Modeling and Identification of Multirate Systems[1)]

Feng DING Tongwen CHEN

(*Department of Electrical and Computer Engineering University of Alberta, Edmonton, Alberta, T6G 2V4 Canada*)

(E-mail: fding@ece.ualberta.ca, tchen@ece.ualberta.ca)

Abstract Multirate systems are abundant in industry; for example, many soft-sensor design problems are related to modeling, parameter identification, or state estimation involving multirate systems. The study of multirate systems goes back to the early 1950's, and has become an active research area in systems and control. This paper briefly surveys the history of development in the area of multirate systems, and introduces some basic concepts and latest results on multirate systems, including a polynomial transformation technique and the lifting technique as tools for handling multirate systems, lifted state space models, parameter identification of dual-rate systems, how to determine fast single-rate models from dual-rate models and directly from dual-rate data, and a hierarchical identification method for general multirate systems. Finally, some further research topics for multirate systems are given.

Key words Dual-rate systems, multirate systems, identification, parameter estimation, systems representation, hierarchical identification, the lifting technique, polynomial transformation, least squares, state space models, auxiliary models, convergence properties

1 Introduction

Systems with two or more operating frequencies are called *multirate* systems. For example, consider a discrete-time system in which the control updating period T_1 is not equal to the output sampling period T_2 ($T_1 \neq T_2$); this gives rise to a simple multirate system—a *dual-rate* system, as shown in Fig. 1. Here P_c is a continuous-time process; the control input $u_c(t)$ to P_c is produced by a zero-order hold H_{T_1} with period T_1; and the output $y_c(t)$ of P_c is sampled by a sampler S_{T_2} with period T_2, yielding a discrete-time signal $y_c(kT_2)$. The signals $u_c(t)$ and $u_c(kT_1)$ satisfy: $u_c(t) = u_c(kT_1),\ kT_1 \leqslant t < (k+1)T_1,\ k = 0, 1, 2, \cdots$. Note that multirate systems normally are *time-varying* even if their underlying continuous-time processes are linear time-invariant.

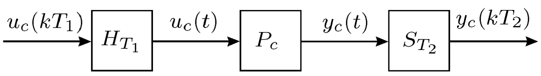

Fig. 1 The dual-rate sampled-data system

When $T_1 = T_2$, we obtain a conventional sample-data system—single-rate system[1], which is a special case of multirate systems. Identification and control of single-rate systems are well studied, e.g., see the books entitled Process Identification[2], System Identification: Theory for the User[3], Adaptive Filtering, Prediction and Control[4], and Adaptive Control Systems[5].

For a multi-input multi-output system, when each input/output channel has different updating/sampling period, we obtain a multirate system. Multirate systems are abundant in industrial processes[6~9], mostly due to sensor and actuator speed constraints; for example, in polymer reactors[6,8], fermentation processes[7] and petroleum production[9], the composition, density or molecular weight distribution, and gasoline octane quality measurements are typically obtained after several minutes of analysis, whereas the manipulated variables can be adjusted at relatively fast rate. Also, many soft-sensor design problems in chemical processes are related to modeling, parameter identification, or state estimation involving multirate systems.

The study of multirate systems goes back to the early 1950′s. The first important work was performed by Kranc on the switch decomposition technique[10], later termed as the lifting technique, which has become a standard tool to transform a periodically time-varying system into a time-invariant

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one. The lifting results in a multivariable system with a causality constraint even if the corresponding continuous-time process is a scalar one. Multirate system identification requires dealing with this causality constraint and extracting single-rate models from the estimated multirate models[11,12].

For decades, modeling and control of multirate systems have had many successful applications in chemical and petroleum processes[6~9], and a series of results have been achieved in theory, including controllability and observability[13,14], optimal control[15,16], robust control[17~22], adaptive control[8,23~26], predictive control[14,27~29], inferential control[9,30,31], signal processing[32~36], modeling and identification[11,14,37~46], and so on.

The objective of this paper is to establish the mapping relationship between available multirate input and output data by using a polynomial transformation technique and the lifting technique, namely, to derive general model descriptions for multirate systems, and to survey some identification methods for multirate systems.

The paper is organized as follows. In Section 2, an identification model of a simple dual-rate system is derived by using the polynomial transformation technique; the properties of the parameter and missing output estimation in the stochastic framework are studied; a single-rate model is extracted from the obtained dual-rate model and is identified directly from dual-rate data. In Section 3, a lifted state-space model of a simple dual-rate system is derived by using the lifting technique; the identifiability problem of dual-rate systems is discussed, and so is the relationship between the dual-rate model and single-rate model. In Section 4, a lifted state-space model and a time-varying state-space representation of general dual-rate systems are established; hierarchical identification methods of combined state and parameter estimation based on the lifted model are introduced; and algorithms of determining the single-rate models with different sampling periods are discussed. In Section 5, a lifted state-space model for a general multirate system is given. Finally, some concluding remarks are collected in Section 6.

2 Simple dual-rate systems

2.1 Dual-rate models

Fig. 2 is a simple dual-rate system, where H_h is a zero-order hol with period h, S_{qh} a sampler with period qh ($q \geqslant 2$ being an integer). For convenience, we often omit h and write $u(k) := u_c(kh)$, $y(kq) := y_c(kqh)$, $\cdots$. The dual-rate input-output data available are

- $\{u(k),\ k = 0, 1, 2, \cdots\}$ at the fast rate, and
- $\{y(kq),\ k = 0, 1, 2, \cdots\}$ at the slow rate.

Thus, the intersample outputs (also called missing outputs), $y_c(kqh+ih) =: y(kq+i)$, $i = 1, 2, \cdots, q-1$, are unavailable due to hardware limitation.

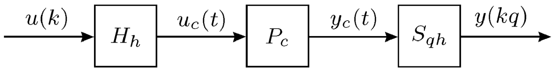

Fig. 2 A simple dual-rate system

Therefore, the objectives of modeling and identification of multirate systems are two-fold: To establish the mapping relationship between available input and output data, and to estimate the intersample (missing) output samples by using the obtained model. If we identify the models obtained by the lifting technique or by the polynomial transformation technique, then it is required to find the corresponding fast single-rate models since they can be used to design, e.g., the inferential control scheme [9,31] and self-tuning control strategy [43,44] for multirate systems.

Let us introduce some notation first. The symbol I stands for an identity matrix of appropriate sizes; the superscript T denotes the matrix transpose; $|X| = \det[X]$ represents the determinant of the matrix X; the norm of a matrix X is defined by $\|X\|^2 = \mathrm{tr}[XX^{\mathrm{T}}]$; $\lambda_{\max}[X]$ and $\lambda_{\min}[X]$ represent the maximum and minimum eigenvalues of X, respectively; $f(k) = o(g(k))$ represents $f(k)/g(k) \to 0$ as $t \to \infty$; for $g(k) \geqslant 0$, we write $f(k) = O(g(k))$ or $f(k) \sim g(k)$ if there exists a positive constant δ_m such that $|f(k)| \leqslant \delta_m g(k)$; $\hat{\theta}(*)$ denotes the estimate of the parameter vector θ.

Assume that P_c is a continuous-time linear time-invariant system with the following state-space representation

$$\dot{x}_c(t) = A_c x_c(t) + B_c u_c(t)$$
$$y_c(t) = C x_c(t) + D u_c(t)$$

where $x_c(t) \in R^n$ is the state vector, A_c, B_c, C and D are constant matrices of appropriate sizes. Discretizing P_c with sampling period h yields a single-rate system $P_1 = S_h P_c H_h$:

$$x(k+1) = Ax(k) + Bu(k) \tag{1}$$
$$y(k) = Cx(k) + Du(k) \tag{2}$$

where $x(k) := x_c(kh)$, matrices A, B, A_c and B_c have the following relations[1]:

$$A = \mathrm{e}^{A_c h},\ B = \int_0^h \mathrm{e}^{A_c t}\mathrm{d}t B_c$$

In general, P_1 has a rational transfer function:

$$P_1(z) = D + C(zI - A)^{-1}B =: \frac{b(z)}{a(z)} \tag{3}$$

where z^{-1} represents a unit backward operator at the fast rate $[z^{-1}x(k) = x(k-1)]$, $a(z)$ and $b(z)$ are polynomials in z^{-1} and

$$a(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n},\ b(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n}$$

Hence, we obtain a fictitious single-rate model:

$$y(k) = P_1(z)u(k) = \frac{b(z)}{a(z)}u(k)$$

However, this model is not suitable for system identification using the dual-rate data $\{u(k), y(kq)\}$. For example, take $q = 3$, it is easy to get the recursive equation:

$$y(k) + a_1 y(k-1) + a_2 y(k-2) + \cdots + a_n y(k-n) = b_0 u(k) + b_1 u(k-1) + \cdots + b_n u(k-n)$$

When k is an integer multiple of q, $y(k)$ is available, but $y(k-1)$ and $y(k-2)$ are not. In order to obtain the model we can use with the dual-rate data, the polynomial transformation technique and the lifting technique can be used to [12,37].

