) is asymptotically stable at the origin for any $\tau \leqslant \tau_0$ under compensator gain $E_c = ZS^{-1}$ for any initial condition $\boldsymbol{\phi}_\xi(\theta) \in \mathcal{X}_\delta$ with $\delta = 1/\rho\delta_{T_2}$ and

$$\delta_{T_2} = \bar{\sigma}(P_1) + \tau_0\bar{\sigma}(Q_1) + \frac{9\tau_0^2}{2}[\bar{\sigma}(\bar{A}^{\mathrm{T}} Q_2 \bar{A}) + \bar{\sigma}(\bar{A}_\tau^{\mathrm{T}} Q_2 \bar{A}_\tau) + \bar{\sigma}((\bar{B} + \bar{R}E_c)^{\mathrm{T}} Q_2 (\bar{B} + \bar{R}E_c))|G|^2] \tag{22}$$

where $P_1 = X_2^{-\mathrm{T}} X_1 X_2^{-1}, Q_1 = X_2^{-\mathrm{T}} H_1 X_2^{-1}, Q_2 = X_2^{-\mathrm{T}} H_2 X_2^{-1}$, and $G = YX_2^{-1}$.

Proof. The derivative of $V(\boldsymbol{\xi}(t))$ along the trajectory of system (4) can be computed as follows

$$\dot{V}_2 = \boldsymbol{\xi}^{\mathrm{T}}(t) Q_1 \boldsymbol{\xi}(t) - \boldsymbol{\xi}^{\mathrm{T}}(t-\tau) Q_1 \boldsymbol{\xi}(t-\tau), \quad \dot{V}_3 = \tau \boldsymbol{z}^{\mathrm{T}}(t) Q_2 \boldsymbol{z}(t) - \int_{t-\tau}^{t} \boldsymbol{z}^{\mathrm{T}}(s) Q_2 \boldsymbol{z}(s) \mathrm{d}s$$

Using the Leibniz-Newton formula, we obtain $\boldsymbol{\xi}(t-\tau) = \boldsymbol{\xi}(t) - \int_{t-\tau}^{t} \dot{\boldsymbol{\xi}}(s)\mathrm{d}s = \boldsymbol{\xi}(t) - \int_{t-\tau}^{t} \boldsymbol{z}(s)\mathrm{d}s$.

Then equality (16) can be rewritten as

$$2\tilde{\boldsymbol{\xi}}^{\mathrm{T}}(t)\mathcal{P}^{\mathrm{T}}((\bar{A} + \bar{A}_\tau)\boldsymbol{\xi}(t) - \boldsymbol{z}(t) - \bar{A}_\tau \int_{t-\tau}^{t} \boldsymbol{z}(s)\mathrm{d}s - (\bar{B} + \bar{R}E_c)\boldsymbol{\psi}) = 0 \tag{23}$$

By equality (23), we obtain

$$\begin{aligned} \dot{V}(\boldsymbol{\xi}) =& 2\boldsymbol{\xi}^{\mathrm{T}} P_1 \dot{\boldsymbol{\xi}} + \dot{V}_2 + \dot{V}_3 = 2\boldsymbol{\xi}^{\mathrm{T}} P_1 \boldsymbol{z} + 2\tilde{\boldsymbol{\xi}}^{\mathrm{T}}(t)\mathcal{P}^{\mathrm{T}}\tilde{A}\tilde{\boldsymbol{\xi}} + 2\tilde{\boldsymbol{\xi}}^{\mathrm{T}}(t)\mathcal{P}^{\mathrm{T}}\bar{A}_\tau(\boldsymbol{\xi} - \boldsymbol{\xi}) - \\ & 2\tilde{\boldsymbol{\xi}}^{\mathrm{T}}(t)\mathcal{P}^{\mathrm{T}}\bar{A}_\tau \int_{t-\tau}^{t} \boldsymbol{z}(s)\mathrm{d}s + 2\tilde{\boldsymbol{\xi}}^{\mathrm{T}}(-\bar{B} - \bar{R}E_c)\boldsymbol{\psi} + \dot{V}_2 + \dot{V}_3 \end{aligned}$$

