Study on the Application of Iterative Learning Control to Terminal Control of Linear Time-varying Systems[1)]

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Abstract An iterative learning control algorithm based on shifted Legendre orthogonal polynomials is proposed to address the terminal control problem of linear time-varying systems. First, the method parameterizes a linear time-varying system by using shifted Legendre polynomials approximation. Then, an approximated model for the linear time-varying system is deduced by employing the orthogonality relations and boundary values of shifted Legendre polynomials. Based on the model, the shifted Legendre polynomials coefficients of control function are iteratively adjusted by an optimal iterative learning law derived. The algorithm presented can avoid solving the state transfer matrix of linear time-varying systems. Simulation results illustrate the effectiveness of the proposed method.

Key words Iterative learning control, orthogonal polynomials, terminal control, linear time-varying system

1 Introduction

Systems requiring change of states or outputs in a finite time constitute a special class of control problems and are often called the terminal (or point-to-point) control problems[1]. The precise terminal control finds increasing applications in large space apparatus, robotic manipulator and small scale apparatus like computer disk drives. When a system executes a given task repeatedly, this repeatability can be utilized to improve the system control performance by the iterative learning control method[2]. Learning method has been applied to terminal control problems successfully in [3∼7]. We refer to this special learning scheme as terminal iterative learning control[7]. The key technique of the method is to represent the control function as a linear combination of a pre-determined piecewise continuous functional basis and then to update the coefficient vector based on the terminal output error. However, a drawback of the existing algorithm is that the error convergence checking condition contains the system state transfer matrix. It is difficult to obtain the precise solution to the state transfer matrix for linear time-varying systems[8]. As a consequence, it is not easy to check a priori whether the linear time-varying system satisfies the convergence condition so that the existing method can apply.

In this paper, we develop an iterative learning control algorithm based on shifted Legendre orthogonal polynomials for the application to the terminal control of linear time-varying systems. The approach starts using finite shifted Legendre polynomials expansion to parameterize a linear time-varying system. Then, the dynamical equation of the plant is reduced to a set of linear algebraic equations by employing the orthogonality relations and boundary values of shifted Legendre polynomials. The control problem becomes one of finding the shifted Legendre polynomials coefficients of control function, which are iteratively adjusted by learning algorithm. The algorithm is derived by minimizing the quadratic performance index in the form of algebraic forms via shifted Legendre polynomials expansion. Compared to the existing method, the method presented can avoid solving the state transfer matrix of linear time-varying systems. Simulation results show the effectiveness of the proposed scheme.

2 Brief review on shifted Legendre polynomials

The well-known Legendre differential equation is described by

$$[(1-z^2)P_i'(z)]' + i(i+1)P_i'(z) = 0,\ -\leqslant z \leqslant 1,\ i = 0, 1, \cdots, \infty \tag{1}$$

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First we try to transform the independent variable into values between 0 and T, and let

$$z = (2t/T) - 1, \quad t \in [0, T] \tag{2}$$

The shifted Legendre polynomials in t satisfy the following recurrence relation

$$\begin{cases} P_0(t) = 1 \\ P_1(t) = 2(t/T) - 1 \\ (i+1)P_{i+1}(t) = (2i+1)P_1(t)P_i(t) - iP_{i-1}(t), \ i \geqslant 1 \end{cases} \tag{3}$$

We mention the following properties, which are essential in the subsequent development

a) Orthogonality relations

$$\int_0^{\mathrm{T}} P_m(t)P_n(t)\mathrm{d}t = \begin{cases} 0, & m \neq n \\ T/(2n+1), & m = n \end{cases} \tag{4}$$

b) Boundary values

$$P_i(T) = 1 \tag{5}$$

An arbitrary function $f(t)$, which is absolutely integrable in $t \in [0, T]$ can be expressed in terms of shifted Legendre polynomials

$$f(t) = \sum_{i=0}^{\infty} f_i P_i(t) \tag{6}$$

In general, we may obtain an approximate expression of $f(t)$, by truncating the series (6) up to m terms (m is determined by the approximation accuracy requirement)

