Why is the Danger Cylinder Dangerous in the P3P Problem?1)

ZHANG Cai-Xia[1,2] HU Zhan-Yi[1,2]

[1](*National Laboratory of Pattern Recognition, Institute of Automation, Chinese Academy of Sciences, Beijing* 100080)

[2](*Institute of Image Processing and Pattern Recognition, North China University of Technology, Beijing* 100041)

(E-mail: caixiazhang@gmail.com, huzy@nlpr.ia.ac.cn)

Abstract The PnP problem is a widely used technique for pose determination in computer vision community, and finding out geometric conditions of multiple solutions is the ultimate and most desirable goal of the multi-solution analysis, which is also a key research issue of the problem. In this paper, we prove that given 3 control points, if the camera′s optical center lies on the so-called "danger cylinder" and is enough far from the supporting plane of control points, the corresponding P3P problem must have 3 positive solutions. This result can bring some new insights into a better understanding of the multi-solution problem. For example, it is shown in the literature that the solution of the P3P problem is instable if the optical center lies on this danger cylinder, we think such occurrence of triple-solution is the primary source of this instability.

Key words The P3P problem, the danger cylinder, instability of solution

1 Introduction

The Perspective-n-Point problem, or the PnP problem in short, is a classical problem in computer vision, photogrammetry, and even in mathematics. It was first formally introduced by Fishler and Bolles in 1981[1], and later extensively studied by others, *e.g.* [1 ∼ 11] to cite a few. The PnP problem is meaningful only when "n", *i.e.*, the number of the point correspondences from space control points to their projected image ones, belongs to $\{3, 4, 5\}$, or the P3P problem, the P4P problem, and the P5P problem. This is because when $n < 3$, the problem is not well defined, and when $n > 5$, the problem can be linearly solved, for example, by the direct linear transformation method[12].

Among the P3P problem, the P4P problem, and the P5P problem, the most fundamental one is the P3P problem due to its wide applicability as well as its pivotal role-played for the other two problems. In the literature, the P3P problem has been extensively studied. Su *et al.*[5] showed that the necessary and sufficient condition for the P3P problem having an infinitely large number of solutions is the co-circularity of its three control points with the camera′s optical center. Fishler and Bolles[1] showed that the P3P problem has at most 4 positive solutions and this upper bound is also attainable via a concrete example. Haralick *et al.*[2] reviewed and compared 6 different direct approaches to solve the P3P problem. Gao *et al.*[9] gave a complete solution classification of the P3P problem. Their results are purely algebraic, and seem less instructive in real applications than directly solving a 4^{th} degree polynomial as originally stated. Hence, it is desirable to give a geometrical interpretation of multiple solutions of the P3P problem. In [13], a general condition for the P3P problem to have 4 solutions is obtained. In this paper, we give a sufficient condition for the P3P problem to have 3 solutions.

Although the P3P problem has been extensively studied in the literature, we think the present study is worth reporting: Firstly, these results can bring some new insights into a better understanding of the multi-solution problem. For example, it is shown in [14] that the solution of the P3P problem is instable if the optical center lies on the danger cylinder, we think such occurrence of triple-solution is the primary source of this instability. Secondly, the results are purely geometric and can be used as theoretical guides in real applications to avoid harmful multiple solution problem by properly arranging the control points.

In this paper, a bold capital letter denotes a matrix or a non-homogeneous 3-vector, a bold small letter denotes homogeneous 3-vector of image points, τ stands for transpose, $|\bullet|$ means absolute value for real numbers, $\|\bullet\|$ means Frobenius norm for vectors or matrices.

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2 Preliminaries

For the clarity and convenience, the two kinds of definitions of the P3P problem, their relations, and the fundamental constraints will be listed in this section at first. They are necessary elements for us to proceed to our main conclusions, which will be elaborated in Section 3.

2.1 The distance-based definition

It is defined as follows: given the relative spatial locations of 3 control points and given the angle to every pair of control points from the perspective center, find the distance of each of the control points from the perspective center.