The polynomial transformation technique[12]. Let z_i be the root of $a(z)$, then

$$a(z) = \prod_{i=1}^{n}(1 - z_i z^{-1})$$

Define

$$\phi_q(z) = \prod_{i=1}^{n}(1 + z_i z^{-1} + z_i^2 z^{-2} + \cdots + z_i^{q-1} z^{-q+1}) =: 1 + f_1 z^{-1} + f_2 z^{-2} + \cdots + f_{n_f} z^{-n_f}$$

Multiplying the denominator and numerator of $P_1(z)$ by $\phi_q(z)$ gets a new model P_2:

$$P_2(z) = \frac{b(z)\phi_q(z)}{a(z)\phi_q(z)} =: \frac{\beta(z)}{\alpha(z^q)},\ \text{or}\ \alpha(z^q)y(k) = \beta(z)u(k) \tag{4}$$

where

$$\alpha(z^q) = a(z)\phi_q(z) = 1 + \alpha_1 z^{-q} + \alpha_2 z^{-2q} + \cdots + \alpha_n z^{-qn}$$
$$\beta(z) = b(z)\phi_q(z) = \beta_0 + \beta_1 z^{-1} + \beta_2 z^{-2} + \cdots + \beta_{qn} z^{-qn}$$

Of course, the two models P_1 and P_2 are equivalent but P_2 is more complicated than P_1 because of the common factor $\phi_q(z)$. However, the advantage with the model P_2 is that the denominator is a polynomial of z^{-q}; arising from here is a recursive equation using only slowly sampled outputs. The system identification algorithm we propose later for dual-rate systems will be based on this model which does not involve the unavailable intersample output data. The price we paid is to estimate more parameters in $\beta(z)$ than in $b(z)$.

2.2 Parameter and output estimation of ARX dual-rate models

Here we discuss the identification problem of the model in (4). From (4), introducing a random noise term $v(k)$ with zero mean, we have

$$\alpha(z^q)y(k) = \beta(z)u(k) + v(k) \tag{5}$$

This is a dual-rate stochastic system described by the equation-error model or the ARX (Auto-Regression with eXternal input)/CAR (Controlled Auto-Regression) model. Define the parameter vector θ and information vector $\varphi(k)$ as

$$\theta = [\alpha_1, \alpha_2, \cdots, \alpha_n, \beta_0, \beta_1, \cdots, \beta_{qn}]^{\mathrm{T}} \in R^N, \quad N =: qn + n + 1$$
$$\varphi(k) = [-y(k-q), -y(k-2q), \cdots, -y(k-qn), u(k), u(k-1), \cdots, u(k-qn)]^{\mathrm{T}}$$

Then it is not difficult to get

$$y(k) = \varphi^{\mathrm{T}}(k)\theta + v(k) \tag{6}$$

The parameter identification of the model in (6) is relatively simple because when k is an integer multiply of q, $\varphi(k)$ uses only available dual-rate data, *i.e.*, the past slow rate outputs and the past and current fast rate inputs. Thus, many conventional identification methods, e.g., stochastic gradient[5,47] least squares[40,48,49], and multi-innovation[50~54] algorithms, can be applied to the model in (6). Recently, the authors studied the parameter and intersample output estimation problems based on (6)[12,40,43].

Replace k in (6) with kq and let $\hat{\theta}(kq)$ be the estimate of θ at time kq. The least squares algorithm of estimating the parameter vector θ of the dual-rate systems in (6) may be expressed as (DR-LS algorithm for short):

$$\hat{\theta}(kq) = \hat{\theta}(kq-q) + P(kq)\varphi(kq)[y(kq) - \varphi^{\mathrm{T}}(kq)\hat{\theta}(kq-q)] \tag{7}$$
$$\hat{\theta}(kq+i) = \hat{\theta}(kq), \quad i = 0, 1, \cdots, q-1 \tag{8}$$
$$P^{-1}(kq) = P^{-1}(kq-q) + \varphi(kq)\varphi^{\mathrm{T}}(kq), \quad P(0) = P^{\mathrm{T}}(0) > 0 \tag{9}$$
$$\varphi(kq) = [-y(kq-q), \cdots, -y(kq-qn), u(kq), \cdots, u(kq-qn)]^{\mathrm{T}} \tag{10}$$

Initially, we may take $P(0) = p_0 I$ with p_0 a positive number, e.g., $p_0 = 10^6$, and $\hat{\theta}(0)$ a small real vector, e.g., $\hat{\theta}(0) = 10^{-6}\mathbf{1}_N$ with $\mathbf{1}_N$ being an N-dimensional vector whose elements are 1. Notice that the parameter estimate $\hat{\theta}$ is updated every q fast samples, namely, at the slow rate; so is the covariance matrix P; in between the slow samples, we simply hold $\hat{\theta}$ unchanged. Thus, every time $\hat{\theta}$ is updated, we have q new input samples and one new output sample.

The goal here is, under weak conditions, to study the properties of the parameter estimation by the DR-LS algorithm and the intersample output estimation error. By the stochastic process theory and martingale convergence theorem (Lemma D.5.3 in [4]), we easily establish the following theorems[12] under weaker conditions than [55, 56].

Theorem 1. For the dual-rate stochastic system in (6) and DR-LS algorithm in (7)~(10), assume that $\{v(k)\}$ is independent white noise sequence with zero mean and bounded time-varying variance $\sigma_v^2(k)$, *i.e.*,

$(A1)$ $\quad \mathrm{E}[v(k)] = 0, \text{ a.s.}; \quad \mathrm{E}[v^2(k)] = \sigma_v^2(k) \leqslant \bar{\sigma}_v^2 < \infty, \text{ a.s.}$

and that there exist positive constants c_0, c_1, c_2 and k_0 such that, for $k \geqslant k_0$, the following generalized persistent excitation condition (unbounded condition number) holds:

$(C1)$ $\quad c_1 I \leqslant \frac{1}{k}\sum_{i=1}^{k}\varphi(iq)\varphi^{\mathrm{T}}(iq) \leqslant c_2 k^{c_0} I, \text{ a.s.}$

Then, for any $c > 1$, the parameter estimation error $\|\hat{\theta}(kq) - \theta\|^2$ satisfies

1) $\|\hat{\theta}(kq) - \theta\|^2 = O\left(\dfrac{[\ln k]^c}{k}\right) \to 0, \text{ a.s.};$

2) $\|\hat{\theta}(kq) - \theta\|^2 = O\left(\dfrac{\ln k[\ln\ln k]^c}{k}\right) \to 0, \text{ a.s.};$

3) $\|\hat{\theta}(kq)-\theta\|^2=O\left(\frac{\ln k(\ln\ln k)[\ln\ln\ln k]^c}{k}\right)\rightarrow 0$, a.s.;

4) $\|\hat{\theta}(kq)-\theta\|^2=O\left(\frac{\ln k(\ln\ln k)(\ln\ln\ln k)[\ln\ln\ln\ln k]^c}{k}\right)\rightarrow 0$, a.s.

Theorem 1 shows that for the noise sequence $\{v(k)\}$ with a bounded time-varying variance, the parameter estimates by the DR-LS algorithm converges to the true parameters at the rate of at least $\sqrt{(\ln k)^c/k}$. For an arbitrary small positive real ε, we have $[\ln k]^{\beta}=o(k^{\varepsilon})$. Hence

$$\|\hat{\theta}(kq)-\theta\|^2=O\left(\frac{1}{k^{1-\varepsilon}}\right)\rightarrow 0,\ \text{a.s.}$$

The following theorem indicates that the parameter estimation error also converges to zero for unbounded noise variance.

Theorem 2. For the dual-rate stochastic system in (6) and DR-LS algorithm in (7)~(10), assume that the noise sequence $\{v(k)\}$ satisfies:

($A2$) $\quad \mathrm{E}[v(k)]=0$, a.s.; $\mathrm{E}[v^2(k)]=\sigma_v^2(k)\leqslant\delta_{\varepsilon}k^{\varepsilon}$, a.s., $0\leqslant\delta_{\varepsilon}<\infty,\ 0\leqslant\varepsilon<1$

that is, $\{v(k)\}$ is an independent random noise sequence with unbounded variance, and that there exist positive constants c_3, c_4 and k_0 such that, for $k\geqslant k_0$, the following weak persistent excitation condition (bounded condition number) holds:

($C2$) $\quad c_3I\leqslant\frac{1}{k}\sum_{i=1}^{k}\varphi(iq)\varphi^{\mathrm{T}}(iq)\leqslant c_4I$, a.s.

