Optimal Control for Nonlinear Interconnected Large-scale Systems: A Successive Approximation Approach[1)]

TANG Gong-You SUN Liang

(*College of Information Science and Engineering, Ocean University of China, Qingdao* 266071)
(E-mail: gtang@ouc.edu.cn)

Abstract The optimal control problem for nonlinear interconnected large-scale dynamic systems is considered. A successive approximation approach for designing the optimal controller is proposed with respect to quadratic performance indexes. By using the approach, the high order, coupling, nonlinear two-point boundary value (TPBV) problem is transformed into a sequence of linear decoupling TPBV problems. It is proven that the TPBV problem sequence uniformly converges to the optimal control for nonlinear interconnected large-scale systems. A suboptimal control law is obtained by using a finite iterative result of the optimal control sequence.

Key words Nonlinear large-scale systems, optimal control, suboptimal control, successive approximation approach

1 Introduction

Many physical systems are composed of interconnections of lower-dimensional subsystems with interconnections among them. It is unpractical to analyse and synthesize interconnected large-scale systems with general control theories and methods, because that will lead to huge computation time and memory. During the past few decades, the problem of analysis and synthesis for dynamic large-scale systems has received considerable attention. Based on the characteristics of large-scale systems many results on this subject have been proposed, mainly about modeling[1,2], stability[3], stabilization[4,5], robust control[6,7], decentralized control[6~8], hierarchical control[9], and so on. It is useful to reduce computation through modeling and decomposition techniques, but the simplified computation often brings a conservative result. Therefore, how to find a simpler analysis method or control strategy for large-scale systems with a better result is faced by researchers. The optimal control problem of a general nonlinear system always leads to a high order, coupling, nonlinear TPBV problem. But for the general regulation problem of nonlinear systems, with the exception of simplest cases, they are impossible to be solved analytically[12]. This has inspired researchers to look for some approaches to approximately obtain the solution to the nonlinear TPBV problem as well as obtain a suboptimal feedback control for nonlinear interconnected large-scale dynamic systems.

The main objective of this article is to address the problem of optimally controlling a nonlinear interconnected large-scale dynamic system modeled as an interconnection of subsystems. A successive approximation approach for designing the optimal controller is proposed with respect to quadratic performance indexes. By using the approach, a high order, coupling, nonlinear TPBV problem is transformed into a sequence of linear decoupling TPBV problems. We prove that the TPBV problem sequence uniformly converges to the optimal control for nonlinear interconnected large-scale systems. A suboptimal control law is obtained by using a finite iterative result of the optimal control sequence. A simulation example shows that the successive approximation optimal control algorithm is effective.

2 Problem statement

Consider a nonlinear interconnected large-scale systems described by

$$\begin{aligned} &\dot{\boldsymbol{x}}_i(t) = A_i\boldsymbol{x}_i(t) + B_i\boldsymbol{u}_i(t) + \boldsymbol{f}_i(\boldsymbol{x}), \quad t > t_0 \\ &\boldsymbol{x}_i(t_0) = \boldsymbol{x}_{i0}, \quad i = 1, 2, \cdots, N \end{aligned} \tag{1}$$

1) Supported by National Natural Science Foundation of P. R. China (60074001) and the Natural Science Foundation of Shandong Province (Y2000G02)
Received February 24, 2003; in revised form June 29, 2004

which can be decomposed into N subsystems, where $\boldsymbol{x}_i \in R^{n_i}, \boldsymbol{u}_i \in R^{m_i}$ are the state vectors and the control vectors, respectively; $\boldsymbol{f}_i : C^1(R^n) \rightarrow U_i \subset R^{n_i}$; A_i and B_i are constant matrices of appropriate dimensions; $\boldsymbol{x} = (\boldsymbol{x}_1^{\mathrm{T}}, \boldsymbol{x}_2^{\mathrm{T}}, \cdots, \boldsymbol{x}_N^{\mathrm{T}})^{\mathrm{T}}, n_1 + n_2 + \cdots + n_N = n$. Assume that (A_i, B_i) are controllable and nonlinear interconnected terms $\boldsymbol{f}_i(\boldsymbol{x})$ satisfy the conditions described by

$$\|\boldsymbol{f}_i(\boldsymbol{x})\| \leqslant c\|\boldsymbol{x}\|, \|\boldsymbol{f}_i(\boldsymbol{x}) - \boldsymbol{f}_i(\boldsymbol{y})\| \leqslant h\|\boldsymbol{x} - \boldsymbol{y}\|, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in R^n \tag{2}$$