$$f(t) \approx \sum_{i=0}^{m-1} f_i P_i(t) = \boldsymbol{f}^{\mathrm{T}} \boldsymbol{D}_m(t) \tag{7}$$

where $\boldsymbol{f}$ is called the shifted Legendre polynomials coefficient vector, and $\boldsymbol{D}_m(t)$ is the shifted Legendre polynomials vector. The two vectors are defined as

$$\boldsymbol{f} = [f_0 \quad f_1 \quad f_2 \cdots f_{m-1}]^{\mathrm{T}}, \quad \boldsymbol{D}_m(t) = [P_0(t) \quad P_1(t) \quad P_2(t) \cdots P_{m-1}(t)]^{\mathrm{T}}$$

The orthogonal coefficients can be computed by

$$f_i = \int_0^{\mathrm{T}} f(t)P_i(t)\mathrm{d}t \Big/ \int_0^{\mathrm{T}} P_i(t)^2 \mathrm{d}t \tag{8}$$

which shows (7) is a least-squares approximation.

3 Terminal control problem for linear time-varying systems

Consider the repeatable linear time-varying system described by

$$\begin{cases} \dot{\boldsymbol{x}}_k(t) = A(t)\boldsymbol{x}_k(t) + B(t)\boldsymbol{u}_k(t) \\ \boldsymbol{y}_k(t) = C(t)\boldsymbol{x}_k(t) \end{cases}, \quad 0 \leqslant t \leqslant T \tag{9}$$

where subscript k indicates the system repetition number; $\boldsymbol{x} \in R^{n\times 1}, \boldsymbol{u} \in R^{l\times 1}$ and $\boldsymbol{y} \in R^{r\times 1}$ are state, input and output, respectively. The time-varying matrices $A(t) \in R^{n\times n}, B(t) \in R^{n\times l}$ and $C(t) \in R^{n\times r}$ are assumed to be smooth and square integrable on the time interval $[0, T]$. The control task is to find the control function $\boldsymbol{u}_k(t)$ in an iterative learning manner such that $\boldsymbol{y}_k(T)$ approaches to a given desired terminal output y_d as k increases. To restrict our discussion, we assume that system (9) is completely controllable and observable.

Integrating both sides of (9) from 0 to T, we obtain

$$\boldsymbol{x}_k(T) - \boldsymbol{x}_k(0) = \int_0^{\mathrm{T}} (A(t)\boldsymbol{x}_k(t) + B(t)\boldsymbol{u}_k(t))\mathrm{d}t \tag{10}$$

Let $A(t), B(t), \boldsymbol{x}_k(t)$ and $\boldsymbol{u}_k(t)$ be approximated in terms of $m-$term shifted Legendre polynomials

$$A(t)=\sum_{i=0}^{m-1}A_iP_i(t),\ B(t)=\sum_{i=0}^{m-1}B_iP_i(t),\ \boldsymbol{x}_k(t)=\sum_{i=0}^{m-1}\boldsymbol{x}_{ki}P_i(t),\ \boldsymbol{u}_k(t)=\sum_{i=0}^{m-1}\boldsymbol{u}_{ki}P_i(t) \tag{11}$$

where the coefficients A_i and B_i can be determined from (8). Substituting (11) into (10) and using the properties of shifted Legendre polynomials mentioned in the last section, we have

$$\bar{\boldsymbol{x}}_k-\bar{\boldsymbol{x}}_{0k}=\left(\sum_{i=0}^{m-1}\bar{E}_i^{\mathrm{T}}\otimes A_i\right)\bar{\boldsymbol{x}}_k+\left(\sum_{i=0}^{m-1}\bar{E}_i^{\mathrm{T}}\otimes B_i\right)\bar{\boldsymbol{u}}_k \tag{12}$$

