# An ILMI Approach to RFDF for Uncertain Linear Systems with Nonlinear Perturbations[1)]

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**Abstract** The robust fault detection filter design for uncertain linear systems with nonlinear perturbations is formulated as a two-objective optimization problem. Solvable conditions for the existence of such a robust fault detection filter are given in terms of matrix inequalities (MIs), which can be solved by applying iterative linear matrix inequality (ILMI) techniques. Particularly, compared with two existing LMI methods, the developed algorithm is more generalized and less conservative. An illustrative example is given to show the effectiveness of the proposed method.

**Key words** Fault detection filter, LMI, nonlinear perturbation, observer, residual

## 1 Introduction

Much attention in research has been devoted in recent years to the design of observer-based fault detection systems and a great number of results have been achieved, see [1~5] and references therein. Most of these aforementioned works are about systems without modeling errors. The problem of FDI for systems with modelling errors has not been fully investigated yet, which is not only theoretically interesting and challenging, but also very important in practical applications[6~8]. In [6~7], the $H_\infty$-filtering formulation of robust fault detection filter (RFDF) was proposed, but the determination of reference residual model is a little difficult. With an introduced residual model[8], formulated the RFDF as an $H_\infty$ model-matching problem and, based on this, an LMI approach was proposed.

The main purpose of this research is to study the RFDF problem for a class of uncertain systems with non-linear perturbations. Contributions consist of the formulation of a two-objective optimization problem, the derivation of solvable conditions, and further the development of an LLMI algorithm. A numerical example is given to demonstrate the feasibility and effectiveness of the proposed algorithm.

## 2 Problem formulation

Consider a class of uncertain systems described by

$$\begin{aligned}&\triangle j(\dot{\boldsymbol{x}},t)+\dot{\boldsymbol{x}}=A\boldsymbol{x}+\triangle g(\boldsymbol{x},t)+B\boldsymbol{u}+B_f\boldsymbol{f}+B_d\boldsymbol{d}\\&\boldsymbol{y}=C\boldsymbol{x}+D\boldsymbol{u}+D_f\boldsymbol{f}+D_d\boldsymbol{d}\end{aligned}\tag{1}$$

where $\boldsymbol{x}\in R^n, \boldsymbol{u}\in R^p$ and $\boldsymbol{y}\in R^q$ denote state, control input and measurement output, respectively. $\boldsymbol{d}\in R^m$ denotes $L_2$-norm bounded unknown input. $\boldsymbol{f}\in R^l$ is the fault to be detected. $\triangle j(\dot{\boldsymbol{x}},t): R^n\times n\to R^n$, $\triangle g(\boldsymbol{x},t): R^n\times R\to R^n$ describe the non-linear unknown perturbations, and

$$\begin{aligned}&\triangle j(\dot{\boldsymbol{x}},t)=E_j\delta_j(\dot{\boldsymbol{x}},t), \|\delta_j(\dot{\boldsymbol{x}},t)\|^2\leqslant\|W_j\dot{\boldsymbol{x}}\|^2, \forall\dot{\boldsymbol{x}}\in R^n, \forall t\in R\\&\triangle g(\boldsymbol{x},t)=E_g\delta_g(\boldsymbol{x},t), \|\delta_g(\boldsymbol{x},t)\|^2\leqslant\|W_g\boldsymbol{x}\|^2, \forall\boldsymbol{x}\in R^n, \forall t\in R\\&\Omega_j=\{\delta_j(\dot{\boldsymbol{x}},t)\Big|\ \|\delta_j(\dot{\boldsymbol{x}},t)\|^2\leqslant\|W_j\dot{\boldsymbol{x}}\|^2, \forall t\in R\}\\&\Omega_g=\{\delta_g(\boldsymbol{x},t)\Big|\ \|\delta_g(\boldsymbol{x},t)\|^2\leqslant\|W_g\boldsymbol{x}\|^2, \forall t\in R\}\end{aligned}\tag{2}$$

where $E_j, E_g, W_j, W_g, A, B, C, D, B_f, B_d, D_f$ and $D_d$ are known matrices with appropriate dimensions. It is assumed that

