On Asymptotic Stabilization of Linear Discrete-time Systems with Saturated State Feedback[1)]

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Abstract As saturation is involved in the stabilizing feedback control of a linear discrete-time system, the original global-asymptotic stabilization (GAS) may drop to region-asymptotic stabilization (RAS). How to test if the saturated feedback system is GAS or RAS? The paper presents a criterion to answer this question, and describes an algorithm to calculate an invariant attractive ellipsoid for the RAS case. At last, the effectiveness of the approach is shown with examples.

Key words Actuator saturation, stability, discrete-time system, invariant ellipsoid, domain of attraction

1 Introduction and problem statement

For convenience, the nomenclature and symbols we will use are list below

GAS global asymptotic stabilization

RAS region asymptotic stabilization

R real number field

Z nonnegative integer set

I unit matrix

A' transpose of matrix A

$|\cdot|$ absolute value for numbers, or determinant for matrices

$\lambda_{\min}(A)$ minimum eigenvalue of matrix A

sup supremum

Let a time-invariant saturated state feedback system be as follows

$$\begin{cases} \boldsymbol{x}_{t+1} = A\boldsymbol{x}_t + bu_t \\ u_t = sat_d(k'\boldsymbol{x}_t),\ t \in Z \end{cases} \tag{1}$$

where state $\boldsymbol{x}_t \in R^n$, control $u_t \in R$, A, b and k' are real constant matrices with appropriate dimensions. Notice that $u_t = sat_d(k'\boldsymbol{x}_t) := sign(k'\boldsymbol{x}_t)\min\{d, |k'\boldsymbol{x}_t|\}$, where size $d > 0$ and gain $\boldsymbol{k} \in R^n$ are given beforehand.

Assumption 1. The feedback gain $k' \neq 0$, and $A + bk'$ is Schur stable, *i.e.*, all the eigenvalues of $A + bk'$ lie in the open unit circle.

Assumption 1 implies that there exists a symmetric positive-definite matrix P so that

$$(A + bk')'P(A + bk') - P = -I \tag{2}$$

With this P and a positive number r, we define an open ellipsoid

$$\Omega(P, r) := \{\boldsymbol{x} \in R^n : \boldsymbol{x}'P\boldsymbol{x} < r\}$$

Given an initial state $\boldsymbol{x}_0$, denote the evaluation of (1) by $\psi(t, \boldsymbol{x}_0)$. A domain $\Psi \subset R^n$ is called to be invariant if $\psi(t, \boldsymbol{x}_0) \in \Psi$ for all $t \in Z$ and all $\boldsymbol{x}_0 \in \Psi$. A domain $\Xi \subset R^n$ is attractive if $\lim_{t\to\infty} \psi(t, \boldsymbol{x}_0) = 0$ for all $\boldsymbol{x}_0 \in \Xi$. We call the system of (1) to be GAS if the whole state space is attractive (naturally it is invariant in this case), or RAS if its attractive domain is a subregion of the state space. However in general cases, an attractive domain may not be invariant, also an invariant domain may not be attractive.

1) Supported by National Natural Science Foundation of P. R. China (60174040)

Received December 15, 2003; in revised form June 5, 2004

Let

$$\mathcal{L}(k') := \{\boldsymbol{x} \in R^n : |k'\boldsymbol{x}| \leqslant d\}$$

obviously, $sat_d(k'\boldsymbol{x}) = k'\boldsymbol{x}\ \ \forall \boldsymbol{x} \in \mathcal{L}(k')$, thus we call $\mathcal{L}(k')$ the unsaturated region of the feedback control, or the linear region of the saturated feedback $sat_d(k'\boldsymbol{x})$, or simply, the linear region[1,2].

This paper will consider the following problem.

Problem 1. Under Assumption 1, how to test the stability of the saturated feedback system of (1)? Calculate a large invariant attractive ellipsoid in the RAS case.

