Multi-step Predictive Control of a PDF-shaping Problem[1)]

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Abstract A predictive control strategy is proposed for the shaping of the output probability density function (PDF) of linear stochastic systems. The B-spline neural network is used to set up the output PDF model and therefore converts the PDF-shaping into the control of B-spline weights vector. The Diophantine equation is then introduced to formulate the predictive PDF model, based on which a moving-horizon control algorithm is developed so as to realize the predictive PDF tracking performance.

Key words Probability density function (PDF), predictive control, B-spline functions, stochastic distribution control

1 Introduction

In many stochastic systems, it is required to control the whole shape of the output probability density function (PDF) instead of its mean and variance. Typical examples include the molecular weight distribution control[1~3] or particle size distribution control[4~6] in polymerization processes, particle size distribution control in crystallization and powder industries[7~10], pulp fibre length distribution control in paper industries[11], *etc.* The mean-variance control strategies, such as minimum variance control, self-tuning control, stochastic control for systems with Markovian jumping parameters and LQG control, *etc.*, are mainly used for systems subjected to Gaussian inputs. They may not work well for non-Gaussian stochastic systems due to the fact that the mean and variance are not adequate to decide the PDF shape for non-Gaussian or non-symmetric distributions. Motivated by the control requirements of general stochastic systems, a PDF-shaping method was proposed by Wang in 1998, the main idea of which is to design a crisp control input so that the system output PDF will follow the target PDF[12]. This method is more general than the traditional mean-variance control simply because the PDF contains more information than the mean and variance. A series of algorithms have been developed for the PDF-shaping control based on the B-spline models, input-output ARMAX models and multi-input-multi-output neural network models. So far the B-spline models are most commonly used for PDF control because they have general model formulation and therefore make the controller design convenient[12~14]. The B-spline methods have been proved to be efficient for the static PDF-shaping problem, however, for the dynamic systems, the PDF tracking performance is difficult to achieve especially when the non-linear dynamic is simplified to be a linear model.

Model predictive control (MPC) has the merits of model prediction, moving-horizon optimization and feedback correction[15]. The moving-horizon operation is to choose the control action by repeatedly solving an on-line optimal control problem, therefore compensates for model mismatch, time variation and disturbance, *etc.* in time. Feedback correction compares the actual output and the predictive model output so that the predictive model can be on-line modified. In this paper, the main features of MPC are embedded into the PDF-shaping control in order to improve the efficiency of the dynamic PDF control.

2 Predictive model of output PDF

Consider the following discretized model for a stochastic distribution system

$$x_{k+1} = f_1(x_k, u_k) \tag{1}$$

$$\gamma(y, u_k) = g_1(y, x_k, u_k), \quad y \in [a, b] \tag{2}$$

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where x_k is the state variable, u_k is the control input, $\gamma(y, u_k)$ is the PDF of the system output with y defined on a bounded interval $[a, b]$, k is the current sampling time, $f_1(\cdot)$ and $g_1(\cdot)$ are the functions of system dynamics and PDF expression. From (1) and (2) it can be seen that the shape of the output PDF can be adjusted by the control input u_k. Using the well-known B-spline neural network[16], $\gamma(y, u_k)$ can be approximated as follows[12~14],

$$\gamma(y, u_k) = \sum_{i=1}^{n} \omega_i(u_k) B_i(y) \tag{3}$$

where $B_i(y)(i = 1, \cdots, n)$ are the pre-specified basis functions defined on $[a, b]$, $\omega_i(i = 1, \cdots, n)$ are the corresponding weight coefficients. Consider the linear dynamics of the system and denote $b_i = \int_a^b B_i(y)\mathrm{d}y$, $L(y) = \dfrac{B_n(y)}{b_n}$, and $C_i(y) = B_i(y) - \dfrac{B_n(y)}{b_n} b_i$, the following decoupled state-space model of the output PDF can be established

$$\boldsymbol{\nu}_{k+1} = A\boldsymbol{\nu}_k + Bu_k \tag{4}$$

$$\gamma(y, u_k) = C(y)\boldsymbol{\nu}_k + L(y) \tag{5}$$

where A, B are the parameter matrices with proper dimensions; the vector $C(y) = [C_1(y), C_2(y), \cdots, C_{n-1}(y)]$ and the variable $L(y)$ are formulated from the basis functions; $\boldsymbol{\nu}_k = [\omega_1(u_k) \quad \omega_2(u_k) \quad \cdots \quad \omega_{n-1}(u_k)]^{\mathrm{T}}$ is the weight vector. As $L(y)$ is known when the basis functions are chosen, denote