where $\|\cdot\|$ represents the form of vectors; c and h are known positive constants. The problem is to find a control law that minimizes the quadratic cost functional described by

$$J = \frac{1}{2}\sum_{i=1}^{N}\Big\{\boldsymbol{x}_i^{\mathrm{T}}(t_f)F_i\boldsymbol{x}_i(t_f) + \int_{t_0}^{t_f}[\boldsymbol{x}_i^{\mathrm{T}}(t)Q_i\boldsymbol{x}_i(t) + \boldsymbol{u}_i^{\mathrm{T}}(t)R_i\boldsymbol{u}_i(t)]\mathrm{d}t\Big\} \tag{3}$$

where matrices F_i, Q_i and R_i satisfy general linear-quadratic regulator conditions.

Applying the maximum principle to (1) and (3), the necessary condition of the optimal control problem is described by

$$\begin{aligned} &\dot{\boldsymbol{x}}_i(t) = A_i\boldsymbol{x}_i(t) - S_i\boldsymbol{\lambda}_i(t) + \boldsymbol{f}_i(\boldsymbol{x}), \quad t_0 < t \leqslant t_f \\ &-\dot{\boldsymbol{\lambda}}_i(t) = Q_i\boldsymbol{x}_i(t) + A_i^{\mathrm{T}}\boldsymbol{\lambda}_i(t) + \sum_{j=1}^{N}\boldsymbol{\sigma}_{ij}\boldsymbol{\lambda}_j(t), \quad t_0 \leqslant t < t_f \\ &\boldsymbol{x}_i(t_0) = \boldsymbol{x}_{i0}, \quad \boldsymbol{\lambda}_i(t_f) = F_i\boldsymbol{x}_i(t_f), \quad i = 1, 2, \cdots, N \end{aligned} \tag{4}$$

where $S_i = B_iR_i^{-1}B_i^{\mathrm{T}}, \boldsymbol{\sigma}_{ij} = \partial\boldsymbol{f}_j/\partial\boldsymbol{x}_i$. The optimal control law can be described by

$$\boldsymbol{u}_i^*(t) = -R_i^{-1}B_i^{\mathrm{T}}\boldsymbol{\lambda}_i(t), \quad t_0 < t \leqslant t_f, i = 1, 2, \cdots, N \tag{5}$$

(4) are nonlinear interconnected n-th order large-scale TPBV problems, which can be decomposed into N subproblems, where nonlinear interconnected function vectors $\boldsymbol{f}_i$ are the nonlinear relating terms of nonlinear large-scale system (1). For general nonlinear function vectors $\boldsymbol{f}_i$, it is very difficult to solve this problem accurately. On the other hand, even if the large-scale interconnected nonlinear TPBV problems in (4) are theoretically solvable, the computation load can be extremely tremendous because of large-scale system′s regular characteristic of high dimensions. Therefore, it is necessary to find some approximate approaches for solving the nonlinear large-scale interconnected TPBV problems in (4).

3 Preliminaries

Consider the nonlinear interconnected time-varying autonomous large-scale system that can be decomposed into N subsystems, which can be described by

$$\begin{aligned} &\dot{\boldsymbol{x}}_i(t) = G_i(t)\boldsymbol{x}_i(t) + \boldsymbol{f}_i(\boldsymbol{x}), \quad t_0 < t \leqslant t_f \\ &\boldsymbol{x}_i(t_0) = \boldsymbol{x}_{i0}, \quad i = 1, 2, \cdots, N \end{aligned} \tag{6}$$

where $\boldsymbol{x}_i \in R^{n_i}$ are state vectors, $\boldsymbol{f}_i : C^1(R^n) \rightarrow U_i \subset R^{n_i}$ satisfy the conditions in (2), $G_i(t)$ are continuous time-varying matrices of appropriate dimensions.