Before solving the problem above, it is necessary to list some known facts[1~5]: 1) Even if a linear discrete-time system is stabilized by a linear state feedback control, as the saturation is involved in the control, the original GAS may be dropped to RAS. 2) For the saturated system of (1) with Assumption 1, its linear region may be neither invariant nor attractive. 3) $\Omega(P,r)$ is invariant and attractive whenever $\Omega(P,r) \subset \mathcal{L}(k')$. 4) Let $V(\boldsymbol{x}_t) := x_t'P\boldsymbol{x}_t$ where $P' = P > 0$ is the solution matrix of (2). $\Omega(P,r)$ is (contractively) invariant for the system of (1) if $\Delta V(\boldsymbol{x}_t) = V(\boldsymbol{x}_{t+1}) - V(\boldsymbol{x}_t) \leqslant (<) 0$ for all nonzero $\boldsymbol{x}_t \in \Omega(P,r)$ and $t \in Z$. Clearly, if $\Omega(P,r)$ is contractively invariant, then it is inside the domain of attraction.

Actuator saturation is a nonlinear problem that needs to be dealt with in all practical control systems, especially in servo or robot control systems[6]. In the literature, invariant attractive ellipsoids have been used to estimate the domain of attraction for linear discrete-time systems with saturated state feedback under either some restrictive conditions or overelaborate procedure. However, different from the known results, we will present a clear and simple procedure to solve Problem 1 based on the discrete-type Lyapunov equation, but without restriction on matrix A.

2 Main result

Define a so-called saturation level function, $\mu : R^n \to [0,1)$

$$\mu(\boldsymbol{x}) := \begin{cases} 0, & \boldsymbol{x} \in \mathcal{L}(k') \\ 1 - \dfrac{d}{|k'\boldsymbol{x}|}, & \boldsymbol{x} \notin \mathcal{L}(k') \end{cases} \tag{3}$$

We simply denote $\mu(\cdot)$ by μ if no confusion occurs. It is easy to see that $sat_d(k'\boldsymbol{x}) = (1-\mu)k'\boldsymbol{x}$ for all $x \in R^n$. Thus the system of (1) can be rewritten as

$$\boldsymbol{x}_{t+1} = (A + (1-\mu)bk')\boldsymbol{x}_t \tag{4}$$

For any positive number r and the solution matrix $P > 0$ in (2), it is easy to prove that

$$\sup\{|k'\boldsymbol{x}| : \boldsymbol{x} \in \Omega(P,r)\} = \sqrt{r(k'P^{-1}k)}$$

and that the supremum is achievable as $\boldsymbol{x} \to \pm\sqrt{\dfrac{r}{(k'P^{-1}k)}}$, the boundary points of $\Omega(P,r)$. Thus we have

$$\mu^+ = \sup\{\mu(\boldsymbol{x}) : \boldsymbol{x} \in \Omega(P,r)\} = \begin{cases} 0, & \Omega(P,r) \subset \mathcal{L}(k') \\ 1 - \dfrac{d}{\sqrt{r(k'P^{-1}k)}} > 0, & \Omega(P,r) \not\subset \mathcal{L}(k') \end{cases}$$

Conversely, if $\mu^+ \in [0,1)$ is known in advance, then

$$r = \frac{d^2}{(k'P^{-1}k)(1-\mu^+)^2} \tag{5}$$

With this r and matrix $P' = P > 0$ in (2) the ellipsoid $\Omega(P,r)$ can be uniquely determined.

Lemma 1. i) If $\mu^+ = 0$, then $\Omega(P,r) \subset \mathcal{L}(k')$ and $\mu(\boldsymbol{x}) \equiv 0\ \forall \boldsymbol{x} \in \Omega(P,r)$. ii) If $\mu^+ > 0$, then $\Omega(P,r) \not\subset \mathcal{L}(k')$ and $\mu(x) < \mu^+\ \forall x \in \Omega(P,r)$.