As shown in Fig. 1(a), A, B, and C are three control points, O is the optical center, by the law of Cosines, we have the following familiar system of constraints:

$$\begin{cases} z^2 + y^2 - 2zy\cos\alpha = a^2 \\ x^2 + z^2 - 2xz\cos\beta = b^2 \\ x^2 + y^2 - 2xy\cos\gamma = c^2 \end{cases} \tag{1}$$

where $a = |BC|$, $b = |AC|$, and $c = |AB|$. $x = |OA|$, $y = |OB|$, and $z = |OC|$ are the three distances to determine under the distance-based definition.

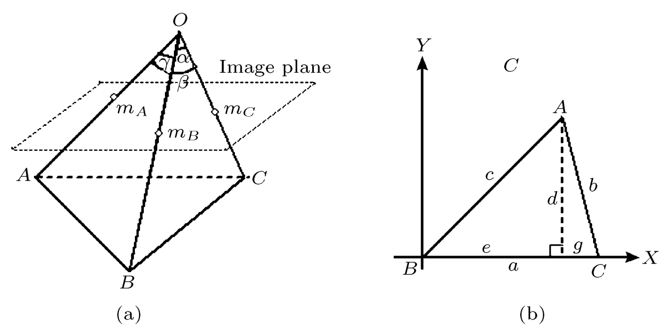

Fig. 1 The geometry of the P3P problem and the object system setup

2.2 The transformation-based definition

It is defined as follows: given 3 control points with known coordinates in an object-centered frame and their corresponding projections onto an image plane and given the intrinsic camera parameters, find the transformation matrix between the object frame and the camera frame[3].

As shown in Fig. 1(b), A, B, and C are three control points, whose non-homogenous coordinates are $\boldsymbol{M}_A, \boldsymbol{M}_B$, and $\boldsymbol{M}_C$, respectively. To simplify the problem, a special object frame is setup such that

$$\boldsymbol{M}_A = (e \quad d \quad 0)^\tau, \quad \boldsymbol{M}_B = (0 \quad 0 \quad 0)^\tau, \quad \boldsymbol{M}_C = (a \quad 0 \quad 0)^\tau$$

with $d = c\sin\angle ABC$, $e = c\cos\angle ABC$.

Without loss of generality, let the intrinsic camera parameter matrix be the identity matrix, and the three corresponding image points in homogenous coordinates be $\boldsymbol{m}_A, \boldsymbol{m}_B, \boldsymbol{m}_C$ respectively, which are all unit vectors, *i.e.* $\|\boldsymbol{m}_A\| = 1$, $\|\boldsymbol{m}_B\| = 1$, and $\|\boldsymbol{m}_C\| = 1$; then the perspective imaging process can be expressed as:

$$s_i\boldsymbol{m}_i = (R \quad \boldsymbol{t})\begin{pmatrix} \boldsymbol{M}_i \\ 1 \end{pmatrix}, \quad i = A, B, C$$

where s_A, s_B, s_C are three unknown scale factors, $R, \boldsymbol{t}$ are the 3×3 rotation matrix and the translation vector, respectively, hence we have the following constraints:

$$\begin{cases} s_B\boldsymbol{m}_B = t \\ s_C\boldsymbol{m}_C = ar_1 + t \\ s_A\boldsymbol{m}_A = er_1 + dr_2 + t \end{cases} \tag{2}$$

In (2), $R = (r_1, r_2, r_1\times r_2)$ and t are transformation parameters to determine under the transformation-based definition.

