一种采用方块脉冲函数求解延时最优控制问题的算法[1)]

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摘　　要

本文采用方块脉冲函数方法，将线性延时系统二次型最优控制问题转化为函数极值问题，得到了最优控制规律的分段恒定解答，导出了一种不用迭代求解的简便算法. 所提算法对小延时和大延时系统均有效.

对于线性延时系统二次型最优控制问题，不少文献（如[1]）从泛函寻优的角度出发导出了求解方法. 但这些方法一般要近似迭代求解延时微分方程的两点边值问题，计算比较繁琐，且对于大延时系统精度不易保证. 本文在文献[2]的基础上，采用方块脉冲函数方法，将上述延时最优控制问题转化为函数极值问题，直接得到控制和状态的分段恒定解答，避免了繁琐的迭代求解过程，并且对大延时系统也能保证精度.

一、方块脉冲函数的性质

$t\in[0,T)$ 上的m个分量的方块脉冲函数族表达式为

$$\Pi_i(t)=\begin{cases}1, & (i-1)T/m\leqslant t<iT/m,\\ 0, & \text{其它}\ i=1,2,\cdots,m.\end{cases} \tag{1}$$

性质　若记 $\Pi(t)=[\Pi_1(t),\Pi_2(t),\cdots,\Pi_m(t)]^T$,

则

$$\int_0^T \Pi(\tau)d\tau\cong\frac{T}{m}H\Pi(t)=E\Pi(t),\ 0\leqslant t<T. \tag{2}$$

其中

$$E=\frac{T}{m}H=\frac{T}{m}\begin{bmatrix}\frac{1}{2} & 1 & 1 & \cdots & 1\\ & \frac{1}{2} & 1 & \cdots & 1\\ & & \ddots & \ddots & \vdots\\ & 0 & & \ddots & 1\\ & & & & \frac{1}{2}\end{bmatrix}. \tag{3}$$

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二、线性定常延时状态方程的求解

考虑含有延时环节的线性定常系统

$$\begin{cases}\dot{\boldsymbol{x}}(t)=A\boldsymbol{x}(t)+B\boldsymbol{u}(t)+C\boldsymbol{x}(t-\tau_l)+D\boldsymbol{u}(t-\tau_h),\\ \boldsymbol{x}(t)=\hat{\boldsymbol{x}}(t),\quad -\tau_l\leqslant t\leqslant 0,\\ \boldsymbol{u}(t)=\hat{\boldsymbol{u}}(t),\quad -\tau_h\leqslant t\leqslant 0.\end{cases}\tag{4}$$

式中 $\boldsymbol{x}(t)=[x_i(t)]_{n\times1}$ 为状态向量，$\boldsymbol{u}(t)=[u_j(t)]_{r\times1}$ 为控制向量。τ_l, $\tau_h\geqslant 0$ 分别为状态和控制的延时量。

设 $\tau_l=l\cdot\frac{T}{m}$, $\tau_h=h\cdot\frac{T}{m}$, 其中 l, h 为正整数。

采用方块脉冲函数近似展开，得

$$\begin{cases}\dot{\boldsymbol{x}}(t)\cong\sum_{i=1}^{m}\boldsymbol{v}_i\Pi_i(t)=V\Pi(t),\\ \boldsymbol{x}(t)\cong\sum_{i=1}^{m}\boldsymbol{x}_i\Pi_i(t)=X\Pi(t),\\ \boldsymbol{x}(t-\tau_l)\cong\sum_{i=1}^{m}\tilde{\boldsymbol{x}}_i\Pi_i(t)=\tilde{X}\Pi(t),\\ \boldsymbol{u}(t)\cong\sum_{i=1}^{m}\boldsymbol{u}_i\Pi_i(t)=U\Pi(t),\\ \boldsymbol{u}(t-\tau_h)\cong\sum_{i=1}^{m}\tilde{\boldsymbol{u}}_i\Pi_i(t)=\tilde{U}\Pi(t).\end{cases}\tag{5}$$

式中

$$V=\begin{bmatrix}v_{11} & v_{12} & \cdots v_{1m}\\ v_{21} & v_{22} & \cdots v_{2m}\\ \vdots & \vdots & \vdots\\ v_{n1} & v_{n2} & \cdots v_{nm}\end{bmatrix}\triangleq[\boldsymbol{v}_1,\boldsymbol{v}_2,\cdots,\boldsymbol{v}_m].$$