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A1). $(C, A)$ is detectable;

A2). For $\forall \delta_j(\dot{\boldsymbol{x}}, t) \in \Omega_j,\ \forall \delta_g(\boldsymbol{x}, t) \in \Omega_g$, system (1) is asymptotically stable;

A3). $(A, B_f, C, D_f)$ has no zeros on the imaginary axis and at infinity, and rank $D_f = l$.

**Remark 1.** Assumption A1 is necessary for FDI. Assumption A2 is without loss of generality, and if it is not met then one can first design a controller to ensure the robust stability of system (1) by applying the method in [9]. Assumption A3 ensures that $\|T_{rf}(s)\|_-$ defined in Section 3 is non-zero.

The RFDF considered in this paper is given by

$$\begin{aligned}&\dot{\hat{\boldsymbol{x}}} = A\hat{\boldsymbol{x}} + B\boldsymbol{u} + H(\boldsymbol{y} - \hat{\boldsymbol{y}})\\&\hat{\boldsymbol{y}} = C\hat{\boldsymbol{x}} + D\boldsymbol{u},\ \boldsymbol{r} = V(\boldsymbol{y} - \hat{\boldsymbol{y}})\end{aligned} \tag{3}$$

where $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ denote estimation errors of the state and output, respectively. $\boldsymbol{r}$ is the so-called residual. $H \in R^{n\times q}$ and $V \in R^{l\times q}$ are parameters to be determined. Defining $\boldsymbol{e} = \boldsymbol{x} - \hat{\boldsymbol{x}}$, it is obtained from (1) and (3) that

$$\begin{aligned}&\dot{\boldsymbol{e}} = (A - HC)\boldsymbol{e} - E_j\delta_j(\dot{\boldsymbol{x}}, t) + E_g\delta_g(\boldsymbol{x}, t) + (B_f - HD_f)\boldsymbol{f} + (B_d - HD_d)\boldsymbol{d}\\&\boldsymbol{r} = V(C\boldsymbol{e} + D_f\boldsymbol{f} + D_d\boldsymbol{d})\end{aligned} \tag{4}$$

An ideal residual generator should be

$$\boldsymbol{r} = \boldsymbol{0},\ \textit{when}\ \boldsymbol{f} = \boldsymbol{0};\ \ \boldsymbol{r} \neq \boldsymbol{0},\ \textit{when}\ \boldsymbol{f} \neq \boldsymbol{0} \tag{5}$$

However, (5) is difficult to be achieved. Without loss of generality, the main task of this paper is to design a stable FDF to enhance the robustness of residual to unknown input and the nonlinear perturbations, and to maximize the sensitivity to fault, *i.e.*, the RFDF issue.

## 3 Design of RFDF

### 3.1 Basic ideas of this study

Notice that if system (1) is asymptotically stable, and $\boldsymbol{u}, \boldsymbol{d}, \boldsymbol{f} \in L_2$, then $\dot{\boldsymbol{x}}$ and $\boldsymbol{x}$ are $L_2$-norm bounded, and further $\delta_j(\dot{\boldsymbol{x}}, t)$ and $\delta_g(\boldsymbol{x}, t)$. Denote

$$\boldsymbol{w} = [\boldsymbol{d}^{\mathrm{T}}\quad \rho_j\delta_j^{\mathrm{T}}(\dot{\boldsymbol{x}}, t)\quad \rho_g\delta_g^{\mathrm{T}}(\boldsymbol{x}, t)]^{\mathrm{T}},\ B_w = [B_d\quad -\rho_j^{-1}E_j\quad \rho_g^{-1}E_g],\ D_w = [D_d\quad 0\quad 0]$$

where $\rho_j > 0$ and $\rho_g > 0$ are weighting scalars. Then system (4) can be re-written into

$$\begin{aligned}&\dot{\boldsymbol{e}} = (A - HC)\boldsymbol{e} + (B_f - HD_f)\boldsymbol{f} + (B_w - HD_w)\boldsymbol{w}\\&\boldsymbol{r} = V(C\boldsymbol{e} + D_f\boldsymbol{f} + D_w\boldsymbol{w})\end{aligned} \tag{6}$$