$$f(y, k) = \gamma(y, u_k) - L(y) = C(y)\boldsymbol{\nu}_k \tag{6}$$

then (6) can be used for PDF control instead of (5). Using the back-shift operator z^{-1} in (4) and substituting the input-output form into (6), the following representation can be obtained

$$f(y, k) = C(y)(I - z^{-1}A)^{-1}Bu_{k-1} \tag{7}$$

where I is a unit matrix with dimension $(n - 1)$. According to the matrix theory, (7) can be expanded to

$$f(y, k) = \sum_{i=1}^{n-1} a_i f(y, k - i) + \sum_{j=0}^{n-2} C(y) D_j u_{k-j-1} \tag{8}$$

The second term in (8) can be further expanded to $C(y)D_j u_{k-j-1} = \sum_{i=1}^{n-1} d_{j,i} u_{k-j-1} C_i(y)$. Introducing the back-shift operator z^{-1}, (8) can be represented as

$$\alpha(z^{-1}) f(y, k) = \beta(z^{-1}, y) u_{k-1} \tag{9}$$

where

$$\alpha(z^{-1}) = 1 - \sum_{i=1}^{n-1} a_i z^{-i}, \quad \beta(z^{-1}, y) = \sum_{j=0}^{n-2} C(y) D_j z^{-j} \tag{10}$$

(9) is the input-output model of the output PDF. All the coefficients in (10) can be estimated by the least square identification algorithms when the data pairs $(u_k, \gamma(y, u_k))$ are available[14].

3 Output PDF predictive control algorithm

The following Diophantine function is introduced to construct the predictive PDF model

$$1 = G_q(z^{-1})\alpha(z^{-1}) + H_q(z^{-1})z^{-q} \tag{11}$$

where q is the step for prediction,

$$G_q(z^{-1}) = 1 + \sum_{i=1}^{q-1} g_{q,i} z^{-i} \tag{12}$$

$$H_q(z^{-1}) = \sum_{i=0}^{n-2} h_{q,i} z^{-i} \tag{13}$$

The coefficients in the Diophantine equation can be obtained by recursive development[15]. Multiplying $G_q(z^{-1})$ to both sides of (9) yield

$$G_q(z^{-1})\alpha(z^{-1})f(y,k) = G_q(z^{-1})\beta(z^{-1},y)u_{k-1} \tag{14}$$

Taking (12) into (14), the following form is obtained

$$f(y,k+q) = H_q(z^{-1})f(y,k) + G_q(z^{-1})\beta(z^{-1},y)u_{k+q-1} \tag{15}$$

Denote

$$S_q(z^{-1})G_q(z^{-1})\beta(z^{-1},y) = \sum_{i=0}^{n-2+q-1} s_{q,i}z^{-i} \tag{16}$$

and take $q = 1,2,\cdots,p$, the multi-step predictive PDF model can be established from (15) in the following matrix form

$$\boldsymbol{\Pi}(y,k,p) = \hat{\boldsymbol{H}}f(y,k) + \Omega(y)\boldsymbol{U}_k + \Phi\boldsymbol{\eta}_k \tag{17}$$

where

$$\hat{\boldsymbol{H}} = \begin{bmatrix} H_1(z^{-1}) \\ H_2(z^{-1}) \\ \vdots \\ H_p(z^{-1}) \end{bmatrix},\ \Omega(y) = \begin{bmatrix} s_{1,0} & 0 & 0 & 0 \\ s_{2,1} & s_{2,0} & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ s_{p,p-1} & \cdots & s_{p,1} & s_{p,0} \end{bmatrix},\ \Phi = \begin{bmatrix} s_{1,1} & s_{1,2} & \cdots & s_{1,n-2} \\ s_{2,2} & s_{2,3} & \cdots & s_{2,n-2+1} \\ \vdots & \vdots & \vdots & \vdots \\ s_{p,p} & s_{p,p+1} & \cdots & s_{p,n-2+p-1} \end{bmatrix}$$

$$\boldsymbol{\Pi}(y,k,p) = \begin{bmatrix} f(y,k+1) \\ f(y,k+2) \\ \vdots \\ f(y,k+p) \end{bmatrix},\ \boldsymbol{U}_k = \begin{bmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+p-1} \end{bmatrix},\ \boldsymbol{\eta}_k = \begin{bmatrix} u_{k-1} \\ u_{k-2} \\ \vdots \\ u_{k-(n-2)} \end{bmatrix}$$