Define the function vector sequence $\{\boldsymbol{x}_i^{(k)}(t)\}$ as the solution sequence of vector integral equation sequence described by

$$\begin{aligned} &\boldsymbol{x}_i^{(0)}(t) = \varPhi_i(t, t_0)\boldsymbol{x}_i(t_0) \\ &\boldsymbol{x}_i^{(k)}(t) = \varPhi_i(t, t_0)\boldsymbol{x}_i(t_0) + \int_{t_0}^{t}\varPhi_i(t, \tau)\boldsymbol{f}_i(\boldsymbol{x}^{(k-1)}(\tau))\mathrm{d}\tau \\ &t_0 < t \leqslant t_f, \quad i = 1, 2, \cdots, N, \quad k = 1, 2, \cdots \end{aligned} \tag{7}$$

where $\varPhi_i(t, t_0)$ are state transition matrices of time-varying matrices $G_i(t)$.

Lemma 1. The solution sequence of vector integral equation sequence (7) uniformly converges to solution of time-varying nonlinear large-scale system (6).

Proof. Consider $\{\boldsymbol{x}_i^{(k)}(t)\}$ as a sequence in $C^N[t_0,t_f]$. From (7)

$$\boldsymbol{x}_i^{(1)}(t)-\boldsymbol{x}_i^{(0)}(t)=\int_{t_0}^{t}\Phi_i(t,\tau)\boldsymbol{f}_i(\boldsymbol{x}^{(0)}(\tau))\mathrm{d}\tau \tag{8}$$

Letting $a=\sup\limits_{t_0\leqslant t\leqslant t_f}\|\Phi_i(t,t_0)\|, b=\sup\limits_{t_0\leqslant t\leqslant t_f}\|\boldsymbol{x}^{(0)}(t)\|$, and noting that $\boldsymbol{f}_i$ satisfies inequalities in (2), we can obtain

$$\|\boldsymbol{x}_i^{(1)}(t)-\boldsymbol{x}_i^{(0)}(t)\|\leqslant\int_{t_0}^{t}ac\|\boldsymbol{x}^{(0)}(\tau)\|\mathrm{d}\tau\leqslant abc\int_{t_0}^{t}\mathrm{d}\tau=abc(t-t_0),\quad t_0<t\leqslant t_f,\ i=1,2,\cdots,N \tag{9}$$

Similarly, from (7), we obtain

$$\boldsymbol{x}_i^{(2)}(t)-\boldsymbol{x}_i^{(1)}(t)=\int_{t_0}^{t}\Phi_i(t,\tau)[\boldsymbol{f}_i(\boldsymbol{x}^{(0)}(\tau))-\boldsymbol{f}_i(\boldsymbol{x}^{(0)}(\tau))]\mathrm{d}\tau,\quad t_0<t\leqslant t_f,\ i=1,2,\cdots,N \tag{10}$$

Moreover, from (2) one gets

$$\begin{aligned}&\|\boldsymbol{x}_i^{(2)}(t)-\boldsymbol{x}_i^{(1)}(t)\|\leqslant ah\int_{t_0}^{t}\|\boldsymbol{x}^{(1)}(\tau)-\boldsymbol{x}^{(0)}(\tau)\|\mathrm{d}\tau\leqslant\\&ah\sum_{j=1}^{N}\int_{t_0}^{t}\|\boldsymbol{x}_j^{(1)}(\tau)-\boldsymbol{x}_j^{(0)}(\tau)\|\mathrm{d}\tau\leqslant ah\sum_{j=1}^{N}\int_{t_0}^{t}abc(t-t_0)\mathrm{d}\tau\leqslant\\&Na^2bch\frac{(t-t_0)^2}{2},\quad t_0<t\leqslant t_f,\quad i=1,2,\cdots,N\end{aligned}$$

By mathematical induction, we can obtain

$$\|\boldsymbol{x}_i^{(k)}(t)-\boldsymbol{x}_i^{(k-1)}(t)\|\leqslant N^{k-1}a^kbch^{k-1}\frac{(t-t_0)^k}{k!},\quad t_0<t\leqslant t_f, i=1,2,\cdots,N \tag{12}$$