Furthermore, $\|T_{rw}(s)\|_\infty$ is used to measure the robustness of residual to unknown input. While the sensitivity of residual to fault is measured by $\|T_{tf}(s)\|_- = \inf_{\omega\in(0,\infty)} \underline{\sigma}[T_{rf}(j\omega)], \underline{\sigma}[T_{rf}(j\omega)]$, denotes the non-zero singular value of $T_{rf}(j\omega)$. Similar to [5], the RFDF problem can then be formulated as finding $H$, and $V$ such that system (6) remains asymptotically stable for $\forall \delta_j(\dot{\boldsymbol{x}}, t) \in \Omega_j,\ \forall \delta_g(\boldsymbol{x}, t) \in \Omega_g$, and performance index $J_- = \frac{\|T_{rw}(s)\|_\infty}{\|T_{rf}(s)\|_-}$ is minimized. Or more loosely, for given $\gamma > 0$, it follows that

$$\|T_{rw}(s)\|_\infty < \gamma,\ \|T_{rf}(s)\|_- \to \max \tag{7}$$

### 3.2 An ILMI algorithm of RFDF design

**Lemma 1 (Bounded real lemma).** Given $\gamma > 0$, the system

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{w},\ \boldsymbol{z} = C\boldsymbol{x} + D\boldsymbol{w},\ \boldsymbol{x}(0) = \boldsymbol{0} \tag{8}$$

is asymptotically stable and satisfies $\|T_{rw}(s)\|_\infty < \gamma$, or $\int_0^{+\infty} z^{\mathrm{T}}z\mathrm{d}t < \gamma^2\int_0^{+\infty} w^{\mathrm{T}}w\mathrm{d}t$, iff there exists matrix $P > 0$ satisfying LMI

$$\begin{bmatrix} A^{\mathrm{T}}P + PA + C^{\mathrm{T}}C & PB + C^TD\\ B^{\mathrm{T}}P + D^{\mathrm{T}}C & -\gamma^2I + D^{\mathrm{T}}D\end{bmatrix} < 0$$

**Lemma 2**[10]. Given $\beta > 0$, if system (8) is asymptotically stable, has no zeros on the imaginary axis and at infinity, $\boldsymbol{w} \in L_2$, and $D$ is full column rank, then $\|T_{zw}(s)\|_- > \beta$ or $\int_0^{+\infty} z^{\mathrm{T}} z \mathrm{d}t > \beta^2 \int_0^{+\infty} w^{\mathrm{T}} w \mathrm{d}t$, iff there exists matrix $Q < 0$ satisfying LMI

$$\begin{bmatrix} A^{\mathrm{T}}Q + QA + C^{\mathrm{T}}C & QB + C^{\mathrm{T}}D \\ B^{\mathrm{T}}Q + D^{\mathrm{T}}C & -\beta^2 I + D^{\mathrm{T}}D \end{bmatrix} > 0 \qquad \square$$

By using Lemmas 1 and 2, the following Theorem 1 is obtained, with proof being omitted.