where $\otimes$ stands for Kronecker product and

$$\bar{\boldsymbol{x}}=[\boldsymbol{x}_{k0}^{\mathrm{T}}\quad \boldsymbol{x}_{k1}^{\mathrm{T}}\quad\cdots\quad \boldsymbol{x}_{k,m-1}^{\mathrm{T}}]^{\mathrm{T}},\ \bar{\boldsymbol{x}}_{0k}=[\boldsymbol{x}_k(0)^{\mathrm{T}}\quad 0^{\mathrm{T}}\quad\cdots\quad 0^{\mathrm{T}}]^{\mathrm{T}},\ \bar{\boldsymbol{u}}_k=[\boldsymbol{u}_{k0}^{\mathrm{T}}\quad \boldsymbol{u}_{k1}^{\mathrm{T}}\quad\cdots\quad \boldsymbol{u}_{k,m-1}^{\mathrm{T}}]^{\mathrm{T}}$$
$$\bar{E}_i=(a_{pq})_{m\times m},\ \text{if } p=q=i+1,\ \text{then } a_{pq}=T/(2i-1),\ \text{else } a_{pq}=0\ (p,q=1,2,\cdots,m)$$

If matrix $I_{nm}-\sum_{i=0}^{m-1}\bar{E}_i^{\mathrm{T}}\otimes A_i$ is nonsingular, $\bar{\boldsymbol{x}}_k$ can be expressed by

$$\bar{\boldsymbol{x}}_k=\bar{A}^{-1}\bar{B}\bar{\boldsymbol{u}}_k+\bar{A}^{-1}\bar{\boldsymbol{x}}_{0k} \tag{13}$$

where I_{nm} is the $nm-$dimensional unity matrix, $\bar{A}^{-1}=(I_{nm}-\sum_{i=0}^{m-1}\bar{E}_i^{\mathrm{T}}\otimes A_i)^{-1}, \bar{B}=\sum_{i=0}^{m-1}\bar{E}_i^{\mathrm{T}}\otimes B_i$

The approximate solution to $\boldsymbol{x}_k(t)$ and $\boldsymbol{u}_k(t)$ can be written as

$$\bar{\boldsymbol{x}}_k(t)=(\boldsymbol{D}_m(t)^{\mathrm{T}}\otimes I_n)\bar{\boldsymbol{x}}_k \tag{14a}$$
$$\bar{\boldsymbol{u}}_k(t)=(\boldsymbol{D}_m(t)^{\mathrm{T}}\otimes I_l)\bar{\boldsymbol{u}}_k \tag{14b}$$

Definition 1. We refer to (13) and (14) as $m-$degree ($m\geqslant 1$) approximated model of linear time-varying system (9) with respect to $\{P_i(t)\}$. The initial condition of the approximated model is $\bar{\boldsymbol{x}}_k(0)=\boldsymbol{x}_k(0)$.

Remark 1. The approximated model presented here is more accurate and simpler than the conventional approximated model concerning $\{P_i(t)\}$[9]. Furthermore, the deduction of (13) avoids using the complex integration operational matrix of shifted Legendre polynomials. The role $\bar{A}^{-1}$ plays in (13) is equivalent to that state transfer matrix Φ plays in (9).

Introducing (14a) into (9), we have the approximate value of the terminal output $\boldsymbol{y}_k(T)$

$$\bar{\boldsymbol{y}}_k(T)=\bar{C}(\bar{A}^{-1}\bar{B}\bar{\boldsymbol{u}}_k+\bar{A}^{-1}\bar{\boldsymbol{x}}_{0k})=G\bar{\boldsymbol{u}}_k+\boldsymbol{\eta}_k \tag{15}$$

where $\bar{C}=C(T)(\boldsymbol{D}_m(T)^{\mathrm{T}}\otimes I_n), G=\bar{C}\bar{A}^{-1}\bar{B}, \boldsymbol{\eta}_k=\bar{C}\bar{A}^{-1}\bar{\boldsymbol{x}}_{0k}$. So the expression for the approximate value of the terminal output error is

$$\bar{\boldsymbol{e}}_k(T)=\boldsymbol{y}_d-\bar{\boldsymbol{y}}_k(T)=\boldsymbol{y}_d-G\bar{\boldsymbol{u}}_k-\boldsymbol{\eta}_k \tag{16}$$