When k is sufficiently large, for any positive integer M, one gets

$$\begin{aligned}&\|\boldsymbol{x}_i^{(k+M)}(t)-\boldsymbol{x}_i^{(k-1)}(t)\|\leqslant\sum_{j=k}^{k+M}N^{j-1}a^jbch^{j-1}\frac{t-t_0}{j!}=\\&\frac{bc}{Nh!}\left(\frac{(Nah(t-t_0))^k}{k!}+\frac{(Nah(t-t_0))^{k+1}}{(k+1)!}+\cdots+\frac{(Nah(t-t_0))^{k+M}}{(k+M)!}\right)\leqslant\\&\frac{bc(Nah(t-t_0))^k}{Nhk!}\left(1+\frac{Nah(t-t_0)}{k+1}+\frac{(Nah(t-t_0))^2}{(k+1)(k+2)}+\cdots+\frac{(Nah(t-t_0))^M}{(k+1)(k+2)\cdots(k+M)}\right)\leqslant\\&\frac{bc(Nah(t-t_0))^k}{Nhk!}\left(1+Nah(t-t_0)+\frac{(Nah(t-t_0))^2}{2!}+\cdots+\frac{(Nah(t-t_0))^M}{M!}\right)\leqslant\\&\frac{bc(Nah(t-t_0))^k}{Nhk!}\sum_{k=0}^{\infty}\frac{(Nah(t-t_0))^j}{j!}=\frac{bc(Nah(t-t_0))^k}{Nhk!}e^{Nah(t-t_0)},\quad t_0<t\leqslant t_f, i=1,2,\cdots,N\end{aligned} \tag{13}$$

Inequalities in (13) imply

$$\lim_{k\to\infty}\|\boldsymbol{x}_i^{(k+M)}(t)-\boldsymbol{x}_i^{(k)}(t)\|=0,\quad\forall M>0,\ t_0<t\leqslant t_f,\ i=1,2,\cdots,N \tag{14}$$

This means that $\{\boldsymbol{x}_i^{(k)}(t)\}$ is a Cauchy sequence in $C^N[t_0,t_f]$. Therefore, this sequence is uniformly convergent. Since M is arbitrary, the limit of this sequence is clearly the solution of large-scale system (6). The proof is complete.

4 Main results

We now consider a sequence of linear TPBV problems described by

$$\dot{\boldsymbol{x}}_i^{(k)}(t)=A_i\boldsymbol{x}_i^{(k)}(t)-S_i\boldsymbol{\lambda}_i^{(k)}(t)+\boldsymbol{f}_i(\boldsymbol{x}^{(k-1)}),\quad t_0<t\leqslant t_f$$

$$-\dot{\boldsymbol{\lambda}}_i^{(k)}(t) = Q_i\boldsymbol{x}_i^{(k)}(t) + A_i^{\mathrm{T}}\boldsymbol{\lambda}_i^{(k)}(t) + \sum_{j=1}^{N}\boldsymbol{\sigma}_{ij}^{(k-1)}\boldsymbol{\lambda}_j^{(k-1)}, \quad t_0 \leqslant t < t_f \tag{15}$$
$$\boldsymbol{x}_i^{(k)}(t_0) = \boldsymbol{x}_{i0}, \quad \boldsymbol{\lambda}_i^{(k)}(t_f) = F_i\boldsymbol{x}_i(t_f), \quad i = 1, 2, \cdots, N, \quad k = 1, 2, \cdots$$

where $\boldsymbol{f}_i(\boldsymbol{x}^{(0)}) = 0, \boldsymbol{\lambda}_i^{(0)}(t) = 0, \boldsymbol{\sigma}_{ij}^{(k)} = [\partial\boldsymbol{f}_j/\partial\boldsymbol{x}_i]_{\boldsymbol{x}_i = \boldsymbol{x}_i^{(k)}}$. And define the control vector sequence as

$$\boldsymbol{u}_i^{(k)}(t) = -R_i^{-1}B_i^{\mathrm{T}}\boldsymbol{\lambda}_i^{(k)}(t), \quad t_0 < t \leqslant t_f, \quad i = 1, 2, \cdots, N, \quad k = 1, 2, \cdots \tag{16}$$

Theorem 1. The control vector sequence in (16) that satisfies linear TPBV problem sequence (15) uniformly converges to the optimal control law $\boldsymbol{u}_i^*(t)$ for large-scale system (1) with quadratic cost functional (3).