The following Kullback-Leibler distance based performance function is constructed to realize the PDF-shaping control

$$J = \int_a^b \boldsymbol{\Pi}(y,k,p)^{\mathrm{T}}\log\frac{\boldsymbol{\Pi}(y,k,p)}{\boldsymbol{\Gamma}(y)}\mathrm{d}y + \boldsymbol{U}_k^{\mathrm{T}}Q\boldsymbol{U}_k \tag{18}$$

where $\boldsymbol{\Gamma}(y) = [g(y)-L(y), g(y)-L(y),\cdots,g(y)-L(y)]^{\mathrm{T}} \in R^{p\times 1}$ is the target vector corresponding to the given PDF $g(y)$, when $\boldsymbol{\Pi}(y,k,p) = \boldsymbol{\Gamma}(y)$, the first term in (18) is zero; the second term in (18) is the constraint to the control input, $Q \in R^{M\times M}$ is a diagonal weighting matrix. As the predictive step is normally different from the control step in real applications, and the control step should be shorter than the predictive one, the control step in the performance function is taken as M, $M \leqslant p$, that is, $\boldsymbol{U}_k = [u_k, u_{k+1},\cdots,u_{k+M-1}]^{\mathrm{T}}$. Accordingly,

$$\Omega(y) = \begin{bmatrix} s_{1,0} & 0 & \cdots & 0 \\ s_{2,1} & s_{2,0} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ s_{p,p-1} & s_{p,p-2} & \cdots & s_{p,p-M} \end{bmatrix}_{p\times M} \tag{19}$$

Denoting $\xi(y) = \hat{H}f(y,k) + \Phi\boldsymbol{\eta}_k$ as the known term at sampling time k and substituting (17) into (18), it can be obtained that

$$J = \int_a^b (\Omega(y)\boldsymbol{U}_k + \xi(y))^{\mathrm{T}}\log\frac{\Omega(y)\boldsymbol{U}_k + \xi(y)}{\boldsymbol{\Gamma}(y)}\mathrm{dy} + \boldsymbol{U}_k^{\mathrm{T}}Q\boldsymbol{U}_k \tag{20}$$

$$\frac{\partial J}{\partial \boldsymbol{U}_k} = \int \Big(\Omega(y)^{\mathrm{T}}\log\frac{\Omega(y)\boldsymbol{U}_k + \xi(y)}{\boldsymbol{\Gamma}(y)} + D\cdot(\Omega(y)\boldsymbol{U}_k + \xi(y))\Big)\mathrm{d}y + 2Q\boldsymbol{U}_k \tag{21}$$

$$\frac{\partial^2 J}{\partial \boldsymbol{U}_k^2} = \int_a^b (D\Omega(y))\mathrm{d}y + 2Q \tag{22}$$

where $D = \dfrac{\partial \log \Pi(y,k,p)}{\partial U} = \left[\dfrac{\Omega(1,:)}{\Omega(1,:)U+\xi(1)}; \dfrac{\Omega(2,:)}{\Omega(2,:)U+\xi(2)}; \cdots; \dfrac{\Omega(p,:)}{\Omega(p,:)U+\xi(p)}\right]^{\mathrm{T}}$, $\xi(i)$ and $\Omega(i,:)$, $(i = 1, 2, \ldots, p)$ are the ith row of Ω and the ith component of ξ, respectively. When $\dfrac{\partial^2 J}{\partial \boldsymbol{U}_k^2} > 0$, calculating the control input from $\partial J/\partial \boldsymbol{U}_k = 0$ can minimize J, and the predictive control input can be obtained in the following gradient form

$$\boldsymbol{U}_k = \boldsymbol{U}_{k-1} - \mu\Big(\int_a^b \Big(\Omega(y)^{\mathrm{T}}\log\frac{\Omega(y)U+\xi(y)}{\boldsymbol{\Gamma}(y)} + D\cdot(\Omega(y)U+\xi(y))\Big)\mathrm{d}y + 2QU\Big)|_{U=U_{k-1}} \tag{23}$$

In the control phase, $\boldsymbol{U}_k$ is calculated using equation (23), with $\dfrac{\partial^2 J}{\boldsymbol{U}_k^2} > 0$ being continuously validated. If $\dfrac{\partial^2 J}{\partial \boldsymbol{U}_k^2} > 0$ is not true, Q should be tuned to satisfy this condition again. The process goes on until the objective horizon is achieved.