4 Optimal iterative learning control algorithm

Now the control task is converted into finding an iterative scheme such that $\bar{\boldsymbol{u}}_k$ converges as k increases. To derive the learning controller, we consider the following performance index.

$$J_{k+1}=\boldsymbol{e}_{k+1}(T)^{\mathrm{T}}F\boldsymbol{e}_{k+1}(T)+\int_0^{\mathrm{T}}\Delta\boldsymbol{u}_{k+1}(t)^{\mathrm{T}}R(t)\Delta\boldsymbol{u}_{k+1}(t)\mathrm{d}t \tag{17}$$

where $\Delta\boldsymbol{u}_{k+1}(t)=\boldsymbol{u}_{k+1}(t)-\boldsymbol{u}_k(t), \boldsymbol{e}_{k+1}(t)=\boldsymbol{y}_d-\boldsymbol{y}_{k+1}(T), F$ and $R(t)$ are symmetric positive definitive weight matrices. We assume $R(t)$ is constant matrix for convenience.

The shifted Legendre polynomials expansion of index function (17) is

$$\begin{aligned}J_{k+1}=&\bar{\boldsymbol{e}}_k(T)^{\mathrm{T}}F\bar{\boldsymbol{e}}_k(T)-2\bar{\boldsymbol{e}}_k(T)FG\Delta\bar{\boldsymbol{u}}_{k+1}-2\bar{\boldsymbol{e}}_k(T)^{\mathrm{T}}F\Delta\boldsymbol{\eta}_{k+1}+\\&2\Delta\boldsymbol{\eta}_{k+1}^{\mathrm{T}}FG\Delta\bar{\boldsymbol{u}}_{k+1}+\Delta\boldsymbol{\eta}_{k+1}^{\mathrm{T}}F\Delta\boldsymbol{\eta}_{k+1}+\Delta\bar{\boldsymbol{u}}_{k+1}^{\mathrm{T}}(G^{\mathrm{T}}FG+\hat{R})\Delta\bar{\boldsymbol{u}}_{k+1}\end{aligned} \tag{18}$$

where $\hat{R}=\int_0^{\mathrm{T}}[(\boldsymbol{D}_m(t)\otimes I_l)R(\boldsymbol{D}_m(t)^{\mathrm{T}}\otimes I_l)]\mathrm{d}t, \Delta\boldsymbol{\eta}_{k+1}=\boldsymbol{\eta}_{k+1}-\boldsymbol{\eta}_k, \Delta\bar{\boldsymbol{u}}_{k+1}=\bar{\boldsymbol{u}}_{k+1}-\bar{\boldsymbol{u}}_k$

Minimizing the performance index (18) with respect to $\Delta\bar{\boldsymbol{u}}_{k+1}$, one can obtain the optimal learning law

$$\bar{\boldsymbol{u}}_{k+1} = \bar{\boldsymbol{u}}_k + [G^{\mathrm{T}}FG + \hat{R}]^{-1}G^{\mathrm{T}}F[\bar{\boldsymbol{e}}_k(T) - \Delta\boldsymbol{\eta}_{k+1}] \tag{19}$$

The convergence condition for the above optimal learning scheme is presented in the following theorem.

Theorem 1. Consider the repetitive linear time-varying system (9) with a given achievable terminal output $\boldsymbol{y}_d$. By applying the control functional parameterization (14b) and the iterative learning law (19), if 1) $\| I + G\hat{R}^{-1}G^{\mathrm{T}}F \| > 1$, 2) $\boldsymbol{x}_k(0) = \boldsymbol{x}^0, k = 0, 1, 2, \cdots$ are satisfied, then when $k \to \infty$, the terminal tracking error will converge to a bound.