同样，

$$X\triangleq[\boldsymbol{x}_1,\boldsymbol{x}_2,\cdots,\boldsymbol{x}_m]_{n\times m},\quad \tilde{X}\triangleq[\tilde{\boldsymbol{x}}_1,\tilde{\boldsymbol{x}}_2,\cdots,\tilde{\boldsymbol{x}}_m]_{n\times m}$$
$$U\triangleq[\boldsymbol{u}_1,\boldsymbol{u}_2,\cdots,\boldsymbol{u}_m]_{r\times m},\quad \tilde{U}\triangleq[\tilde{\boldsymbol{u}}_1,\tilde{\boldsymbol{u}}_2,\cdots,\tilde{\boldsymbol{u}}_m]_{r\times m}.$$

将(5)式代入(4)式，有

$$V\Pi(t)\cong AX\Pi(t)+BU\Pi(t)+C\tilde{X}\Pi(t)+D\tilde{U}\Pi(t).\tag{6}$$

易证

$$\tilde{X}\Pi(t)=XH_l\Pi(t)+\hat{X}\Pi(t),\tag{7}$$
$$\tilde{U}\Pi(t)=UH_h\Pi(t)+\hat{U}\Pi(t).\tag{8}$$

其中 $H_l=\Delta^l$, $H_h=\Delta^h$.

$$\Delta = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ & 0 & 1 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & \text{0} & & \ddots & 1 \\ & & & & 0 \end{bmatrix},$$

$$\hat{\boldsymbol{x}} = (\hat{\boldsymbol{x}}_l, \hat{\boldsymbol{x}}_{l-1}, \cdots, \hat{\boldsymbol{x}}_1, 0, \cdots, 0), \tag{9}$$

$$\hat{U} = (\hat{\boldsymbol{u}}_h, \hat{\boldsymbol{u}}_{n-1}, \cdots, \hat{\boldsymbol{u}}_1, 0, \cdots, 0), \tag{10}$$

$$\hat{\boldsymbol{x}}_i = \frac{m}{T}\int_{-iT/m}^{-(i-1)T/m} \hat{\boldsymbol{x}}(t)dt, \quad i = 1, 2, \cdots, l, \tag{11}$$

$$\hat{\boldsymbol{u}}_j = \frac{m}{T}\int_{-jT/m}^{-(j-1)T/m} \hat{\boldsymbol{u}}(t)dt, \quad j = 1, 2, \cdots, h. \tag{12}$$

$$X\Pi(t) \cong VE\Pi(t) + [\boldsymbol{x}_0, \boldsymbol{x}_0, \cdots, \boldsymbol{x}_0]\Pi(t), \tag{13}$$

其中 $\boldsymbol{x}_0 = \boldsymbol{x}(0)$.

将(7),(8)和(13)式代入(6)式,经整理可得

$$V \cong AVE + CVEH_l + BU + DUH_h + \widetilde{W}. \tag{14}$$

其中

$$\widetilde{W} = A(\boldsymbol{x}_0, \boldsymbol{x}_0, \cdots, \boldsymbol{x}_0) + C(\boldsymbol{x}_0, \boldsymbol{x}_0, \cdots, \boldsymbol{x}_0)H_l + C\hat{\boldsymbol{x}} + D\hat{U}$$
$$\triangleq [\widetilde{\boldsymbol{w}}_1, \widetilde{\boldsymbol{w}}_2, \cdots, \widetilde{\boldsymbol{w}}_m]_{n\times m}. \tag{15}$$

运用矩阵的 kronecker 乘积,(14)式可转化为

$$\bar{V} \cong [I_{mn} - A\otimes E^T - C\otimes(EH_l)^T]^{-1}[(B\otimes I_m - D\otimes H_h^T)\bar{U} + \bar{W}]. \tag{16}$$

其中 I_{mn} 和 I_m 分别为 $m\times n$ 阶和 m 阶的单位阵.