Assuming that $M \geqslant 0$ and N are matrices of appropriate dimensions, and satisfy $\begin{bmatrix} M & N^{\mathrm{T}} \\ * & H_2 \end{bmatrix} \geqslant 0$. Thus, under the relax condition[3], we obtain

$$\begin{aligned} -2\tilde{\boldsymbol{\xi}}^{\mathrm{T}}(t)\mathcal{P}^{\mathrm{T}}\bar{A}_\tau \int_{t-\tau}^{t} \boldsymbol{z}(s)\mathrm{d}s \leqslant& \tau\tilde{\boldsymbol{\xi}}^{\mathrm{T}} M\tilde{\boldsymbol{\xi}} + 2\tilde{\boldsymbol{\xi}}^{\mathrm{T}}(N^{\mathrm{T}} - \mathcal{P}^{\mathrm{T}}\bar{A}_\tau)\int_{t-\tau}^{t} \boldsymbol{z}(s)\mathrm{d}s + \int_{t-\tau}^{t} \boldsymbol{z}^{\mathrm{T}}(s) Q_2 \boldsymbol{z}(s)\mathrm{d}s = \\ & \tau\tilde{\boldsymbol{\xi}}^{\mathrm{T}} M\tilde{\boldsymbol{\xi}} + 2\tilde{\boldsymbol{\xi}}^{\mathrm{T}}(N^{\mathrm{T}} - \mathcal{P}^{\mathrm{T}}\bar{A}_\tau)(\boldsymbol{\xi} - \boldsymbol{\xi}_\tau) + \int_{t-\tau}^{t} \boldsymbol{z}^{\mathrm{T}}(s) Q_2 \boldsymbol{z}(s)\mathrm{d}s \end{aligned}$$

Together with the application of sector condition (6), the derivative of $V(\boldsymbol{\xi}(t))$ can be relaxed as

$$\begin{aligned} \dot{V}(\boldsymbol{\xi}) \leqslant& \tilde{\boldsymbol{\xi}}^{\mathrm{T}}(\mathcal{P}^{\mathrm{T}}\tilde{A} + \tilde{A}^{\mathrm{T}}\mathcal{P} + \tau M + N^{\mathrm{T}}\Pi + \Pi^{\mathrm{T}} N)\tilde{\boldsymbol{\xi}} + 2\boldsymbol{\xi}^{\mathrm{T}} P_1 \boldsymbol{z} + \\ & \tau \boldsymbol{z}^{\mathrm{T}} Q_2 \boldsymbol{z} + \boldsymbol{\xi}^{\mathrm{T}} Q_1 \boldsymbol{\xi} - \boldsymbol{\xi}_\tau^{\mathrm{T}} Q_1 \boldsymbol{\xi}_\tau + 2\boldsymbol{\psi}^{\mathrm{T}} T G \boldsymbol{\xi} - 2\boldsymbol{\psi}^{\mathrm{T}} T \boldsymbol{\psi} \end{aligned}$$

Similar to the proof of Theorem 1, we obtain inequalities (19), (20) and (21). By equality (15), we obtain

$$\boldsymbol{z}^{\mathrm{T}}Q_2\boldsymbol{z} = \boldsymbol{\xi}^{\mathrm{T}}\bar{A}^{\mathrm{T}}Q_2\bar{A}\boldsymbol{\xi} + \boldsymbol{\xi}_\tau^{\mathrm{T}}\bar{A}_\tau^{\mathrm{T}}Q_2\bar{A}_\tau\boldsymbol{\xi}_\tau + \boldsymbol{\psi}^{\mathrm{T}}(\bar{B}+\bar{R}E_c)^{\mathrm{T}}Q_2(\bar{B}+\bar{R}E_c)\boldsymbol{\psi} + \nu_1+\nu_2+\nu_3$$