Proof. We first prove the solvability of TPBV problem sequence (15). When $k = 1$, from linear TPBV problem sequence (15), we can obtain

$$\begin{aligned}
&\dot{\boldsymbol{x}}_i^{(1)}(t) = A_i\boldsymbol{x}_i^{(1)}(t) - S_i\boldsymbol{\lambda}_i^{(1)}(t), \quad t_0 < t \leqslant t_f \\
&-\dot{\boldsymbol{\lambda}}_i^{(1)}(t) = Q_i\boldsymbol{x}_i^{(1)}(t) + A_i^{\mathrm{T}}\boldsymbol{\lambda}_i^{(1)}(t), \quad t_0 \leqslant t < t_f \\
&\boldsymbol{x}_i^{(1)}(t_0) = \boldsymbol{x}_{i0}, \boldsymbol{\lambda}_i^{(1)}(t_f) = F_i\boldsymbol{x}_i(t_f), \quad i = 1, 2, \cdots, N
\end{aligned} \tag{17}$$

This is a linear TPBV problem. Letting $\boldsymbol{\lambda}_i^{(1)}(t) = P_i(t)\boldsymbol{x}_i^{(1)}(t)$, $\boldsymbol{x}_i^{(1)}(t)$ can be obtained from

$$\begin{aligned}
&\dot{\boldsymbol{x}}_i^{(1)}(t) = (A_i - S_iP_i(t))\boldsymbol{x}_i^{(1)}(t), \quad t_0 < t \leqslant t_f \\
&\boldsymbol{x}_i^{(1)}(t_0) = \boldsymbol{x}_{i0}, \quad i = 1, 2, \cdots, N
\end{aligned} \tag{18}$$

where $P_i(t)$ is the unique semi-positive definite matrix of the following *Riccati* matrix differential equation

$$\begin{aligned}
&\dot{P}_i(t) + P_i(t)A_i + A_i^{\mathrm{T}}P_i(t) - P_i(t)S_iP_i(t) + Q_i = 0, \quad t_0 \leqslant t < t_f \\
&P_i(t_f) = Q_f, \quad i = 1, 2, \cdots, N
\end{aligned} \tag{19}$$

From (18), we can obtain $\boldsymbol{x}_i^{(1)}$. Therefore, we can easily obtain $\boldsymbol{\lambda}_i^{(1)}$, $\boldsymbol{f}_i(\boldsymbol{x}^{(1)})$ and $\boldsymbol{\sigma}_{ij}^{(1)}$.

Assume that function vectors $\boldsymbol{x}_i^{(k-1)}(t)$ and $\boldsymbol{\lambda}_i^{(k-1)}(t)$ have been obtained in the $(k-1)$-th iteration. Hence, we can easily obtain $\boldsymbol{f}_i(\boldsymbol{x}^{(k-1)})$ and $\boldsymbol{\sigma}_{ij}^{(k-1)}$. In the k-th iteration, from (15) we can obtain

$$\begin{aligned}
&\dot{\boldsymbol{x}}_i^{(k)}(t) = A_i\boldsymbol{x}_i^{(k)}(t) - S_i\boldsymbol{\lambda}_i^{(k)}(t) + \boldsymbol{f}_i(\boldsymbol{x}^{(k-1)}), \quad t_0 < t \leqslant t_f \\
&-\dot{\boldsymbol{\lambda}}_i^{(k)}(t) = Q_i\boldsymbol{x}_i^{(k)}(t) + A_i^{\mathrm{T}}\boldsymbol{\lambda}_i^{(k)}(t) + \sum_{j=1}^{N}\boldsymbol{\sigma}_{ij}^{(k-1)}\boldsymbol{\lambda}_j^{(k-1)}, \quad t_0 \leqslant t < t_f \\
&\boldsymbol{x}_i^{(k)}(t_0) = \boldsymbol{x}_{i0}, \quad \boldsymbol{\lambda}_i^{(k)}(t_f) = F_i\boldsymbol{x}_i(t_f), \quad i = 1, 2, \cdots, N
\end{aligned} \tag{20}$$

Note that $\boldsymbol{f}_i(\boldsymbol{x}^{(k-1)}$, $\boldsymbol{\lambda}_j^{(k-1)}$ and $\boldsymbol{\sigma}_{ij}^{(k-1)}$ are known. Therefore, (20) is a linear nonhomogeneous TPBV problem. Let

$$\boldsymbol{\lambda}_i^{(k)}(t) = P_i(t)\boldsymbol{x}_i^{(k)}(t) + \boldsymbol{g}_i^{(k)}(t), \quad i = 1, 2, \cdots, N \tag{21}$$