Then $\lim\limits_{k\rightarrow\infty}\|\hat{\theta}(kq)-\theta\|^2=0$, a.s.

Condition (C1) is weaker than Condition (C2), because setting $c_0=0$ in Condition (C1), we obtain Condition (C2).

Since single-rate systems belong to a special class of dual-rate systems with $q=1$, the results of Theorems 1 and 2 still hold for single-rate systems. Here, unlike in [57,58], there is no assumption that the high-order moments of the noise $\{v(k)\}$ exist, *i.e.*, we do not assume that $\mathrm{E}[|v(k)|^{\gamma}|\mathcal{F}_{k-1}]<\infty$, a.s. for some $\gamma>2$.

Based on the input data and past and current output data $\{y(kq),y(kq-q),\cdots\}$, define the prediction of the output at time $kq+q$,

$$\hat{y}(kq+q)=\varphi^{\mathrm{T}}(kq+q)\hat{\theta}(kq)$$

and the intersample outputs,

$$\hat{y}(kq+i)=\begin{cases}y(kq), & i=0\\ \hat{\varphi}^{\mathrm{T}}(kq+i)\hat{\theta}(kq), & i=1,2,\cdots,q-1\end{cases} \tag{11}$$

$$\hat{\varphi}(kq+i)=[-\hat{y}(kq-q+i),-\hat{y}(kq-2q+i),\cdots,-\hat{y}(kq-qn+i),$$
$$u(kq+i),u(kq+i-1),\cdots,u(kq+i-qn)]^{\mathrm{T}}$$

The following results on the output prediction/estimation can be established.

Theorem 3. Assume that (A1) and (C2) hold, then the output prediction (or sometimes called output estimation) at sampling instants has the minimum variance property, *i.e.*,

$$\lim_{k\rightarrow\infty}\frac{1}{k}\sum_{i=1}^{k}[\hat{y}(iq)-y(iq)+v(iq)]^2=0,\ \text{a.s., or}\quad \lim_{k\rightarrow\infty}\frac{1}{k}\sum_{i=1}^{k}\mathrm{E}\{[\hat{y}(iq)-y(iq)]^2\}\leqslant\bar{\sigma}_v^2$$

Further, assume that $\alpha(z)$ is strictly stable, *i.e.*, all zeros of $\alpha(z)$ are strictly inside the unit circle, then the bounded input assumption implies that the output estimation error is uniformly bounded, *i.e.*,

$$\lim_{k\rightarrow\infty}\frac{1}{k}\sum_{i=1}^{k}\mathrm{E}\{[\hat{y}(i)-y(i)]^2\}=O\left(\frac{(\ln k)^{c+1}}{k}+\bar{\sigma}_v^2\right)$$

2.3 Determination of equivalent single-rate models

Next we give a way to determine the single-rate model $P_1(z)$ from the obtained dual-rate model $P_2(z)$. The parameter vector $\hat{\theta}(kq)$ is used to form the estimate of $P_2(z)$,

$$\hat{P}_2(z)=\frac{\hat{\beta}(kq,z)}{\hat{\alpha}(kq,z^q)}=\frac{\hat{\beta}_0(kq)+\hat{\beta}_1(kq)z^{-1}+\hat{\beta}_2(kq)z^{-2}+\cdots+\hat{\beta}_{qn}(kq)z^{-qn}}{1+\hat{\alpha}_1(kq)z^{-q}+\hat{\alpha}_2(kq)z^{-2q}+\cdots+\hat{\alpha}_n(kq)z^{-qn}}$$

where

$$[\hat{\alpha}_1(kq), \hat{\alpha}_2(kq), \cdots, \hat{\alpha}_n(kq), \hat{\beta}_0(kq), \hat{\beta}_1(kq), \cdots, \hat{\beta}_{qn}(kq)]^{\mathrm{T}} = \hat{\theta}(kq)$$

Let the estimate of $P_1(z)$ be

$$\hat{P}_1(z) = \frac{\hat{b}(kq,z)}{\hat{a}(kq,z)} = \frac{\hat{b}_0(kq) + \hat{b}_1(kq)z^{-1} + \hat{b}_2(kq)z^{-2} + \cdots + \hat{b}_n(kq)z^{-n}}{1 + \hat{a}_1(kq)z^{-1} + \hat{a}_2(kq)z^{-2} + \cdots + \hat{a}_n(kq)z^{-n}}$$

According to (4), applying the rational fraction approximation theory or model equivalence principle, we have[12,59]

$$\frac{\hat{b}(kq,z)}{\hat{a}(kq,z)} = \frac{\hat{\beta}(kq,z)}{\hat{\alpha}(kq,z^q)}, \quad \text{or} \quad \hat{\alpha}(kq,z^q)\hat{b}(kq,z) = \hat{\beta}(kq,z)\hat{a}(kq,z)$$

Expanding this into a polynomial equation and comparing the coefficients of z^{-i} term on both sides set up the $(qn+n+1)$ equations with unknowns $\hat{a}_i(kq)$ and $\hat{b}_i(kq)$; solving these equations directly gives the estimate $\hat{P}_1(z)$ of $P_1(z)$.

2.4 Other models

In this section, we discuss identification problems of two class of dual-rate systems, *i.e.*, the corresponding single-rate system is a CAR model or CARMA (Controlled Auto-Regression and Moving Average) model.

1) The single-rate CAR case. The equation is

$$a(z)y(k) = b(z)u(k) + v(k) \tag{12}$$

Notice that we continue using definitions of variables as before. Using the polynomial transformation technique and multiplying both sides by $\phi_q(z)$ give

$$\alpha(z^q)y(k) = \beta(z)u(k) + \phi_q(z)v(k), \quad \text{or} \quad \alpha(z^q)y(kq) = \beta(z)u(kq) + \phi_q(z)v(kq)$$

This is a dual-rate system described by a CARMA model. Let

$$\varphi(kq) = [-y(kq-q), -y(kq-2q), \cdots, -y(kq-qn), u(kq), u(kq-1), \cdots, u(kq-qn),$$
$$\hat{v}(kq-1), \hat{v}(kq-2), \cdots, \hat{v}(kq-n_f)]^{\mathrm{T}} \tag{13}$$

$$\hat{v}(kq-i) = y(kq-i) - \varphi^{\mathrm{T}}(kq-i)\hat{\theta}(kq), \ i = 1, 2, \cdots, n_f \tag{14}$$

$$\hat{\theta}(kq) = [\hat{\alpha}_1(kq), \hat{\alpha}_2(kq), \cdots, \hat{\alpha}_n(kq), \hat{\beta}_0(kq), \hat{\beta}_1(kq), \cdots, \hat{\beta}_{qn}(kq),$$
$$\hat{f}_1(kq), \hat{f}_2(kq), \cdots, \hat{f}_{n_f}(kq)]^{\mathrm{T}} \tag{15}$$

$$\theta = [\alpha_1, \alpha_2, \cdots, \alpha_n, \beta_0, \beta_1, \cdots, \beta_{qn}, f_1, f_2, \cdots, f_{n_f}]^{\mathrm{T}} \tag{16}$$

The recursive extended least squares (RELS)[2,60,61] may be used to identify this dual-rate CARMA model by increasing the dimensions of the parameter vector and information vector, adding the parameters f_i of the noise model $\phi_q(z)$ to the parameter vector and adding the noise terms $v(kq-i), \ i = 1, 2, \cdots, n_f$ to the information vector (since $v(kq-i)$ is unknown, it is replaced by the estimation residual $\hat{v}(kq-i)$). It seems that the RELS algorithm in (7)~(9) and (13)~(15) may compute the extended parameter estimation vector $\hat{\theta}(kq)$ in (16), but actually $\varphi(kq-i)$ on the right-hand side of (14) contains the unknown intersample outputs if i is not an integer multiple of q. Thus, it is impossible to compute the residual by (14).

As in the preceding section, a feasible way is that after computing the parameter estimate $\hat{\theta}(kq)$ at each step, determine the estimate $\hat{P}_1(z) = \hat{b}(kq,z)/\hat{a}(kq,z)$ of $P_1(z) = b(z)/a(z)$, and then with reference to (11), the single-rate model in (12) is used to estimate the intersample outputs:

$$\hat{y}(kq+i) = \begin{cases} y(kq), & i = 0 \\ \hat{\varphi}_a(kq+i)\hat{\theta}_a(kq), & i = 1, 2, \cdots, q-1 \end{cases} \tag{17}$$

$$\hat{\varphi}_a(kq+i) = [-\hat{y}(kq+i-1), -\hat{y}(kq+i-2), \cdots, -\hat{y}(kq+i-n),$$
$$u(kq+i), u(kq+i-1), \cdots, u(kq+i-n)]^{\mathrm{T}}$$

$$\hat{\theta}_a(kq) = [\hat{a}_1(kq), \hat{a}_2(kq), \cdots, \hat{a}_n(kq), \hat{b}_0(kq), \hat{b}_1(kq), \cdots, \hat{b}_n(kq)]^{\mathrm{T}} \in R^{2n+1}$$

After obtaining the intersample output $\hat{y}(k)$, the residual can be computed by (14).