Proof. Recalling the definitions of μ and μ^+ derives assertion i) immediately. ii) From $r = \dfrac{d^2}{(k'P^{-1}k)(1-\mu^+)^2}$, we know $\sup\{|k'\boldsymbol{x}| : \boldsymbol{x} \in \Omega(P,r)\} = \dfrac{d}{1-\mu^+} > d$. This implies that there exists $\hat{\boldsymbol{x}} \in R^n$ such that $|k'\hat{\boldsymbol{x}}| > d$, *i.e.*, $\hat{\boldsymbol{x}} \notin \mathcal{L}(k')$. Thus the lemma is proved. □

The most important thing is how to determine μ^+. To this end, let

$$\mu^+ := \sup\{\mu \in [0,1) : I + \mu E > 0\} \tag{6}$$

where

$$E := \begin{bmatrix} 0 & P^{1/2}bk' \\ kb'P^{1/2} & G+G' \end{bmatrix} \tag{7}$$

$$G := kb'P(A+bk') \tag{8}$$

From (6) it is easy to find

$$\mu^+ = \begin{cases} 1, & \lambda_{\min}(E) \geqslant -1 \\ -\dfrac{1}{\lambda_{\min}(E)}, & \lambda_{\min}(E) < -1 \end{cases}$$

and furthermore, $I + \mu E > 0\ \forall \mu \in [0, \mu^+)$.

Before presenting our main result, we need another useful lemma.

Lemma 2. For some $\nu \in R$, $I + \nu(G+G') - \nu^2 kb'Pbk' > 0$ if and only if $I + \nu E > 0$ for the same ν, where G and E are defined in (8) and (7), respectively.

Proof. Notice that

$$\begin{bmatrix} I & 0 \\ -\nu kb'P^{1/2} & I \end{bmatrix} \begin{bmatrix} I & \nu P^{1/2}bk' \\ \nu kb'P^{1/2} & I+\nu(G+G') \end{bmatrix} \begin{bmatrix} I & -\nu P^{1/2}bk' \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I+\nu(G+G') - \nu^2 kb'Pbk' \end{bmatrix}$$

Hence the result follows. □

Now we are ready to present our main result.

Theorem 1. Suppose that system (1) satisfies Assumption 1. i) If $\lambda_{\min}(E) \geqslant -1$, where E is defined in (7),then system (1) is GAS;

ii) If $\lambda_{\min}(E) < -1$, then system (1) is RAS, and $\Omega(P,r)$ is contractively invariant ellipsoid, where $P' = P > 0$ satisfies (2), μ^+ and r are calculated from (9) and (5), respectively.

Proof. We use the descriptions of (4) instead of (1). Let $V(\boldsymbol{x}_t) = \boldsymbol{x}_t' P \boldsymbol{x}_t$. Along with the trajectory of (4), we have $\Delta V(\boldsymbol{x}_t) = \boldsymbol{x}_{t+1}' P \boldsymbol{x}_{t+1} - \boldsymbol{x}_t' P \boldsymbol{x}_t = -\boldsymbol{x}_t'(I + \mu(G+G') - \mu^2 kb'Pbk')\boldsymbol{x}_t$. i) If $\lambda_{\min}(E) \geqslant -1$, then $1 + \mu\lambda_{\min}(E) > 0\ \forall \mu \in [0,1)$, furthermore, $I + \mu(\boldsymbol{x}_t)E > 0\ \forall \boldsymbol{x}_t \in R^n$, based on Lemma 2, we immediately have $(I + \mu(\boldsymbol{x}_t)(G+G') - \mu(\boldsymbol{x}_t)^2 kb'Pbk') > 0$ for all $\boldsymbol{x}_t \in R^n$. This also means that $\Delta V(\boldsymbol{x}_t) < 0\ \forall \boldsymbol{x}_t \in R^n$, *i.e.*, the saturated system is GAS. ii) if $\lambda_{\min}(E) < -1$, recalling the formula of (9), $\mu^+ = -\dfrac{1}{\lambda_{\min}(E)} < 1$ and $1 + \mu^+\lambda_{\min}(E) = 0$, thus, $I + \mu^+ E \geqslant 0$. From this we know $I + \mu(\boldsymbol{x}_t)E > 0\ \forall \boldsymbol{x}_t \in \Omega(P,r)$. Based on Lemma 2, we have $(I + \mu(\boldsymbol{x}_t)(G+G') - \mu(\boldsymbol{x}_t)^2 kb'Pbk') > 0$ for all $\boldsymbol{x}_t \in \Omega(P,r)$. Obviously, this leads to $\Delta V(\boldsymbol{x}_t) < 0\ \forall \boldsymbol{x}_t \in \Omega(P,r)$, which means that the saturated feedback system of (1) is RAS with $\Omega(P,r)$ as its invariant and attractive ellipsoid. □