Substituting (21) into (20), one gets a sequence of adjoint vector differential equations

$$\begin{aligned}
&\dot{\boldsymbol{g}}_i^{(k)}(t) = (P_i(t)S_i - A_i^{\mathrm{T}})\boldsymbol{g}_i^{(k)}(t) - P_i(t)\boldsymbol{f}_i(\boldsymbol{x}^{(k-1)}) - \sum_{j=1}^{N}\boldsymbol{\sigma}_{ij}^{(k-1)}\boldsymbol{\lambda}_j^{(k-1)}, t_0 \leqslant t, t_f \\
&\boldsymbol{g}_i^{(k)}(t_f) = \mathbf{0}, \quad i = 1, 2, \cdots, N, \quad k = 1, 2, \cdots
\end{aligned} \tag{22}$$

and a sequence of state equations

$$\begin{aligned}
&\dot{\boldsymbol{x}}_i^{(k)}(t) = (A_i - S_iP_i(t))\boldsymbol{x}_i^{(k)}(t) - S_i\boldsymbol{g}_i^{(k)}(t) + \boldsymbol{f}_i(\boldsymbol{x}^{(k-1)}), \quad t_0 < t \leqslant t_f \\
&\boldsymbol{x}_i^{(k)}(0) = \boldsymbol{x}_{i0}, \quad i = 1, 2, \cdots, N, \quad k = 1, 2, \cdots
\end{aligned} \tag{23}$$

From (22), we can obtain $\boldsymbol{g}_i^{(k)}$ by reversing integral. Substituting $\boldsymbol{g}_i^{(k)}$ into (23), $\boldsymbol{x}_i^{(k)}$ can be obtained. Therefore, we can easily obtain $\boldsymbol{\lambda}_i^{(k)}$, $\boldsymbol{f}_i(\boldsymbol{x}^{(k)})$ and $\boldsymbol{\sigma}_{ij}^{(k)}$. We have proved TPBV problem sequence (15) is solvable.

Secondly, we prove TPBV problem sequence (15) is uniformly convergent. According to Lemma 1, the solutions for the state equations in (23) and the adjoint vector differential equations in (22) are uniformly convergent. Similarly, the solution sequence of TPBV problem sequence (15) uniformly converges to the solution of large-scale nonlinear interconnected TPBV problem (4). According to (16), $\{\boldsymbol{u}_i^{(k)}(t)\}$ is also uniformly convergent, and uniformly converges to optimal control $\boldsymbol{u}_i^*(t)$, *i.e.*,

$$\begin{aligned}&\boldsymbol{u}_i^*(t) = \lim_{k\to\infty}\boldsymbol{u}_i^{(k)}(t) = -R_i^{-1}B_i^{\mathrm{T}}\lfloor P_i(t)\boldsymbol{x}_i(t) + \lim_{k\to\infty}\boldsymbol{g}_i^{(k)}(t)\rfloor\\ &t_0 \leqslant t \leqslant t_f, \quad i = 1, 2, \cdots, N\end{aligned} \tag{24}$$

The proof is complete. □

In fact, we can not obtain the solution of this problem in case of $k \to \infty$. We may, in practical applications, obtain a suboptimal control law by replacing $k \to \infty$ with $k = M$ in (24), *i.e.*, consider the $\boldsymbol{g}_i^{(M)}(t)$ approximately as its limit. Therefore, according to cost functional (3) an M-th order suboptimal control law of large-scale system (1) is obtained as follows

$$\boldsymbol{u}_{iM}(t) = -R_i^{-1}B_i^{\mathrm{T}}\lfloor P_i(t)\boldsymbol{x}_i(t) + \boldsymbol{g}_i^{(M)}(t)\rfloor, \quad t_0 \leqslant t \leqslant t_f, \quad i = 1, 2, \cdots, N \tag{25}$$

Remark 1. Note that $\boldsymbol{x}_i(t)$ in the first term of (25) is an accurate solution in case of $k \to \infty$, and only $\boldsymbol{g}_i^{(M)}(t)$ in the second term is replacing its limit with the M-th approximate result. Therefore, suboptimal control law $\boldsymbol{u}_{iM}(t)$ is closer to optimal control law $\boldsymbol{u}_i^*(t)$ than $\boldsymbol{u}_i^{(M)}(t)$.

A successive approximation process to obtain suboptimal control law (25) is proposed as follows.

Step1. Obtain the semi-positive definite matrices $P_i(t)$ from the *Riccati* matrix differential equations in (19). Give a positive constant ε. Let $k = 1$, $M = 1$ and $\boldsymbol{x}_i^{(0)}(t) = \boldsymbol{g}_i^{(0)}(t) = \boldsymbol{g}_i^{(1)}(t) = \mathbf{0}$.