However, how to extend the RELS convergence analysis from the single-rate case[4,60,62] to this dual-rate case in (7)∼(9) and (13)∼(17) is still open and worth further exploration.

2) The single-rate CARMA case. The model is

$$a(z)y(k) = b(z)u(k) + d(z)v(k) \tag{18}$$

with

$$d(z) = 1 + d_1 z^{-1} + d_2 z^{-2} + \cdots + d_n z^{-n}$$

Adopting a similar approach as above, we multiply both sides of (18) by $\phi_q(z)$ to get a new model. Comparing with the CAR case above, the difference is that it requires estimating a noise model with more parameters:

$$\phi_q(z)d(z) =: 1 + e_1 z^{-1} + e_2 z^{-2} + \cdots + e_{n_e} z^{-n_e}$$

Similarly, after obtaining the parameter estimate at each step, determine the estimate $\hat{P}_1(z) = \hat{b}(kq,z)/\hat{a}(kq,z)$ of $P_1(z)$, and with reference to (11), the single-rate model in (18) is used to estimate the intersample outputs:

$$\hat{y}(kq+i) = \begin{cases} y(kq), & i = 0 \\ \hat{\varphi}_a(kq+i)\hat{\theta}_a(kq), & i = 1, 2, \cdots, q-1 \end{cases} \tag{19}$$

$$\begin{aligned} \hat{\varphi}_a(kq+i) = [&-\hat{y}(kq+i-1), -\hat{y}(kq+i-2), \cdots, -\hat{y}(kq+i-n), \\ &u(kq+i), u(kq+i-1), \cdots, u(kq+i-n), \\ &\hat{v}(kq+i-1), \hat{v}(kq+i-2), \cdots, \hat{v}(kq+i-n)]^{\mathrm{T}} \in R^{3n+1} \end{aligned} \tag{20}$$

$$\begin{aligned} \hat{\theta}_a(kq) = [&\hat{a}_1(kq), \hat{a}_2(kq), \cdots, \hat{a}_n(kq), \hat{b}_0(kq), \hat{b}_1(kq), \cdots, \hat{b}_n(kq), \\ &\hat{d}_1(kq), \hat{d}_2(kq), \cdots, \hat{d}_n(kq)]^{\mathrm{T}} \in R^{3n+1} \end{aligned}$$

From here, we can see that the intersample output estimates rely on the noise estimates (residuals), and the noise estimates also rely on the intersample output estimates. This is a hierarchical computation process with interactive computation. The convergence of this approach requires further research.

Finally, how to use the polynomial transformation technique to study identification problems for models in Table 1 based on dual-rate input/output data are also future research topics.

Table 1 Some stochastic system models[5]

Items	Name	Description
1	OE	$y(k) = \frac{b(z)}{a(z)} u(k) + v(k)$
2	CARAR/ARARX	$a(z)y(k) = b(z)u(k) + \frac{1}{c(z)} v(k)$
3	CARARMA/ARARMAX	$a(z)y(k) = b(z)u(k) + \frac{d(z)}{c(z)} v(k)$

2.5 Direct identification of single-rate models from dual-rate data

Although the dual-rate data $\{u(k), y(kq)\}$ may be used to identify a dual-rate model by the polynomial transformation technique, this model has more parameters than the original system, especially for large q; hence the corresponding algorithm requires a large amount of computation. The hierarchical identification principle[63∼67] can be used to reduce the computation, but the best way is to identify single-rate models directly from the dual-rate data. Fortunately, the auxiliary model identification principle[68∼70] is suitable for solving the identification problem with unmeasurable variables, and is here used to study identification of single-rate models for dual-rate sampled-data systems[41,42].

The basic idea of auxiliary-model identification is to replace the unmeasurable variables of the system with the outputs of an auxiliary model and to ensure that the model outputs approach the unmeasurable variables so that the consistent parameter estimation and unmeasurable variable estimation are obtained[41,42,68∼70].

Next, we discuss the identification algorithms of single-rate output-error (OE) and CAR models by using the dual-rate data.

2.5.1 The output error models

The output-error model in Fig. 3 can be expressed as

$$x(k) = P_1(z)u(k) = \frac{b(z)}{a(z)}u(k), \quad y(k) = x(k) + v(k) \tag{21}$$

where $x(k)$ is the true output (noise-free output) but unmeasurable. Our goal is, by means of an auxiliary model, $P_{\rm a}(z)$ (see Fig. 3), to study the estimation problem of $P_1(z)$ and $x(k)$ by using the dual-rate data $\{u(k), y(kq)\}$. Its basic idea is to use the auxiliary model (AM) $P_{\rm a}(z)$ to predict $x(k)$, further use $u(k)$ and AM output $x_{\rm a}(k)$ rather than $x(k)$ for identifying $P_1(z)$. The details are as follows.

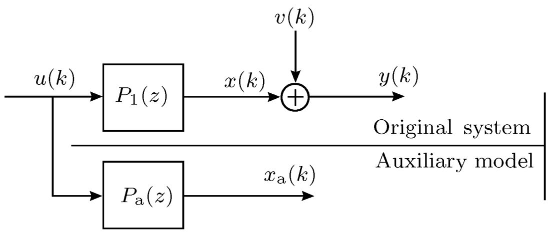

Fig. 3 The single-rate system with an auxiliary model

Define the parameter vector θ_s of $P_1(z)$ and information vector $\varphi_0(k)$ as

$$\theta_s = [a_1, a_2, \cdots, a_n, b_0, b_1, \cdots, b_n]^{\rm T} \in R^{2n+1}$$
$$\varphi_0(k) = [-x(k-1), -x(k-2), \cdots, -x(k-n), u(k), u(k-1), \cdots, u(k-n)]^{\rm T}$$

From (21), we have

$$x(k) = \varphi_0^{\rm T}(k)\theta_s, \quad y(k) = \varphi_0^{\rm T}(k)\theta_s + v(k) \tag{22}$$

A difficulty arises since $x(k)$ is unknown, the standard least squares cannot be applied to (22). In order to recursively estimate θ_s, we replace $x(k-i)$ in $\varphi_0(k)$ with the output $x_{\rm a}(k-i)$ of the auxiliary model:

$$x_{\rm a}(k) = P_{\rm a}(z)u(k) = \frac{b_{\rm a}(z)}{a_{\rm a}(z)}u(k), \quad \text{or} \quad x_{\rm a}(k) = \varphi_{\rm a}^{\rm T}(k)\theta_{\rm a}(k) \tag{23}$$

Here, $\varphi_{\rm a}(k)$ and $\theta_{\rm a}(k)$ are the information vector and parameter vector of the auxiliary model at time k; $x_{\rm a}(k)$ can be referred to as the estimate of $x(k)$.

Therefore, the key is how to establish the auxiliary model in (23) so that the dual-rate data $\{u(k), y(kq)\}$ can be used to obtain the estimate $x_{\rm a}(k)$ of $x(k)$ and to ensure that $x_{\rm a}(k)$ converges to $x(k)$: $x_{\rm a}(k) \to x(k)$; then identification of $P_1(z)$ can be solved by using $x_{\rm a}(k)$ rather than $x(k)$.

There are many ways to choose auxiliary models, e.g., using the estimate $\hat{P}_1(z)$ of the single-rate model[42] $P_1(z)$ and using the finite impulse response (FIR) model of the single-rate system[41].

1) Using the estimate $\hat{P}_1$ as the auxiliary model[42].