$$\bar{V} \triangleq [\boldsymbol{v}_1^T, \boldsymbol{v}_2^T, \cdots, \boldsymbol{v}_m^T]^T, \quad \bar{U} \triangleq [\boldsymbol{u}_1^T, \boldsymbol{u}_2^T, \cdots, \boldsymbol{u}_m^T]^T,$$
$$\bar{W} \triangleq [\widetilde{w}_1^T, \widetilde{w}_2^T, \cdots, \widetilde{w}_m^T]^T.$$

记 $\bar{X} = [\boldsymbol{x}_1^T, \boldsymbol{x}_2^T, \cdots, \boldsymbol{x}_m^T]^T$, 由(13)式可得

$$\bar{X} \cong (I_n\otimes E^T)\bar{V} + \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{x}_0 \\ \vdots \\ \boldsymbol{x}_0 \end{bmatrix}. \tag{17}$$

其中 I_n 为 n 阶单位阵.

将(16)式代入(17)式,可得状态的分段恒定值 $\bar{X}$ 的近似计算式如下:

$$\bar{X} \cong (I_n\otimes E^T)[I_{mn} - A\otimes E^T - C\otimes(EH_l)^T]^{-1}[(B\otimes I_m - D\otimes H)_h^T\bar{U} + \bar{W}] + \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{x}_0 \\ \vdots \\ \boldsymbol{x}_0 \end{bmatrix}$$
$$= \phi\bar{U} + W. \tag{18}$$

其中

$$\begin{cases} \phi = (I_n\otimes E^T)[I_{mn} - A\otimes E^T - C\otimes(EH_l)^T]^{-1}(B\otimes I_m - D\otimes U_n^T), \\ W = (I_n\otimes E^T)[I_{mn} - A\otimes E^T - C\otimes(EH_l)^T]^{-1}\bar{W} + \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{x}_0 \\ \vdots \\ \boldsymbol{x}_0 \end{bmatrix}. \end{cases} \tag{19}$$

三、最优控制规律的确定

下面对(4)式所描述的线性定常延时系统按二次型性能指标

$$J=\int_0^T[\boldsymbol{x}^T(t)Q\boldsymbol{x}(t)+\boldsymbol{u}^T(t)R\boldsymbol{u}(t)]dt \tag{20}$$

为最小的要求确定最优控制规律．式中 Q 为 $n\times n$ 的常数对称半正定矩阵，R 为 $r\times r$ 的常数对称正定矩阵．

采用方块脉冲函数近似展开，可得

$$J\cong\sum_{i=1}^{m}\int_{(i-1)T/m}^{iT/m}[\boldsymbol{x}_i^TQx_i+\boldsymbol{u}_i^TR\boldsymbol{u}_i]dt$$

$$=\frac{T}{m}\sum_{i=1}^{m}[\boldsymbol{x}_i^TQx_i+\boldsymbol{u}_i^TRu_i]=\frac{T}{m}[\bar{X}^T\bar{Q}\bar{X}+\bar{U}^T\bar{R}\bar{U}]. \tag{21}$$

其中，

$$\bar{Q}=\text{block diag }[Q,Q,\cdots,Q],$$
$$\bar{R}=\text{block diag }[R,R,\cdots,R].$$

将(18)式代入(21)式，经整理可得

$$J\cong\frac{T}{m}[\bar{U}^T\bar{G}\bar{U}+\bar{U}^T\bar{F}+\bar{F}^T\bar{U}+W^T\bar{Q}W]. \tag{22}$$

式中 $\bar{G}=\phi^T\bar{Q}\phi+\bar{R}$，$\bar{F}=\phi^T\bar{Q}W$．

(22)式中，J 仅为控制变量 $\bar{U}$ 的函数；于是从 $\frac{dJ}{d\bar{U}}=0$ 得

$$\bar{G}\bar{U}+\bar{F}\cong 0. \tag{23}$$

如设 $\bar{G}$ 的逆存在，则可得求解上述延时最优控制问题的近似算法如下：

$$\begin{cases}\bar{U}^*\cong-\bar{G}^{-1}\bar{F},\\ \bar{X}^*\cong\phi\bar{U}^*+W,\\ J^*\cong\frac{T}{m}[\bar{X}^{*T}\bar{Q}\bar{X}^*+\bar{U}^{*T}\bar{R}\bar{U}^*].\end{cases} \tag{24}$$