where

$$\begin{aligned}
\nu_1 &= 2\boldsymbol{\xi}^{\mathrm{T}}\bar{A}^{\mathrm{T}}Q_2\bar{A}_\tau\boldsymbol{\xi} \leqslant \boldsymbol{\xi}^{\mathrm{T}}\bar{A}^{\mathrm{T}}Q_2\bar{A}\boldsymbol{\xi} + \boldsymbol{\xi}_\tau^{\mathrm{T}}\bar{A}_\tau^{\mathrm{T}}Q_2\bar{A}_\tau\boldsymbol{\xi}_\tau\\
\nu_2 &= 2\boldsymbol{\xi}^{\mathrm{T}}\bar{A}^{\mathrm{T}}Q_2(\bar{B}+\bar{R}E_c)\boldsymbol{\psi} \leqslant \boldsymbol{\xi}^{\mathrm{T}}\bar{A}^{\mathrm{T}}Q_2\bar{A}\boldsymbol{\xi} + \boldsymbol{\psi}^{\mathrm{T}}(\bar{B}+\bar{R}E_c)^{\mathrm{T}}Q_2(\bar{B}+\bar{R}E_c)\boldsymbol{\psi}\\
\nu_3 &= 2\boldsymbol{\xi}_\tau^{\mathrm{T}}\bar{A}_\tau^{\mathrm{T}}Q_2(\bar{B}+\bar{R}E_c)\boldsymbol{\psi} \leqslant \boldsymbol{\xi}_\tau^{\mathrm{T}}\bar{A}_\tau^{\mathrm{T}}Q_2\bar{A}_\tau\boldsymbol{\xi}_\tau + \boldsymbol{\psi}^{\mathrm{T}}(\bar{B}+\bar{R}E_c)^{\mathrm{T}}Q_2(\bar{B}+\bar{R}E_c)\boldsymbol{\psi}
\end{aligned}$$

It follows that

$$\boldsymbol{z}^{\mathrm{T}}Q_2\boldsymbol{z} \leqslant 3(\boldsymbol{\xi}^{\mathrm{T}}\bar{A}^{\mathrm{T}}Q_2\bar{A}\boldsymbol{\xi} + \boldsymbol{\xi}_\tau^{\mathrm{T}}\bar{A}_\tau^{\mathrm{T}}Q_2\bar{A}_\tau\boldsymbol{\xi}_\tau + \boldsymbol{\psi}^{\mathrm{T}}(\bar{B}+\bar{R}E_c)^{\mathrm{T}}Q_2(\bar{B}+\bar{R}E_c)\boldsymbol{\psi}) \tag{24}$$

By the same method used in the proof of Theorem 1, we get that inequality (19) implies $\dot{V}(\boldsymbol{\xi}) < 0$. It follows that $V(\boldsymbol{\xi}(t)) < V(\boldsymbol{\xi}(0))$. Then, together with $|\boldsymbol{\psi}(F\boldsymbol{\xi})| \leqslant |G|\cdot|\boldsymbol{\xi}|$ and inequality (24), we get

$$\boldsymbol{\xi}^{\mathrm{T}}P_1\boldsymbol{\xi}(t) \leqslant V(\boldsymbol{\xi}(t)) < V(\boldsymbol{\xi}(0)) \leqslant |\boldsymbol{\phi}_\xi|^2\delta_{T_2} \leqslant \rho^{-1}$$

Therefore, with any initial condition $\boldsymbol{\phi}_\xi(\theta) \in \mathcal{X}_\delta$, the trajectories $\boldsymbol{\xi}(t,\boldsymbol{\phi}_\xi)$ of the closed-loop system remain within $\Omega(P_1,\rho)$, provided that (10) is satisfied. □

Remark 2. In the case of delay-dependent analysis, a model transformation is adopted, which was also adopted by E. Fridman[4] but restriction on the derivative of initial conditions was imposed. Our approach has removed this restriction and the theorems hold for all the initial conditions in the admissible set $\mathcal{X}_\delta$.