Replacing k in (22) with kq gives

$$y(kq) = x(kq) + v(kq) = \varphi_0^{\rm T}(kq)\theta_s + v(kq) \tag{24}$$

Let $\hat{\theta}_s(kq)$ be the estimate of θ_s. Using $x_{\rm a}(kq-i)$ as $x(kq-i)$ and $\varphi(kq)$ as $\varphi_0(kq)$. we easily get the recursive least squares algorithm of estimating θ_s in (24) based on the auxiliary model:

$$\hat{\theta}_s(kq) = \hat{\theta}_s(kq-q) + P(kq)\varphi(kq)[y(kq) - \varphi^{\rm T}(kq)\hat{\theta}_s(kq-q)] \tag{25}$$
$$\hat{\theta}_s(j) = \hat{\theta}_s(kq), \ j = kq+1, kq+2, \cdots, kq+q-1 \tag{26}$$
$$P^{-1}(kq) = P^{-1}(kq-q) + \varphi(kq)\varphi^{\rm T}(kq) \tag{27}$$
$$\varphi(kq) = [-x_{\rm a}(kq-1), \cdots, -x_{\rm a}(kq-n), u(kq), u(kq-1), \cdots, u(kq-n)]^{\rm T} \tag{28}$$
$$x_{\rm a}(kq+j) = \varphi^{\rm T}(kq+j)\hat{\theta}_s(kq), \ j = 0, 1, \cdots, q-1 \tag{29}$$
$$\hat{\theta}_s(kq) = [\hat{a}_1(kq), \hat{a}_2(kq), \cdots, \hat{a}_n(kq), \hat{b}_0(kq), \hat{b}_1(kq), \cdots, \hat{b}_n(kq)]^{\rm T} \tag{30}$$

Equations (25)∼(29) define our auxiliary model least squares (AMLS) identification algorithm for dual-rate systems, DR-AMLS algorithm for short. Note that the algorithm is combined in the sense that the parameters and true outputs are estimated simultaneously, and can be implemented on-line. Here, the parameter vector and information vector of the auxiliary model are $\theta_{\rm a}(kq) = \hat{\theta}_s(kq)$ and $\varphi_{\rm a}(kq) = \varphi(kq)$.

The properties of the DR-AMLS algorithm are studied in detail in [42]: under certain conditions, we have $\hat{\theta}_s(k) \to \theta_s$ and $x_{\rm a}(k) \to x(k)$.

2) Using FIR as the auxiliary model[41].

Let $\{g_i: \ i = 0, 1, 2, \cdots\}$ be the impulse response parameters of $P_1(z)$. Assuming stability of P_1, we can rewrite (21) as

$$y(k) = (g_0 + g_1 z^{-1} + g_2 z^{-2} + \cdots)u(k) + v(k)$$

Then as $i \to \infty$, $g_i \to 0$. Hence

$$y(k) = G(p, z)u(k) + v(k), \quad G(p, z) = g_0 + g_1 z^{-1} + g_2 z^{-2} + \cdots + g_{p-1} z^{-p+1} \tag{31}$$

As long as p is sufficiently large, the model in (31) is as close to (21) as one desires.

Here, we use the estimate $\hat{G}(p, z)$ of the FIR model $G(p, z)$ as the auxiliary model:

$$x_{\rm a}(k) = P_{\rm a}(z)u(k) = \hat{G}(p, z)u(k)$$

and identify the parameters a_i and b_i of $P_1(z)$ from $u(k)$ and $x_{\rm a}(k)$. The details are as follows.

Define the parameter vector θ and information vector $\varphi(k)$ as

$$\theta = [g_0, g_1, g_2, \cdots, g_{p-1}]^{\rm T} \in R^p$$
$$\varphi(k) = [u(k), u(k-1), u(k-2), \cdots, u(k-p+1)]^{\rm T} \in R^p$$

Hence, (31) may be rewritten as

$$y(k) = \varphi^{\rm T}(k)\theta + v(k), \quad \text{or} \quad y(kq) = \varphi^{\rm T}(kq)\theta + v(kq)$$

Since $v(kq)$ is a white noise and $\varphi(kq)$ and $y(kq)$ both are available, the standard least squares is used to estimate θ:

$$\hat{\theta}(kq) = \hat{\theta}(kq-q) + L(kq)[y(kq) - \varphi^{\rm T}(kq)\hat{\theta}(kq-q)] \tag{32}$$
$$\hat{\theta}(kq+j) = \hat{\theta}(kq), \quad j = 0, 1, \cdots, q-1 \tag{33}$$
$$L(kq) = P(kq-q)\varphi(kq)[1 + \varphi^{\rm T}(kq)P(kq-q)\varphi(kq)]^{-1} \tag{34}$$
$$P(kq) = [I - L(kq)\varphi^{\rm T}(kq)]P(kq-q) \tag{35}$$
$$\varphi(kq) = [u(kq), u(kq-1), u(kq-2), \cdots, u(kq-p+1)]^{\rm T} \in R^p \tag{36}$$
$$\hat{\theta}(kq) = [\hat{g}_0(kq), \hat{g}_1(kq), \hat{g}_2(kq), \cdots, \hat{g}_{p-1}(kq)]^{\rm T} \tag{37}$$

The output $x_{\rm a}(k)$ is given by

$$x_{\rm a}(kq+j) = P_{\rm a}(z)u(kq+j) = \sum_{i=0}^{p-1} \hat{g}_i(kq)u(kq+j-i) = \varphi^{\rm T}(kq+j)\hat{\theta}(kq), \quad j = 0, 1, \cdots, q-1 \tag{38}$$

In the same way, using $x_{\rm a}(k)$ as the estimate of $x(k)$, we can easily compute the parameter vector θ_s in (22) of the single-rate model P_1 by

$$\hat{\theta}_s(k) = \hat{\theta}_s(k-1) + L_s(k)[x_{\rm a}(k) - \varphi_s^{\rm T}(k)\hat{\theta}_s(k-1)] \tag{39}$$
$$L_s(k) = P_s(k-1)\varphi_s(k)[1 + \varphi_s^{\rm T}(k)P_s(k-1)\varphi_s(k)]^{-1} \tag{40}$$
$$P_s(k) = [I - L_s(k)\varphi_s^{\rm T}(k)]P_s(k-1) \tag{41}$$
$$\varphi_s(k) = [-x_{\rm a}(k-1), \cdots, -x_{\rm a}(k-n), u(k), u(k-1), \cdots, u(k-n)]^{\rm T} \tag{42}$$
$$\hat{\theta}_s(k) = [\hat{a}_1(k), \hat{a}_2(k), \cdots, \hat{a}_n(k), \hat{b}_0(k), \hat{b}_1(k), \cdots, \hat{b}_b(k)]^{\rm T} \tag{43}$$

Equations (38)~(39) and (39)~(43) form the FIR model based RLS algorithm of identifying single-rate models by using the dual-rate data, FIR-RLS algorithm for short.

Identification accuracy of the FIR-RLS algorithm depends on the order p in $G(p,z)$. A correlation-analysis based method to identify the FIR model with the order p increasing is provided in [41].

After identifying $G(p,z)$, letting $G(p,z)=b(z)/a(z)$, we may determine the parameters of the single-rate model $P_1(z)$ by the model equivalence principle[12,59].

2.5.2 The equation-error models

The equation-error model is expressed as

$$a(z)y(k)=b(z)u(k)+v(k)$$

Here, we still use the auxiliary model identification methods above to estimate the parameters of this system. The convergence problems are still open. Of course, we may also study identification approaches and their properties of CARAR and CARARMA models in Table 1 based on the dual-rate data.

3 The lifting technique and multirate systems

The standard technique in handling multirate systems[9,11,31,45] is called blocking in signal processing and lifting in control. For a periodically time-varying multirate system, its lifted version is necessarily multivariable, even if the corresponding continuous-time system is a single-input and single-output time-invariant process. The lifting technique transforms a periodically time-varying system into a time-invariant one. In this section, we simply introduce the lifting technique and discuss mathematical models and identifiability of dual-rate systems.

3.1 The lifting technique

Let $u(k)$ be a discrete-time signal defined on the time set $\{0,1,2,\cdots\}$:

$$u=\{u(0),u(1),u(2),\cdots,u(k),\cdots\}$$

For an integer $q>1$, define the q-fold lifting operator, L_q, by the map from $u(k)$ to $\underline{u}(k)$ defined as

$$\underline{u}=\left\{\begin{bmatrix}u(0)\\u(1)\\\vdots\\u(q-1)\end{bmatrix},\begin{bmatrix}u(q)\\u(q+1)\\\vdots\\u(2q-1)\end{bmatrix},\cdots,\begin{bmatrix}u(kq)\\u(kq+1)\\\vdots\\u(kq+q-1)\end{bmatrix},\cdots\right\}$$

We write $\underline{u}=L_q u$. Note that the lifting operation results in no loss of information; the dimension of the lifted signal $\underline{u}(k)$ is q times that of $u(k)$. If the underlying period for $u(k)$ is h, then the underlying period for the lifted signal $\underline{u}(k)$ is qh. It is easy to see that the inverse operator, L_q^{-1}, mapping $\underline{u}(k)$ back to $u(k)$, is well-defined.