Step2. From (18), we obtain $\boldsymbol{x}_i^{(1)}$. And then $\boldsymbol{\lambda}_i^{(1)}$, $\boldsymbol{f}_i(\boldsymbol{x}^{(1)})$ and $\boldsymbol{\sigma}_{ij}^{(1)}$ are obtained. Get $\boldsymbol{u}_{i1}(t)$ from (25) and J_1 from (3). Let $k = k + 1$.

Step3. Letting $M = k$, we find $\boldsymbol{g}_i^{(k)}$ from (22). Get $\boldsymbol{u}_{iM}(t)$ from (25). Calculate J_M according to (3).

Step4. If $\left|\dfrac{J_M - J_{M-1}}{J_M}\right| < \varepsilon$, then stop and put out the suboptimal control law $\boldsymbol{u}_{iM}(t)$.

Step5. Obtain $\boldsymbol{x}_i^{(k)}$ from (23). Consequently, $\boldsymbol{\lambda}_i^{(k)}$, $\boldsymbol{f}_i(\boldsymbol{x}^{(1)})$ and $\boldsymbol{\sigma}_{ij}^{(k)}$ are obtained. Letting $k = k + 1$, go to Step 3.

Remark 2. According to Lemma 1, terminal time t_f of the cost functional (3) may be as long as possible. In practical control systems, we may consider $t_f \to \infty$ when t_f is large enough. Therefore, this approach is also applicable for the case of $t_f \to \infty$. Cost functional (3) becomes

$$J = \frac{1}{2}\sum_{i=1}^{N}\int_{t_0}^{\infty}[\boldsymbol{x}_i^{\mathrm{T}}(t)Q_i\boldsymbol{x}_i(t) + \boldsymbol{u}_i^{\mathrm{T}}(t)R_i\boldsymbol{u}_i(t)]\mathrm{d}t \tag{26}$$

Accordingly, the following algebraic Riccati matrix equation is used instead of the Riccati matrix differential equation (19).

$$P_iA_i + A_i^{\mathrm{T}}P_i - P_iS_iP_i + Q_i = 0, \quad i = 1, 2, \cdots, N \tag{27}$$

where the solution P_i is a unique positive definite constant matrix.

5 Example

Consider the two order nonlinear composite system described by

$$\begin{aligned}&\dot{x}_1 = x_1 + u_1 - x_1^3 + x_2^2\\ &\dot{x}_2 = -x_2 + u_2 + x_1x_2 + x_2^3\\ &x_1(0) = 0, \quad x_2(0) = 0.8\end{aligned} \tag{28}$$

The cost functional is $J = \frac{1}{2}\sum_{i=1}^{2}\int_0^{\infty}(x_i^2 + u_i^2)\mathrm{d}t$.

Note that $A_1 = B_1 = B_2 = 1$, $A_2 = -1$, $Q_1 = Q_2 = R_1 = R_2 = 1$, $f_1(x) = x_2^2 - x_1^3$, $f_2(x) = x_1 x_2 + x_2^3$. From *Riccati* equations in (27), we can obtain $P_1^2 - 2P_1 - 1 = 0$, $P_2^2 + 2P_2 - 1 = 0$. We get $P_1 = 1 + \sqrt{2}$, $P_2 = -1 + \sqrt{2}$. According to the approximation approach, when $k = 1$, substituting P_1, P_2 and the initial conditions of system (28) into (22), one has

$$\begin{aligned}
&\dot{g}_1^{(k)}(t) = \sqrt{2}g_1^{(k)}(t) - (1+\sqrt{2})(x_2^{(k-1)^2} - x_1^{(k-1)^3}) + 3x_1^{(k-1)^2}\lambda_1^{(k-1)} - x_2^{(k-1)}\lambda_2^{(k-1)} \\
&\dot{g}_2^{(k)}(t) = \sqrt{2}g_2^{(k)}(t) + (\sqrt{2}-1)(x_1^{(k-1)}x_2^{(k-1)} + x_2^{(k-1)^3}) - 2x_2^{(k-1)}\lambda_1^{(k-1)} + 3x_2^{(k-1)^2}\lambda_2^{(k-1)} \\
&g_1^{(k)}(\infty) = g_2^{(k)}(\infty) = 0, \quad k = 1, 2, \cdots
\end{aligned} \tag{29}$$

We can obtain $g_i^{(k)}$ from (29). And from (25) we get

$$\begin{aligned}
&u_{1k}(t) = -(1+\sqrt{2})x_1(t) + g_1^{(k)}(t) \\
&u_{2k}(t) = -(1+\sqrt{2})x_2(t) + g_2^{(k)}(t), \quad k = 1, 2, \cdots
\end{aligned} \tag{30}$$

When $k = 1, 2, 3$, the simulation curves of $u_1(t), x_1(t), u_2(t), x_2(t)$ are shown in Fig. 1.