**Theorem 1.** Given $\gamma > 0$ and $\beta > 0$, system (6) is asymptotically stable and satisfies

$$\|T_{rw}(s)\|_\infty < \gamma, \ \|T_{rf}(s)\|_- > \beta \tag{9}$$

iff there exist matrices $P > 0, Q < 0$, $V$ and $H$ satisfying MIs

$$\begin{bmatrix} (A-HC)^{\mathrm{T}}P + P(A-HC) + C^{\mathrm{T}}V^{\mathrm{T}}VC & P(B_w - HD_w) + C^{\mathrm{T}}V^{\mathrm{T}}V^{\mathrm{T}}VD_w \\ (B_w^{\mathrm{T}} - D_w^{\mathrm{T}}H^{\mathrm{T}})P + D_w^{\mathrm{T}}V^{\mathrm{T}}VC & -\gamma^2 I + D_w^{\mathrm{T}}V^{\mathrm{T}}VD_w \end{bmatrix} < 0 \tag{10}$$

$$\begin{bmatrix} (A-HC)^{\mathrm{T}}Q + Q(A-HC) + C^{\mathrm{T}}V^{\mathrm{T}}VC & Q(B_f - HD_f) + C^{\mathrm{T}}V^{\mathrm{T}}VD_f \\ (B_f - HD_f)^{\mathrm{T}}Q + D_f^{\mathrm{T}}V^{\mathrm{T}}VC & -\beta^2 I + D_f^{\mathrm{T}}V^{\mathrm{T}}VD_f \end{bmatrix} > 0 \tag{11}$$

Notice that analytic solutions of MIs (10)∼(11) are difficult to get. By denoting $P = -Q, G = V^{\mathrm{T}}V, PH = Y$, one solution can be obtained from Theorem 1 as in [10], but the result is more conservative. The following Theorem 2 provides equivalent conditions to (10)∼(11). Since the limitation of space, its proof is omitted.

**Theorem 2.** Given $\gamma > 0$ and $\beta > 0$ the RFDF problem is solvable with $H, V = G^{1/2}$, iff there exist matrices $P > 0, Q < 0, G > 0, P_0 > 0, Q_0 < 0$, $H$ and $H_0$ satisfying MIs

$$\begin{bmatrix} M_{11} & PB_w + C^{\mathrm{T}}GD_w & P - C^{\mathrm{T}}H^{\mathrm{T}} & P \\ B_w^{\mathrm{T}}P + D_w^{\mathrm{T}}GC & M_{22} & 0 & -D_w^{\mathrm{T}}H^{\mathrm{T}} \\ P - HC & 0 & I & 0 \\ P & -HD_w & 0 & I \end{bmatrix} < 0 \tag{12}$$

$$\begin{bmatrix} N_{11} & QB_f + C^{\mathrm{T}}GD_f & C^{\mathrm{T}}H^{\mathrm{T}} + Q & Q \\ B_f^{\mathrm{T}}Q + D_f^{\mathrm{T}}GC & N_{22} & 0 & -D_f^{\mathrm{T}}H^{\mathrm{T}} \\ Q + HC & 0 & I & 0 \\ Q & -HD_f & 0 & I \end{bmatrix} > 0 \tag{13}$$

where

$$M_{11} = A^{\mathrm{T}}P + PA + C^{\mathrm{T}}GC - 2P_0P + 2P_0P_0 - C^{\mathrm{T}}H^{\mathrm{T}}H_0C - C^{\mathrm{T}}H_0^{\mathrm{T}}HC + C^{\mathrm{T}}H_0^{\mathrm{T}}H_0C$$
$$M_{22} = -\gamma^2 I + D_w\mathrm{T}GD_w - D_w^{\mathrm{T}}H^{\mathrm{T}}H_0D_w - D_w^{\mathrm{T}}H_0^{\mathrm{T}}HD_w + D_w^{\mathrm{T}}H_0^{\mathrm{T}}HD_w$$
$$N_{11} = A^{\mathrm{T}}Q + QA + C^{\mathrm{T}}GC + 2Q_0Q + 2QQ_0 - 2Q_0Q_0 + C^{\mathrm{T}}H^{\mathrm{T}}H_0C + C^{\mathrm{T}}H_0^{\mathrm{T}}HC - C^{\mathrm{T}}H_0^{\mathrm{T}}H_0C$$
$$N_{22} = -\beta^2 I + D_f^{\mathrm{T}}GD_f + D_f^{\mathrm{T}}H^{\mathrm{T}}H_0D_f + D_f^{\mathrm{T}}H_0^{\mathrm{T}}HD_f - D_f^{\mathrm{T}}H_0^{\mathrm{T}}H_0D_f$$

**Remark 2.** Given $\beta > 0$ and $\gamma > 0$ Theorem 2 provides a solvable condition of the RFDF problem. Although it is difficult to solve MIs (12)∼(13), an iterative LMI (ILMI) algorithm can be obtained by referring to [11,12], which is summarized in the following.