3.2 Lifting a simple dual-rate system

For the dual-rate system in Fig. 2, one can get q inputs and one output in one output (large) sampling interval $T_2=qh$. For convenience, we put the q inputs together to form a lifted input vector $\underline{u}$, and the resulting lifted system has q inputs. Assuming that $\{u(k),y(kq)\}$ is available, and adopting the lifting technique and the zero-order hold property, $u_c(t)=u_c(kh)=:u(k), t\in[kh,(k+1)h)$, it is not difficult to get the lifted state-space model[1,12]:

$$P_q:\ \begin{cases}x((k+1)q)=A^q x(kq)+[A^{q-1}B,\ A^{q-2}B,\ \cdots,\ B]\underline{u}(kq)\\ y(kq)=Cx(kq)+[D,\ 0,\ 0,\ \cdots,\ 0]\underline{u}(kq)\end{cases}\tag{44}$$

where

$$\underline{u}(kq)=\begin{bmatrix}u(kq)\\u(kq+1)\\\vdots\\u(kq+q-1)\end{bmatrix}$$

is the lifted input vector. The resulting system is multi-input, and P_q maps between available input and output data.

For a given single-rate model P_1 in (1)∼(2), one can find its lifted model P_q according to (44). On the other hand, given P_q, whether does there exist a unique P_1 and how is P_1 computed? These are discussed below.

The lifted system in (44) has q-input and P_q has q subsystems:

$$P_q = [P_{q1},\ P_{q2},\ \cdots,\ P_{qq}]$$

From (44), subsystems $P_{q1}, P_{q2}, \cdots, P_{qq}$ have the following transfer functions,

$$P_{q1}(z) = C(zI - A^q)^{-1}A^{q-1}B + D, \quad P_{qi}(z) = C(zI - A^q)^{-1}A^{q-i}B, \quad i = 2, 3, \cdots, q \tag{45}$$

Form a new transfer function

$$P_{1,q}(z) = P_{q1}(z^q) + zP_{q2}(z^q) + z^2P_{q3}(z^q) + \cdots + z^{q-1}P_{qq}(z^q) \tag{46}$$

We can show $P_{1,q}(z) = P_1(z)$. In fact, from (45) and (46), we have

$$P_{1,q}(z) = C(z^qI - A^q)^{-1}(A^{q-1} + zA^{q-2} + \cdots + z^{q-2}A + z^{q-1}I)B + D \tag{47}$$

Since

$$z^qI - A^q = (A^{q-1} + zA^{q-2} + \cdots + z^{q-2}A + z^{q-1}I)(zI - A)$$

Pre-multiplying $(z^qI - A^q)^{-1}$ and post-multiplying $(zI - A)^{-1}$ give

$$(zI - A)^{-1} = (z^qI - A^q)^{-1}(A^{q-1} + zA^{q-2} + \cdots + z^{q-2}A + z^qI)$$

Substituting the last equation into (47) directly leads to $P_{1,q}(z) = C(zI - A)^{-1}B + D = P_1(z)$, which is obtained by canceling common factors of the numerator and denominator of $P_{1,q}(z)$[9,12]. This procedure is theoretically sound, but in practice, if there exist model errors in $P_q(z)$, the required common factor cancelation could be hard to achieve.

3.3 Identifiability of transfer functions for dual-rate systems

Identifiability depends on controllability and observability. Therefore, it is important if the lifted system in (44) is controllable and observable. Under what conditions this is true? To answer this question, we assume that the state-space realization in (1)∼(2) of $P_1(z)$ is minimal, *i.e.*, (A, B) is controllable and (C, A) is observable. Note that this is guaranteed as long as the continuous-time process (A_c, B_c) is controllable and (C, A_c) is observable and if the sampling period h is non-pathological[1]. Controllability of P_q can be achieved under a mild condition[1,11,71]. About the observability of P_q, we have the following lemma.

Lemma 1. If for every eigenvalue λ of A, none of the $q-1$ points $\lambda e^{\mathrm{j}2\pi i/q}$ ($\mathrm{j} = \sqrt{-1}$, $i = 1, 2, \cdots, q-1$) is an eigenvalue of A, observability of (C, A) implies that of (C, A^q). (In turn observability of P_1 implies observability of P_q.)

Lemma 1 (and Lemma 2 in the sequel) can be proved according to the way provided in [11,39,71]; and the proof is omitted here.

For identifiability of the transfer function model $P_2(z)$ in (4), assume that $P_1(z)$ is properly fractional, *i.e.*, $a(z)$ and $b(z)$ have no non-trivial common factor. Although the order qn of $P_2(z)$ becomes greater than the order n of $P_1(z)$ due to the polynomial transformation, $P_2(z)$ is still identifiable because the number of outputs in the information vector does not increase, and the number of inputs increases. This does not affect identifiability since we assume that the input signal is persistently excited.

4 General dual-rate systems

4.1 State-space models of dual-rate systems

For the dual-rate system in Fig. 1, without loss of generality, we assume that $T_1 = ph$ and $T_2 = qh$, p and q are two coprime integers, for otherwise, we can absorb any common factor of p and q into h; h is a positive real number called the *base period*. For example, if $T_1 = 3.09$ s, $T_2 = 4.12$ s, then $p = 3$, $q = 4$, and $h = 1.03$ s.

References [11,45] and [72] exploit different approaches to derive the lifted state-space models for such a general dual-rate system in Fig. 1.

For the dual-rate system in Fig. 1 with $T_1 = ph$ and $T_2 = qh$, let $T := pqh$ be the *frame period* (the smallest common multiply of T_1 and T_2) and $\sigma := T/h = pq$. The coprimeness of p and q implies that for every i, $0 \leqslant i \leqslant p-1$, there exist integers $c_i \geqslant 0$ and $0 \leqslant d_i < p$ such that $iq = c_i p + d_i$. Assume that $[A, B, C, D]$ are the system matrices of P_1 obtained by discretizing P_c with period h – see (1)~(2). Due to adopting the zero-order hold, $u_c(t) = u_c(kT_1) = u_c(kph) =: u(kp)$, $t \in [kph, (k+1)ph)$, $y_c(t)|_{t=kT_2} = y_c(kT_2) =: y(kq)$. Define

$$
\begin{aligned}
&F(i,p) := A^i + A^{i+1} + \cdots + A^{i+p-1}, \quad p \geqslant 0, \quad i \geqslant 0 \\
&A^j F(i,p) = F(i+j, p), \quad j \geqslant 0 \\
&A_p := \exp(A_c T_1) = A^p; \quad A_q := \exp(A_c T_2) = A^q \\
&B_p := \int_0^{T_1 = ph} e^{A_c t} \mathrm{d}t \; B_c = (A^{p-1} + A^{p-2} + \cdots + A + I)B = F(0,p)B \\
&\underline{u}(k\sigma) := \underline{u}_c(kT) = L_q u(k\sigma) = \begin{bmatrix} u(k\sigma) \\ u(k\sigma + p) \\ \vdots \\ u(k\sigma + (q-1)p) \end{bmatrix} \quad \text{(lifted input)} \\
&\underline{y}(k\sigma) := \underline{y}_c(kT) = L_p y(k\sigma) = \begin{bmatrix} y(k\sigma) \\ y(k\sigma + q) \\ \vdots \\ y(k\sigma + (p-1)q) \end{bmatrix} \quad \text{(lifted output)} \\
&A_T := \exp(A_c T) = A^{pq} = A_p^q = A_q^p \in R^{n \times n} \\
&B_T := [A_p^{q-1} B_p, \; A_p^{q-2} B_p, \; \cdots, \; A_p B_p, \; B_p] \\
&\quad\; = [A^{(q-1)p} F(0,p)B, \; A^{(q-2)p} F(0,p)B, \; \cdots, \; A^p F(0,p)B, \; F(0,p)B] \\
&\quad\; = [F(pq-p, p)B, \; F(pq-2p, p)B, \; \cdots, \; F(p,p)B, \; F(0,p)B] \\
&C_T := \begin{bmatrix} C \\ CA_q \\ CA_q^2 \\ \vdots \\ CA_q^{(p-1)} \end{bmatrix}; \; D_T := \begin{bmatrix} D & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ D_{10} & D_{11} & \cdots & D_{1c_1} & 0 & & & \vdots \\ D_{20} & D_{21} & \cdots & \cdots & D_{2c_2} & 0 & & \vdots \\ \vdots & & & & & \ddots & \ddots & \vdots \\ D_{p-1,0} & D_{p-1,1} & \cdots & \cdots & \cdots & \cdots & D_{p-1,c_{p-1}} & 0 \end{bmatrix} \\
&D_{ij} := CF(iq - jp - p, p)B, \quad D_{ic_i} := CF(0, d_i)B + D
\end{aligned}
$$

Theorem 4. The lifted state-space model mapping $\underline{u}$ to $\underline{y}$ is given by

$$
P_T : \begin{cases} x((k+1)\sigma) = A_T x(k\sigma) + B_T \underline{u}(k\sigma) \\ \underline{y}(k\sigma) = C_T x(k\sigma) + D_T \underline{u}(k\sigma) \end{cases}
$$

where $x(k\sigma) := x_c(kT)$.

The controllability and observability of such a model is discussed below.