4.2 Optimization algorithm

As stated in Section 2, our objective is to make the estimation of domain of attraction as large as possible. Theorem 2 gives a sufficient condition allowing to compute the compensation gain E_c such that the closed-loop system is regionally stabilized in the ball $\mathcal{X}_\delta$ of all initial conditions. Obviously, if the delay τ is known, a natural idea is to optimize E_c such that the ball $\mathcal{X}_\delta$ is as large as possible. It is easy to see that this problem can be solved by minimizing $\rho\delta_T$. However, due to the complex representation of $\rho\delta_T$ in Theorem 1 and Theorem 2, we can not reduce the optimization problem of $\min\{\rho\delta_T\}$ to an optimization problem in the form of LMIs directly.

In what follows, we will present an algorithm to solve this problem. Take Theorem 2 for example. To solve the problem, we use the following optimization function

$$\begin{aligned}
&\min\ \{\theta_0\rho + \theta_1 trace(V_{X_1}) + \theta_2 trace(V_{H_1}) + \theta_3 trace(V_{H_2})\}\\
&\text{s.t. } a)\quad X_2^{-\mathrm{T}}X_1X_2^{-1} \leqslant V_{X_1},\ X_2^{-\mathrm{T}}H_1X_2^{-1} \leqslant V_{H_1},\ X_2^{-\mathrm{T}}H_2X_2^{-1} \leqslant V_{H_2}\\
&\qquad b)\quad inequality\ (19), (20), (21)
\end{aligned} \tag{25}$$

where $\theta_0, \theta_1, \theta_2, \theta_3 > 0$ are weighting parameters.

It is easy to see that the optimization objective function in the optimization problem (25) is not equivalent to $\min\{\rho\delta_{T_2}\}$. Here, we try to use the optimization problem (25) to obtain a ball $\mathcal{X}_\delta$ as large as possible, although we may not obtain the optimal solution, *i.e.*, the largest ball. Similarly, we can use the above optimizing method to solve the computation problems for the delay-independent case.

Note that $(X-Y)^{\mathrm{T}}Y^{-1}(X-Y) \geqslant 0$ always holds for any matrixes X and Y of compatible dimensions, and we have $X^{\mathrm{T}}Y^{-1}X \geqslant X^{\mathrm{T}} + X - Y$. By the Schur complement, the optimization problem (25) can be changed to the following LMI optimization problem

$$\begin{aligned}
&\min\ \{\theta_0\rho + \theta_1 trace(V_{X_1}) + \theta_2 trace(V_{H_1}) + \theta_3 trace(V_{H_2})\}\\
&\text{s.t. } a)\quad \begin{bmatrix} V_{X_1} & I\\ I & X_2^{\mathrm{T}}+X_2-X_1\end{bmatrix} \geqslant 0,\ \begin{bmatrix} V_{H_1} & I\\ I & X_2^{\mathrm{T}}+X_2-H_1\end{bmatrix} \geqslant 0\\
&\qquad\quad \begin{bmatrix} V_{H_2} & I\\ I & X_2^{\mathrm{T}}+X_2-H_2\end{bmatrix} \geqslant 0\\
&\qquad b)\quad inequality\ (19), (20), (21)
\end{aligned} \tag{26}$$

where ε, ε_τ, and τ are pre-given, and $\theta_0, \theta_1, \theta_2, \theta_3 > 0$ are weighting parameters.

It is easy to solve the LMI optimization problem (26) by the LMI tools in Matlab. After solving the optimization problem, we can compute the estimation of domain of attraction $\mathcal{X}_\delta$ with $\delta = 1/\rho\delta_{T_2}$. This method can be easily extended to the delay-independent case.