**Algorithm RFDF.**

**Step 1.** Define $\gamma > 0$, iterative steps $L > 0$, and small scalar $\sigma > 0$;

**Step 2.** Let $G = V^{\mathrm{T}}V, Q = -P, H = 0$, maximize $\beta$ subject to (10)∼(11), that is, LMIs in $P, G$. Assign $P_0 = P_{\mathrm{opt}}, H_0 = 0, Q_0 = Q_{\mathrm{opt}}$.

**Step 3.** With known $P_0, H_0, Q_0$, maximize $\beta$ subject to (12)∼(13), that is LMI in $P, Q, G, H$, obtain the new solution $P_{\mathrm{opt}}, Q_{\mathrm{opt}}, H_{\mathrm{opt}}$. Again, let $P_0 = P_{\mathrm{opt}}, H_0 = H_{\mathrm{opt}}, Q_0 = Q_{\mathrm{opt}}$. Denote $\beta^j_{\max} = \beta$, $j$ denotes the $j$-th iteration.

**Step 4.** Repeat Step 3, till $|\beta^j_{\max} - \beta^{j-1}_{\max}| < \sigma$ or $j > L$.

**Remark 3.** i) The proof of convergence of the ILMI algorithm is omitted here, please refer to [11,12]. ii) Under the assumption of $\delta_j(\dot{\boldsymbol{x}}, t) = 0$, $\delta_g(\boldsymbol{x}, t) = 0$ and $d = 0$, the objective function (7) remains only $\|T_{rf}(s)\|_- \to \max$. The worst case observer-based FDF in [11] is in fact a special case of Theorem 2 when $G = I$ and $\|T_{rf}(s)\|_- > \beta$.

## 4 Numerical example

Considering system (1) with parameters:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix},\ B = 0,\ \boldsymbol{B}_d = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix},\ \boldsymbol{B}_f = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ E_j = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},\ E_g = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

$$W_j = W_g = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C = [3 \quad 3],\ D = 0,\ D_d = 0.05,\ D_f = 1$$

Given $\gamma = 0.1$, the following results are obtained:

$$\beta_{\max} = 1.98,\ V = 1.9823,\ \boldsymbol{H} = \begin{bmatrix} 2.1951 \\ 5.1655 \end{bmatrix}$$

To show the effectiveness of the proposed ILMI algorithm, comparisons with early results in [10] and [11] are presented:

$$\text{Method in [10]} : \gamma = 0.1,\ \beta_{\max} = 0.3,\ V = 0.6310,\ \boldsymbol{H} = \begin{bmatrix} 0.4209 \\ 1.5554 \end{bmatrix}$$

$$\text{Method in [11]} : \gamma = 0.3434,\ \beta_{\max} = 0.9,\ V = 1,\ \boldsymbol{H} = \begin{bmatrix} 0.7424 \\ 0.4199 \end{bmatrix}$$

It can be seen from index $\gamma$ and $\beta$ that the ILMI algorithm has a better performance.

## 5 Conclusion

By combing robust control theory and observer-based FDI, this paper dealt with RFDF problem for uncertain linear systems with nonlinear perturbations. The main contributions of this paper include the formulation of the two-objective optimal problem, the derivation of sufficient conditions of solution existence, and the development of an ILMI algorithm. A numerical example is given to show the effectiveness of the proposed algorithm.

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