3 Algorithm and examples

Collecting the results of the preceding sections leads to

Algorithm

Step 1. Select $k' \neq 0$ so that $A + bk'$ is Schur stable, and has satisfactory performances;

Step 2. Solute $P = P' > 0$ in (2);

Step 3. Calculate G, E in (8) and (7), and $\lambda_{\min}(E)$;

Step 4. If $\lambda_{\min}(E) \geqslant -1$, then system (1) is GAS, else it is RAS;

Step 5. Determine μ^+ according to (9);

Step 6. Calculate r according to (5), then ellipsoid $\Omega(P,r)$ is invariant and attractive.

Based on the above algorithm, we calculate the following three examples.

Example 1. Given $x_{t+1} = x_t + sat_\Delta(-1.5x_t)$ with $d = 1$, $x_t \in R$, $t \in Z$.

Solution. $E = \begin{bmatrix} 0 & -1.7321 \\ -1.7321 & 2.0000 \end{bmatrix}$, $\lambda_{\min}(E) = -1$. Hence the system is GAS.

Example 2. Given $\boldsymbol{x}_{t+1} = 2\boldsymbol{x}_t + sat_\Delta(-2.5\boldsymbol{x}_t)$ with $d = 1$, $\boldsymbol{x}_t \in R$, $t \in Z$.

Solution. $E = \begin{bmatrix} 0 & -2.8868 \\ -2.8868 & 3.3333 \end{bmatrix}$, $\lambda_{\min}(E) = -1.6667 < -1$. Hence the system is RAS. Since $P = 1.3333$ and $r = 1.3333$, hence $(-1, 1)$ is its invariant and attractive interval.

Example 3. Given $\boldsymbol{x}_{t+1} = \begin{bmatrix} 1 & 2 \\ 0 & 0.8 \end{bmatrix} \boldsymbol{x}_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} sat_\Delta([-0.8100 \ \ -2.5000]\boldsymbol{x}_t)$, with $\boldsymbol{x}_t \in R^2$, $t \in Z$.

Solution. $E = \begin{bmatrix} 0 & 0 & -1.1310 & -3.4907 \\ 0 & 0 & -3.2499 & -10.0089 \\ -1.1310 & -3.2499 & 10.7955 & 28.6201 \\ -3.4907 & -10.0089 & 28.6201 & 73.8299 \end{bmatrix}$, $\lambda_{\min}(E) = -1.6942 < -1$. Hence the system is RAS, and has an invariant and attractive ellipse $\Omega(P, r)$ with $P = \begin{bmatrix} 4.6827 & 7.8984 \\ 7.8984 & 17.9781 \end{bmatrix}$ and $r = 57.2427(d = 2)$, or $r = 14.8107(d = 1)$, respectively.

4 Conclusions

Stability analysis becomes rather involved as saturation appears in linear discrete-time systems. This paper analyzes the stability of system (1), tests if it is GAS or is dropped to RAS, and calculates its invariant attractive ellipsoid for the latter case. Different from the published results, our method has put no restriction on the system matrix A. Generally speaking, the saturated system can never be GAS if A has any eigenvalue out of the unit circle; however, the invariant attractive ellipsoid will be large enough if i) all $|\lambda(A)| \leqslant 1$, and/or ii) a feedback gain k' is chosen such that all $\lambda(A + bk')$ lie in a suitable region of the unit circle.

Further study will deal with i) multi-input saturated systems, ii) uncertain saturated systems, and iii) finding a good k to guarantee a much larger invariant attractive ellipsoid.

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