5 Numerical examples

Example 1. Firstly we consider an example to illustrate the analysis of delay-independent case. This system is given as follows

$$A = \begin{bmatrix} 1 & 1.5 \\ 0.3 & -2 \end{bmatrix},\ A_\tau = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 10 \\ 1 \end{bmatrix},\ \boldsymbol{C} = [5 \quad 1],\ D = 0,\ u_0 = 15$$

The dynamic controller is given as

$$A_c = \begin{bmatrix} -20.2042 & 2.5216 \\ 2.1415 & -4.4821 \end{bmatrix},\ \boldsymbol{B}_c = \begin{bmatrix} 1.9516 \\ -0.0649 \end{bmatrix},\ \boldsymbol{C}_c = [-0.9165 \quad 0.1091],\ D_c = 0$$

The above example is borrowed from [16]. By Theorem 1 with $\varepsilon = 0.02$, $\varepsilon_\tau = 0$, a feasible solution can be obtained with $\boldsymbol{E}_c = \begin{bmatrix} -21.5148 \\ -4.7793 \end{bmatrix}$. If we set $\tau = 0.5$, then using the algorithm presented in Section 4, δ is computed as $4.8468e+003$ with the above E_c, while δ is $4.7852e+003$ when $E_c = 0$. This implies that the compensator gain has enlarged the estimate of domain of attraction. δ is computed as $4.520e+003$ with $\tau = 0.5$ and $E_c \neq 0$ in [16]. Obviously, our result is better than that in [16]. The detailed comparison is listed in Table 1, where $\delta_{E_c\neq 0}$ denotes δ with E_c as a free parameter, and $\delta_{E_c=0}$ denotes δ obtained with E_c set as zero. Note that the above results were obtained with all the same weighting parameters.

Table 1 δ for different τ

τ	$\delta_{E_c\neq 0}$	$\delta_{E_c=0}$	$\delta_{E_c\neq 0}$ of [16]
0.5	$4.8468e+003$	$4.7852e+003$	$4.5200e+003$
1	$3.7224e+003$	$3.6817e+003$	$2.9860e+003$
2	$2.5427e+003$	$2.5196e+003$	$1.7727e+003$

Example 2. Now we consider an example to illustrate the analysis of delay-dependent case. This example was also used in [16], which is defined as follows

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ A_\tau = \begin{bmatrix} 1 & 1.5 \\ 0.3 & -2 \end{bmatrix},\ \boldsymbol{B} = [10 \quad 0],\ \boldsymbol{C} = [5 \quad 1],\ D = 0,\ u_0 = 15$$

The dynamic controller is also the same as given in Example 1. By Theorem 2 and the algorithm presented in Section 4 with $\varepsilon = 0.15$, $\varepsilon_\tau = 0.02$, the stability of the system can be guaranteed for all the constant time-delay. We list the detailed information of comparison with method in [16] about the δ, E_c in Table 2. Clearly, our result is less conservative than that of [16]. Obviously, for a larger, one obtains a smaller estimation of the region of attraction. These result are obtained with all the same weighting factors in (26).

Table 2 δ and E_c for different values of τ

τ	δ_{E_c}	δ	δ of [16]
0.1	$\begin{bmatrix} 9.0502 \\ -8.6980 \end{bmatrix}$	$8.5167e+003$	$7.6820e+003$
0.2	$\begin{bmatrix} 8.7230 \\ -9.6404 \end{bmatrix}$	$7.2125e+003$	$5.5520e+003$
0.4	$\begin{bmatrix} 7.2257 \\ -8.0450 \end{bmatrix}$	$4.6355e+003$	756.19

6 Conclusion

In this paper, we considered linear systems subject to both time-delays in state and saturation in input signal. A new Lyapunov function approach was presented, and a direct algorithm was introduced to design the anti-windup compensator gain, which can enlarge the domain of attraction of the closed-loop systems. Moreover, with a relax technique, we constructed an algorithm to optimize, such that the estimation of domain of attraction is as large as possible. It is important to note that our method is different from that used in [12,16]. More free parameters were used to reduce the conservatism in this paper. Moreover, the optimization algorithm that is used to compute the maximal estimation of region of attraction is original, which involved less computation than the method used by [12,16]

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