4 Simulation study

Consider the following dynamic system (abstracted from a chemical reaction model)

$$\begin{aligned}
\dot{x}_1 &= k_1\cdot(x_{10}-x_1) - k_2x_1\\
\dot{x}_2 &= k_1(x_{20}-x_2) - k_3x_2x_3 - k_4x_1\\
\dot{x}_3 &= -k_1x_3 + k_4x_1 - k_5x_3^2\\
\dot{x}_4 &= k_1(x_{40}-x_4) + k_6x_2x_3 - k_7(x_4 - x_{400})
\end{aligned} \tag{24}$$

where x_1, x_2, x_3, x_4 are state variables; $k_1 \sim k_{10}, x_{20}, x_{40}, x_{400}$ are known model parameters, and their values are $k_1 = 0.0061, k_2 = 0.0081, k_3 = 26522, k_4 = 0.0049, k_5 = 3.4184e9, k_6 = 1.6290e8, k_7 = 0.0514, k_8 = 1.9777, k_9 = 1.7092e9, k_{10} = 26520, x_{20} = 1.7744, x_{40} = 298.15, x_{400} = 353.15$; the output PDF of the system is

$$\gamma(y,u) = k_1\cdot[k_8\cdot x_2\cdot x_3\cdot\alpha^{-(y-1)} + k_9x_3^2\cdot\alpha^{-(y-2)}(y-1)] \tag{25}$$

where

$$\alpha = 1 + \frac{k_8}{k_{10}} + \frac{k_5}{k_{10}}\frac{x_3}{x_2} + \frac{1}{k_{10}k_1x_2} \tag{26}$$

The control input u directly affects the states of the system, *i.e.*, $x_{10} = 0.0106\cdot(1-u)$; the definition domains for the output PDF and control input are: $y \in [1, 2000]$ and $u \in [0.3, 0.7]$, respectively. The PDF in this case is approximated by seven third-order B-spline functions (modeling details are omitted due to page limit). The target output PDF is chosen to be the one corresponding to $u = 0.6$. In the predictive controller design, both the predictive step and the control step are taken to be 5. The simulation results are shown in Figs. 1$\sim$3. Figs. 1 and 2 show the development of the control input and the output PDFs during the control process. Fig. 3 shows the final tracking effect of the output PDF to the targeted one. From the simulation, it can be concluded that the predictive control algorithm fulfills the PDF-shaping control effectively.