Proof. From (16) and (19), we obtain the relation between two consecutive operation cycles with the aid of the matrix inversion lemma[10]

$$\bar{\boldsymbol{e}}_{k+1}(T) = (I + G\hat{R}^{-1}G^{\mathrm{T}}F)^{-1}[\bar{\boldsymbol{e}}_k(T) - \Delta\boldsymbol{\eta}_{k+1}] \tag{20}$$

Iteratively using (20) leads to

$$\bar{\boldsymbol{e}}_{k+1}(T) = (I + G\hat{R}^{-1}G^{\mathrm{T}}F)^{-(k+1)}\bar{\boldsymbol{e}}_0(T) - \sum_{l=1}^{k+1}(I + G\hat{R}^{-1}G^{\mathrm{T}}F)^{-l}\Delta\boldsymbol{\eta}_{k+2-l} \tag{21}$$

Hence, under the hypothesis of the theorem we have that $\bar{\boldsymbol{e}}_k(T) \to 0$ as $k \to \infty$. Since $\bar{\boldsymbol{e}}_k(T) \to 0, \boldsymbol{e}_k(T) \to \varepsilon$, follows. That is, the terminal output error $\boldsymbol{e}_k(T)$ is bounded and the bound is determined by $\varepsilon = \lim_{k\to\infty} |C(T)[\boldsymbol{x}_k(T) - \bar{\boldsymbol{x}}_k(T)]|$. This completes the proof. □

5 Simulation

Consider the following repeatable linear time-varying system

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} -t^2 & 2t \\ 2t^2 + t & 3t^2 + 5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} t - 1 \\ 3t^2 + 1 \end{bmatrix} u(t), \quad \boldsymbol{x}(0) = 0.1 \\
\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} &= \begin{bmatrix} t & -2 \\ 1 & 3t \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad 0 \leqslant t \leqslant 1
\end{aligned} \tag{22}$$

The desired terminal output is given as $[\boldsymbol{y}_1(1)\ y_2(1)] = [6\ 5]$. We choose $F = R = I$ and $m = 3$ in the numerical simulation.

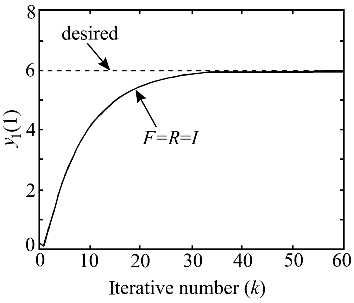

Fig. 1 First terminal output component *vs* iterations

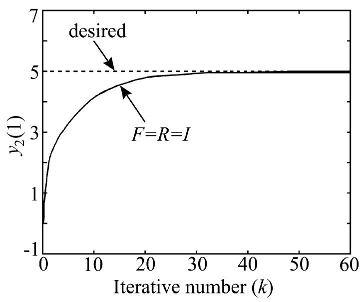

Fig. 2 Second terminal output component *vs* iterations

Terminal output $y_1(1)$ and $y_2(1)$ versus the trial numbers are plotted in Fig. 1 and Fig. 2, respectively. We can learn that the convergence bound ε is related to the approximation accuracy of the approximated model from Theorem 1. To show the approximated model presented here is superior to the conventional approximated model[9], we give the terminal output values based on these two different models in Table 1. It can be seen from Table 1 that the approximated model used in the paper can improve the system control performance.

Table 1 Terminal outputs based on different approximated models

Weight F	Matrices R	The iterative number(k)	Conventional $y_1(1)$	Approximated model[9] $y_2(1)$	Approximated model $y_1(1)$	Presented $y_2(1)$
$3I$	I	20	5.901	4.893	5.989	4.995
I	I	40	5.872	4.803	5.953	4.978
I	$1.5I$	60	5.898	4.812	5.959	4.981

6 Conclusions

This work studies the terminal control problem of linear time-varying systems. An iterative learning algorithm based on shifted Legendre polynomials is developed. An approximated model of linear time-varying systems is deduced by employing orthogonality relations and boundary values of shifted Legendre polynomials. We derive optimal learning scheme on the basis of the approximated model. Finally, an example is presented to demonstrate the effectiveness of the proposed method.

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