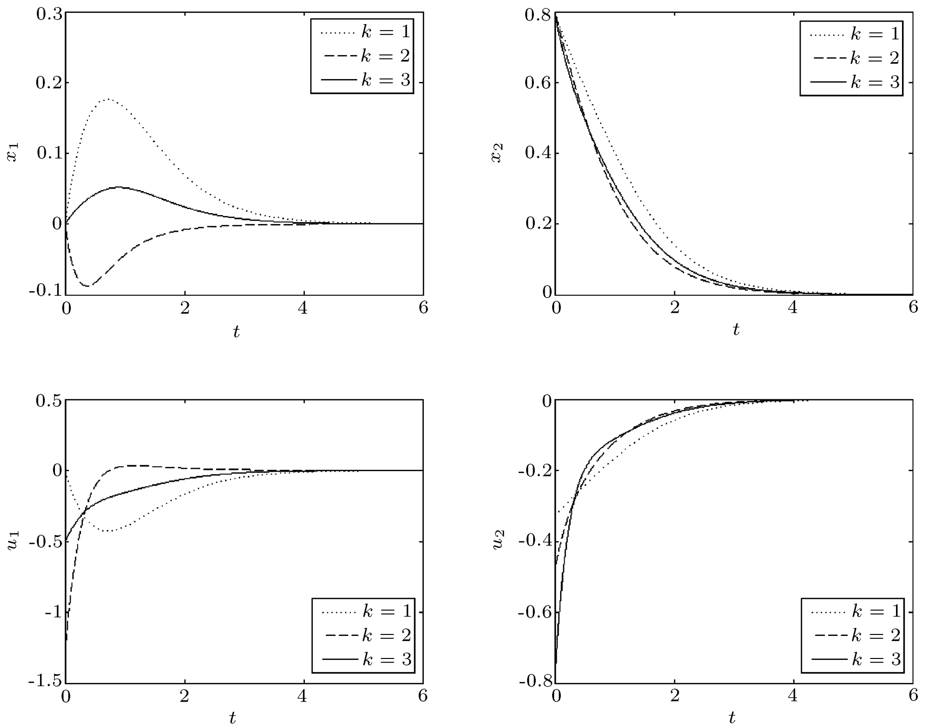

Fig. 1 Simulation curves of the system when $k = 1, 2, 3$

Fig. 1 clearly shows that the more iterative steps, the higher control precision. After the third time of iterative control, we can obtain the cost functionals of composite system (28) $J_1 = 0.7833$, $J_2 = 0.6070$ and $J_3 = 0.5365$. Obviously, $J_1 > J_2 > J_3$. If $\varepsilon = 0.15$, $|(J_3 - J_2)/J_3| \doteq 0.1315 < \varepsilon$. That is, the control precision can be satisfied after 3 times of iteration. Therefore, we can take $\boldsymbol{u} = [u_{13} \quad u_{23}]^{\mathrm{T}}$ as an approximate optimal control law of this composite system.

6 Conclusion

The main result of this article is to develop a successive approximation approach of optimal control for nonlinear interconnected large-scale systems. By using the approach, we have transformed a high order, coupling, nonlinear TPBV problem into a sequence of linear decoupling TPBV problems. The TPBV problem sequence uniformly converges to the optimal control for the nonlinear interconnected large-scale system. A suboptimal control law has been obtained by using a finite iterative result of optimal control law sequence. A simulation example shows that the successive approximation optimal control algorithm is effective.

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TANG Gong-You Professor of the College of Information Science and Engineering, Ocean University of China. He received his Ph. D. degree from South China University of Technology in 1991. His research interests include analysis and control for time-delay systems and nonlinear systems, theory and applications of large-scale systems.

SUN Liang Ph. D. candidate of Ocean University of China. His research interests include analysis and control for time-delay systems and nonlinear systems, theory and applications of large-scale systems.