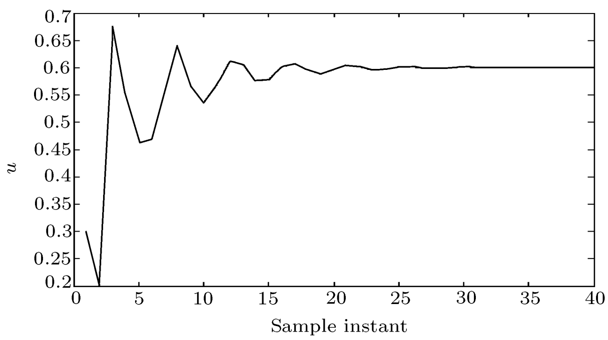

Fig. 1 The control input during the control process

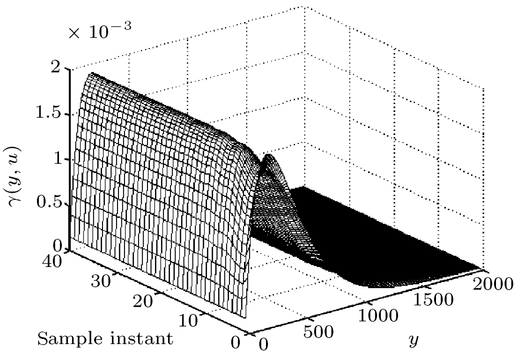

Fig. 2 The output PDFs during the control

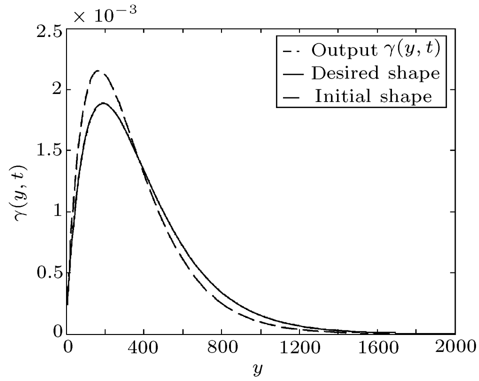

Fig. 3 The final output PDF and the target PDF

5 Conclusions

In this paper, a multi-step predictive control algorithm has been put forward to control the system output PDF. Firstly, the B-spline functions are used to approximate the output PDF of a general stochastic system, then the Diophantine equation is introduced to construct the predictive output PDF model, and a performance function is formulated according to the PDF tracking requirement. Finally, the predictive controller is obtained in the gradient form by a global optimisation operation. Simulation study shows that this controller is effective for the dynamic PDF-shaping. Compared with the previous PDF control algorithm[14], the predictive control strategy has advantages in dealing with PDF-shaping problems of dynamic systems with uncertainties. Analysis of the system stability, robustness, disturbance rejection, *etc.* will be studied in the future work.

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References

1 Crowley T J, Meadows E S, Kostoulas E, Doyle III F J. Control of particle size distribution described by a population balance model of semibatch emulsion polymerisation. *Journal of Process Control*, 2000, **10**(5): 419∼432

2 Flores-Cerrillo J, MacGregor J F. Control of particle size distributions in emulsion semibatch polymerization using mid-course correction polices. *Industry Engineering Chemical Research*, 2002, **41**(7): 1805∼1814

3 Immanuel C D, Doyle III F J. Open-loop control of particle size distribution on semi-batch emulsion copolymerisation using a genetic algorithm. *Chemical Engineering Science*, 2002, **57**(20): 4415∼4427

4 Clarke-Pringle T L, MacGregor J F. Optimization of molecular-weight distribution using batch-to-batch adjustments. *Industry Engineering Chemical Research*, 1998, **37**(9): 3660∼3669

5 Echevarria A, Leiza J R, Cal J C. de la, Asua J M. Molecular weight distribution control in emulsion polymerisation. *American Institute of Chemical Engineers Journal*, 1998, **44**(7): 1667~1679

6 Vicente M, BenAmor S, Gugliotta L M, Leiza J R, Asua J M. Control of molecular weight distribution in emulsion polymerization using on-line reaction calorimetry. *Industry Engineering Chemical Research*, 2001, **40**(1): 218~227

7 Braatz R D. Advanced control of crystallization processes. *Annual Reviews in Control*, 2002, **26**(1): 87~99

8 Eek R A, Bosgra O H. Controllability of particulate processes in relation to the sensor characteristics. *Powder Technology*, 2000, **108**(2~3): 137~146

9 Gommeren H J C, Heitzmann D A, Moolenaar J A C, Scarlett B. Modelling and control of a jet mill plant, *Powder Technology*, 2000, **108**(2~3): 147~154

10 Ma D L, Tafti D K, Braatz R D. Optimal control and simulation of multidimensional crystallization processes. *Computers and Chemical Engineering*, 2002, **26**(7~8): 1103~1116

11 Smook G A. Handbook for Pulp and Paper Technologists, Angus Wilde Publications, 1998

12 Wang H. Control of the output probability density functions for a class of nonlinear stochastic systems. In: Proceedings of the IFAC Workshop on Algorithms and Architectures for Real-time Control, Cancun, Mexico: 1998, 95~99

13 Wang H. Robust control of the output probability density functions for multivariable stochastic systems. *IEEE Transactions on Automatic Control*, 1999, **44**(11): 2103~2107

14 Wang H. Bounded Dynamic Stochastic Systems: Modelling and Control. Springer-Verlag London Ltd, 2000

15 Shu Di-Qian. Predictive Control and Application. Beijing: Mechanical Industry Publishing Company, 1996

16 Cheng Zheng-Xing, Li Shui-Gen. Numerical Approximation and Ordinary Differential Equation Solution. Xi′an: Xi′an Jiaotong University Publishing Company, 2000: 90~91

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