Lemma 2. 1) If for every eigenvalue λ of A_p, none of the $q-1$ points $\lambda e^{\mathrm{j}2\pi i/q}$ ($\mathrm{j} = \sqrt{-1}$, $i = 1, 2, \cdots, q-1$) is an eigenvalue of A_p, controllability of (A_p, B_p) implies that of each $(A_p^q, A_p^i B_p)$ ($i = 0, 1, \cdots, q-1$), which in turn implies controllability of the model P_T. 2) If for every eigenvalue λ of A, none of the $pq-1$ points $\lambda e^{\mathrm{j}2\pi i/(pq)}$ ($i = 1, 2, \cdots, pq-1$)) is an eigenvalue of A, controllability of (A, B) implies that of each $(A_p^q, A^i B)$ ($i = 0, 1, \cdots, pq-1$), which in turn implies controllability of the model P_T; observability of (C, A) implies that of each (CA^i, A_p^q), which in turn implies observability of the model P_T.

4.2 Identification of the lifted systems

Here, we discuss how to identify the parameters and/or states of the lifted system P_T by using the lifted input-output data $\{\underline{u}(k\sigma), \underline{y}(k\sigma)\}$.

Note that D_T is a (block) lower triangular matrix; this structure gives rise to the so-call causality constraint. Since P_T is derived from the causal continuous-time process, the causality constraint is automatically satisfied. However, we need to deal with the causality constraint in identification of lifted systems.

For the case with unmeasurable states, we used the hierarchical identification principle[63∼67] to jointly estimate states and parameters of the lifted systems based on the Kalman filtering[45,46] and on the observability canonical form[72].

Also, taking the z-transform of P_T, we obtain the transfer matrix:

$$P_T(z)=C_T(zI-A_T)^{-1}B_T+D_T=:\frac{Q(z)}{p(z)}\in R^{(pm)\times(qr)}$$

where r and m are the numbers of input and output of the original system, respectively, and $p(z)$ is the characteristic polynomial. Assume that $\{u(kp),\ y(kq):\ k=0,1,2,\cdots\}$ are available; so is $\{\underline{u}(k\sigma),\underline{y}(k\sigma)\}$. Hierarchical identification methods in [64,65] can be applied to estimate the parameters of $P_T(z)$.

4.3 Computation of system matrices

For dual-rate systems, an interesting question is how to find single-rate system matrices $[A_p,B_p,C,D]$, $[A_q,B_q,C,D]$ and $[A,B,C,D]$ with sampling periods $T_1=ph$, $T_2=qh$ and h, respectively, assuming that $[A_T,\ B_T,\ C_T,\ D_T]$ are available. Two ways are given in [11,39] to compute $[A_p,B_p,C,D]$: the controllability and observability approach, and the characteristic roots approach based on the assumption that A_T is diagonalizable.

The algorithms to compute system matrices with different periods are given in [45] which can reduce numerical computation errors. An approach was discussed in [73] to compute the continuous-time models from the obtained discrete-time ones.

4.4 Time-varying state-space representations

For dual-rate systems with $T_1=ph$ and $T_2=qh$, if the frame period is very large, the lifted systems may have a singularity problem: The matrix A_T may be close to zero (assuming system stability), and hence lifted models may lose dynamical properties. For instance, for the scalar system,

$$\dot{x}(t)=ax(t)+u(t),\quad a=-0.125$$

Assume that $T_1=7$s, $T_2=17$s, $h=1$s, we can compute $A=\exp(ah)=0.882497$, $A^p=\exp(aT_1)=0.416862$, $A^q=\exp(aT_2)=0.119433$, but $A_T=\exp(aT)=0.000000346633$. Here, $A_T=A^{pq}$ is very small. In order to avoid this singularity problem, we present a time-varying state-space representation as follows:

$$P_V:\ \left[\begin{array}{c}x_c(kT+(i+1)T_2)\\ \hline y_c(kT+iT_2)\end{array}\right]=\left[\begin{array}{c|ccccccc}A^q & B_1^i & B_2^i & \cdots & B_{\delta_i}^i & 0 & \cdots & 0\\ \hline C & D & 0 & \cdots & \cdots & \cdots & \cdots & 0\end{array}\right]\left[\begin{array}{c}x_c(kT+iT_2)\\ \hline \underline{u}_c(kT)\end{array}\right]$$

Here $i=1,2,\cdots,p-1$, $\delta_i=c_{i+1}-c_i+1$, $\delta=\max[\delta_i:\ i=1,2,\cdots,p-1]$, and

$$B_j^i=F(q-d_i-jp,p)B,\quad j=1,2,\cdots,\delta_i-1$$

$$B_{\delta_i}^i=F(0,d_{i+1})B$$

$$B_j^i=0,\quad j\geqslant\delta_i+1$$

$$\underline{u}_c(kT)=L_\delta u_c(kT)=\begin{bmatrix}u_c(kT)\\ u_c(kT+T_1)\\ \vdots\\ u_c(kT+(\delta-1)T_1)\end{bmatrix}=\begin{bmatrix}u(kpq)\\ u(kpq+p)\\ \vdots\\ u(kpq+(\delta-1)p)\end{bmatrix}$$

This is a periodically time-varying system described by multi-models (p models): For each fixed i, P_V is linear time-invariant. Many methods can be applied to identify P_V, e.g., the combined state and parameter identification approach[5,74].

5 Multirate multivariable systems

A multirate multivariable system is shown in Fig. 4, where P_c is a continuous-time process, $u_c(t) = [u_{c1}(t), u_{c2}(t), \cdots, u_{cr}(t)]^{\mathrm{T}} \in R^r$ the input vector, $y_c(t) = [y_{c1}(t), y_{c2}(t), \cdots, y_{cm}(t)]^{\mathrm{T}} \in R^m$ the output vector. Suppose the updating period in the jth input channel is $p_j h$, and the sampling period in the ith output channel is $q_i h$; $H_{p_j h}$ is a zero-order hold with period $p_j h$: $u_{cj}(t) = u_{cj}(kp_j h) =: u_j(kp_j),\ t \in [kp_j h, (k+1)p_j h)$; $S_{q_i h}$ is a sampler with period $q_i h$: $y_{cj}(kq_i h) =: y_j(kq_i)$. This way we can define the multi-rate hold and sampling blocks as follows:

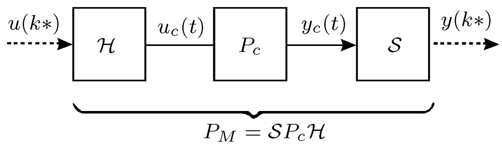

Fig. 4 The multirate multivariable system

$$\mathcal{H} = \begin{bmatrix} H_{p_1 h} & & \\ & \ddots & \\ & & H_{p_r h} \end{bmatrix}, \quad \mathcal{S} = \begin{bmatrix} S_{q_1 h} & & \\ & \ddots & \\ & & S_{q_m h} \end{bmatrix}$$

$$u(k*) = \begin{bmatrix} u_1(kp_1) \\ u_2(kp_2) \\ \vdots \\ u_r(kp_r) \end{bmatrix} \in R^r, \quad y(k*) = \begin{bmatrix} y_1(kq_1) \\ y_2(kq_2) \\ \vdots \\ y_m(kq_m) \end{bmatrix} \in R^m$$

Assume that P_c takes the following form:

$$\begin{cases} \dot{x}_c(t) = A_c x_c(t) + B_c u_c(t) = A_c x(t) + \sum\limits_{j=1}^{r} B_{cj}\, u_{cj}(t) \\ y_c(t) = C x_c(t) + D u_c(t) \end{cases}$$

where $x_c(t) \in R^n$ denotes the state vector, $A_c \in R^{n\times n}$, $B_c := [B_{c1}, B_{c2}, \cdots, B_{cr}] \in R^{n\times r}$,

$$C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_m \end{bmatrix} \in R^{m\times n}, \quad C_i \in R^{1\times n}, \quad D = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1r} \\ d_{21} & d_{22} & \cdots & d_{2r} \\ \vdots & \vdots & & \vdots \\ d_{m1} & d_{m2} & \cdots & d_{mr} \end{bmatrix} \in R^{m\times r}$$