算法的精度可通过增大分段数 m 来保证．此外，由推导过程易见，算法的精度与延时大小无关，因此，上述算法对小延时和大延时系统均有效．

四、举　　例

例． 求解文献[1]中的延时最优控制问题

$$\begin{cases}\dot{x}(t)=x(t-1)+u(t),\ x(t)=1,\ -1\leqslant t\leqslant 0\\ J=\frac{1}{2}\int_0^2(x^2+u^2)dt.\end{cases}$$

解． 由给定条件知 $T=2$，$\tau_i=1$，$\tau_h=0$，$A=D=0$，$B=C=1$，$Q=R=0.5$．取 $m=10$，由(24)式，得最优控制规律和最优状态轨线的分段恒定解答如图 1 虚线所示(图中实线为文献[1]的

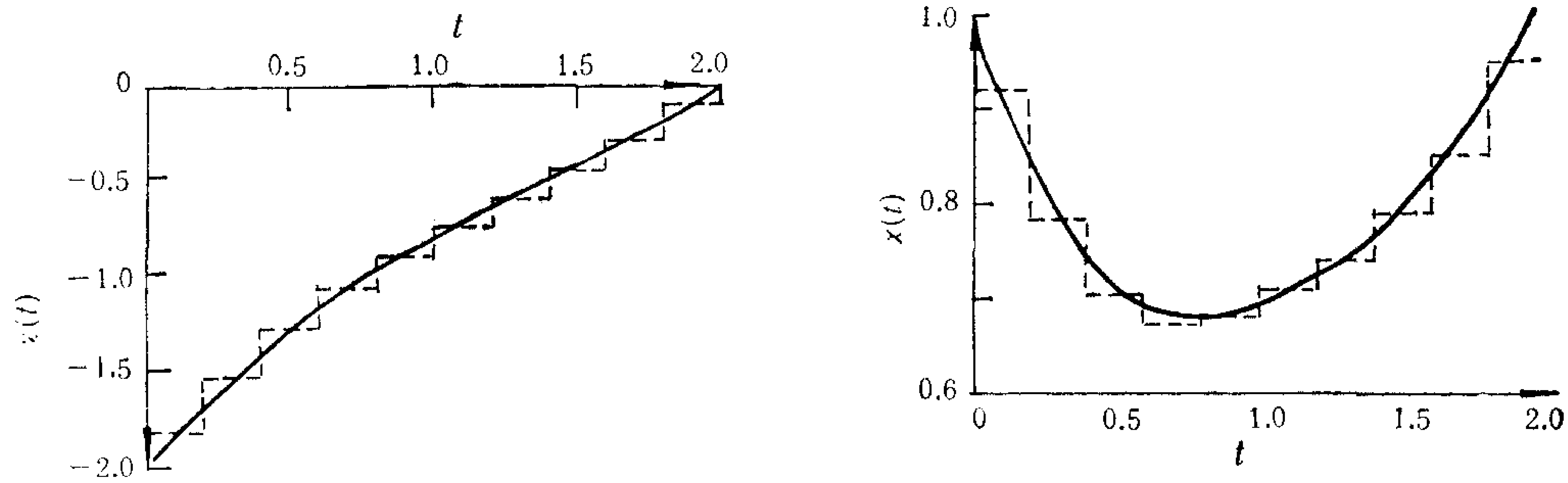

图 1 最优控制规律和最优状态轨线的近似解

结果). 由图可见,结果与文献[1]差别很小,而本文算法更简便. 进一步得 $J^* = 1.6502$,优于文献[1]的结果.

参 考 文 献

[1] Palanisamy, K.R. & Rao, G.P., Optimal Control of Linear System with Delays in State and Control Via Walsh Functions, IEE Proceedings, Vol.**130**, Pt. D, 1983, No. 6, 300—312.

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AN ALGORITHM FOR SOLVING TIME-DELAYED OPTIMAL CONTROL PROBLEMS VIA BLOCK-PULSE FUNCTIONS

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Abstract

In this paper, the linear time-delayed optimal control problem based on quadratic performance is transformed into a simple extremal problem of functions via block-pulse functions. On this basis, the piecewise constant solution of optimal control is obtained and a simple algorithm without iterating is proposed. The algorithm is effective and usable to both small delay systems and large delay systems.