Theorem 5. For the multirate multivariable system in Fig. 4, let σ be the least common multiple of (p_j, q_i), denoted by

$$\sigma := \mathrm{LCM}[p_1, p_2, \cdots, p_r, q_1, q_2, \cdots, q_m]$$

$T := \sigma h$ (the frame period), $\mu_i := \sigma/q_i,\ i = 1, 2, \cdots, m$; $\nu_j := \sigma/p_j,\ j = 1, 2, \cdots, r$. For each i $(i = 1, 2\cdots, \mu_I - 1)$, there exist integers $c_i^{IJ} > 0$ and $0 \leqslant d_i^{IJ} < p_J$ such that $iq_I = c_i^{IJ} p_J + d_i^{IJ}$, $I = 1, 2, \cdots, m,\ J = 1, 2, \cdots, r$. The lifted multirate multivariable systems can be expressed as

$$P_M : \begin{bmatrix} x_c((k+1)T) \\ \hline \underline{y}_{c1}(kT) \\ \underline{y}_{c2}(kT) \\ \vdots \\ \underline{y}_{cm}(kT) \end{bmatrix} = \left[\begin{array}{c|cccc} A^\sigma & \varOmega_1 & \varOmega_2 & \cdots & \varOmega_r \\ \hline \varGamma_1 & H_{11} & H_{12} & \cdots & H_{1r} \\ \varGamma_2 & H_{12} & H_{22} & \cdots & H_{2r} \\ \vdots & \vdots & \vdots & & \vdots \\ \varGamma_m & H_{m1} & H_{m2} & \cdots & H_{mr} \end{array}\right] \begin{bmatrix} x_c(kT) \\ \hline \underline{u}_{c1}(kT) \\ \underline{u}_{c2}(kT) \\ \vdots \\ \underline{u}_{cr}(kT) \end{bmatrix}$$

or equivalently as

$$\begin{bmatrix} x((k+1)\sigma) \\ \hline \underline{y}(k\sigma) \end{bmatrix} = \left[\begin{array}{c|c} A^\sigma & \varOmega \\ \hline \varGamma & H \end{array}\right] \begin{bmatrix} x(k\sigma) \\ \hline \underline{u}(k\sigma) \end{bmatrix}$$

where

$$A = \exp(A_c h) \in R^{n\times n}, \quad B_j = \int_0^h \exp(A_c t)\mathrm{d}t\, B_{cj} \in R^{n\times 1}$$

$$B_{p_j}=\int_0^{p_jh}\exp(A_ct)\mathrm{d}t\,B_{cj}=F(0,p_j)B_j\in R^{n\times 1},\quad A_{p_j}=A^{p_j},\quad A_{q_i}=A^{q_i}$$

$$\Omega=[\Omega_1,\ \Omega_2,\ \cdots,\ \Omega_r]\in R^{n\times(\nu_1+\nu_2+\cdots+\nu_r)}$$

$$\Omega_j=[A_{p_j}^{\nu_j-1}B_{p_j},\ A_{p_j}^{\nu_j-2}B_{p_j},\ \cdots,\ B_{p_j}]=$$
$$[F(\sigma-p_j,p_j)B_j,\ F(\sigma-2p_j,p_j)B_j,\ \cdots,\ F(0,p_j)B_j]\in R^{n\times\nu_j}$$

$$\Gamma=\begin{bmatrix}\Gamma_1\\ \Gamma_2\\ \vdots\\ \Gamma_m\end{bmatrix}\in R^{(\mu_1+\mu_2+\cdots+\mu_m)\times n},\quad \Gamma_i=\begin{bmatrix}C_i\\ C_iA_{q_i}\\ \vdots\\ C_iA_{q_i}^{(\mu_i-1)}\end{bmatrix}\in R^{\mu_i\times n}$$

$$H=\begin{bmatrix}H_{11} & H_{12} & \cdots & H_{1r}\\ H_{12} & H_{22} & \cdots & H_{2r}\\ \vdots & \vdots & & \vdots\\ H_{m1} & H_{m2} & \cdots & H_{mr}\end{bmatrix}\in R^{(\mu_1+\mu_2+\cdots+\mu_m)\times(\nu_1+\nu_2+\cdots+\nu_r)}$$

$$H_{ij}=\begin{bmatrix}d_{ij} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0\\ h_{10}^{ij} & h_{11}^{ij} & \cdots & h_{1c_1^{ij}}^{ij} & 0 & & & \vdots\\ h_{20}^{ij} & h_{21}^{ij} & \cdots & \cdots & h_{2c_2^{ij}}^{ij} & 0 & & \vdots\\ \vdots & \vdots & & & & \ddots & \ddots & \vdots\\ h_{(\mu_i-1),0}^{ij} & h_{(\mu_i-1),0}^{ij} & \cdots & \cdots & \cdots & \cdots & h_{(\mu_i-1),c_{\mu_i-1}^{ij}}^{ij} & 0\end{bmatrix}\in R^{\mu_i\times\nu_j}$$

$$h_{il}=C_IF(iq_I-lp_J-p_J,p_J)B_J,\quad l=1,2,\cdots,c_i^{IJ}-1,\quad h_{ic_i}^{IJ}=C_IF(0,d_i^{IJ})B_J+d_{IJ}$$

$$\underline{u}_j(k\sigma)=\underline{u}_{cj}(kT)=\begin{bmatrix}u_j(k\sigma)\\ u_j(k\sigma+p_j)\\ \vdots\\ u_j(k\sigma+(\nu_j-1)p_j)\end{bmatrix},\quad \underline{u}(k\sigma)=\begin{bmatrix}\underline{u}_1(k\sigma)\\ \underline{u}_2(k\sigma)\\ \vdots\\ \underline{u}_r(k\sigma)\end{bmatrix}$$

$$\underline{y}_i(k\sigma)=\underline{y}_{ci}(kT)=\begin{bmatrix}y_i(k\sigma)\\ y_i(k\sigma+q_i)\\ \vdots\\ y_i(k\sigma+(\mu_i-1)q_i)\end{bmatrix},\quad \underline{y}(k\sigma)=\begin{bmatrix}\underline{y}_1(k\sigma)\\ \underline{y}_2(k\sigma)\\ \vdots\\ \underline{y}_m(k\sigma)\end{bmatrix}$$

In order to save space, the proof is omitted here but available from the authors. A similar result can be obtained from Lemma 7 in [18].

In principle, existing identification methods of multi-input multi-output systems can be applied to the lifted multirate systems P_M based on the multirate data $\{u_j(kp_j), y_i(kq_i):\ j=1,2,\cdots,r;\ i=1,2,\cdots,m;\ k=0,1,2,\cdots\}$, but dealing with causality constraints is necessary. In general, the lifted systems have very high input-output dimensions; so developing identification algorithms with less computational effort is still an important research problem.

6 Conclusions

Mathematical models for dual-rate/multirate systems are derived by using a polynomial transformation technique and the lifting technique. Several parameter and intersample output estimation algorithms are studied, and ways to determine single-rate models from lifted dual-rate models are discussed. The auxiliary model methods of identifying single-rate models are introduced directly from dual-rate sampled data; also discussed are the hierarchical schemes of combined state and parameter estimation for lifted systems. These are fundamental for multirate system modeling and identification. There are many topics requiring further research, for example, how to derive mathematical models for dual-rate/multirate systems by using the polynomial transformation technique, the relationship between the polynomial transformation and lifting techniques, existence and uniqueness of single-rate models with different sampling periods and how to determine these single-rate models. Because lifted multirate multivariable systems have high dimensions, how to develop fast identification algorithms is yet unsolved.

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Feng DING Was born in Guangshui, Hubei Province. He received the bachelor degree in electrical engineering from the Information Engineering College, Hubei University of Technology (Wuhan, P.R. China) in 1984, and the master and Ph.D. degrees in automatic control both from the Department of Automation, Tsinghua University in 1990 and 1994, respectively. From 1984 to 1988, he was an electrical engineer at the Hubei Pharmaceutical Factory. Since 1994 he has been with Department of Automation, Tsinghua University. He is now a Visiting Professor at the University of Alberta, Edmonton, Canada. His current research interests include identification and adaptive control. He co-authored the book *Adaptive Control Systems* (Tsinghua University Press, Beijing, 2002), and published over eighty papers on modelling and identification as the first author.

Tongwen CHEN Received the bachelor degree from Tsinghua University (Beijing) in 1984, and the master and Ph.D. degrees from the University of Toronto in 1988 and 1991, respectively, all in electrical engineering. From 1991 to 1997, he was an Assistant/Associate Professor in the Department of Electrical and Computer Engineering at the University of Calgary, Canada. Since 1997, he has been with the Department of Electrical and Computer Engineering at the University of Alberta, Edmonton, Canada, and is presently a Professor of Electrical Engineering. He held visiting positions at the Hong Kong University of Science and Technology, Tsinghua University and Kumamoto University. His current research interests include process control, multirate systems, robust control, network based control, digital signal processing, and their applications to industrial problems. He co-authored with B.A. Francis the book *Optimal Sampled-Data Control Systems* (Springer, 1995). Dr. Chen received a University of Alberta McCalla Professorship for 2000/2001, and a Fellowship from the Japan Society for the Promotion of Science for 2004. He was an Associate Editor for IEEE Transactions on Automatic Control during 1998-2000. Currently he is an Associate Editor for Automatica, Systems & Control Letters, and DCDIS Series B. He is a registered Professional Engineer in Alberta, Canada.