2.3 The equivalence of the two definitions

At first, we show that $x = s_A$, $y = s_B$, $z = s_C$ in (1) and (2). This is because according to the projection theory [15, p. 144], if O is the optical center, then

$$(R \quad \boldsymbol{t}) \begin{pmatrix} \boldsymbol{O} \\ 1 \end{pmatrix} = \boldsymbol{0}$$

and

$$\boldsymbol{O} = -R^{\tau}\boldsymbol{t}$$

Then

$$x = \sqrt{(\boldsymbol{A}-\boldsymbol{O})^{\tau}(\boldsymbol{A}-\boldsymbol{O})} = \sqrt{(R^{\tau}(s_A\boldsymbol{m}_A-\boldsymbol{t})+R^{\tau}\boldsymbol{t})^{\tau}(R^{\tau}(s_A\boldsymbol{m}_A-t)+R^{\tau}\boldsymbol{t})} = s_A$$

Similarly, $y = s_B$, $z = s_C$.

In the next, we show that these two definitions are indeed equivalent. This is quite evident because from (2), we know that if $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{t}$ are known, s_A, s_B, s_C can be uniquely determined. Conversely, if s_A, s_B, s_C are known, r_1, r_2, t can also be uniquely determined. Hence in this paper, the two definitions will be used indistinguishably.

2.4 The main constraints

The constraints (2) can be further simplified as:

$$\frac{s_C}{s_B}\boldsymbol{m}_C = a\frac{\boldsymbol{r}_1}{s_B} + \boldsymbol{m}_B \tag{3}$$

$$\frac{s_A}{s_B}\boldsymbol{m}_A = e\frac{\boldsymbol{r}_1}{s_B} + d\frac{\boldsymbol{r}_2}{s_B} + \boldsymbol{m}_B \tag{4}$$

Let $s'_A = \dfrac{s_A}{s_B}$, $s'_C = \dfrac{s_C}{s_B}$, $\boldsymbol{r}'_1 = \dfrac{r_1}{s_B}$, $\boldsymbol{r}'_2 = \dfrac{r_2}{s_B}$. From (3), we have

$$r'_1 = \frac{s'_C\boldsymbol{m}_C - \boldsymbol{m}_B}{a}, \quad r'_2 = \frac{as'_A\boldsymbol{m}_A - es'_C\boldsymbol{m}_C - g\boldsymbol{m}_B}{ad}, \text{ with } g = b\cos\angle ACB$$

By $(\boldsymbol{r}'_1)^{\tau}\cdot\boldsymbol{r}'_2 = 0$, we have the first main constraint equation

$$e(s'_C)^2 - a\cos\beta\cdot s'_A s'_C - (2e-a)\cos\alpha\cdot s'_C + a\cos\gamma\cdot s'_A - g = 0 \tag{5}$$

By $\|\boldsymbol{r}'_1\| = \|\boldsymbol{r}'_2\|$, we have the second main constraint equation

$$(c^2-2e^2)\cdot(s'_C)^2 + 2ae\cos\beta\cdot s'_A s'_C - a^2\cdot(s'_A)^2 + 2(2e^2-c^2-ae)\cos\alpha\cdot s'_C + 2ag\cos\gamma\cdot s'_A + (b^2-2g^2) = 0 \tag{6}$$

where $\cos\alpha = \boldsymbol{m}_B^{\tau}\boldsymbol{m}_C$, $\cos\beta = \boldsymbol{m}_A^{\tau}\boldsymbol{m}_C$, and $\cos\gamma = \boldsymbol{m}_B^{\tau}\boldsymbol{m}_A$.

These two constraint equations are the fundamental basis for the proofs of our main results presented in Section 3, where s_A and s_C are two unknowns to determine.

Before ending this section, a further point should be discussed, *i.e.*, how to derive s_A, s_B, s_C from s'_A and s'_C. This can be done as follows.

Firstly, due to $r_1 = s_B\cdot\dfrac{1}{a}(\dfrac{s_C}{s_B}\boldsymbol{m}_C - \boldsymbol{m}_B)$ and $r_1^{\tau}\cdot r_1 = 1$, we have

$$s_B^2\cdot\frac{1}{a^2}(s'_C\boldsymbol{m}_C - \boldsymbol{m}_B)^{\tau}\cdot(s'_C\boldsymbol{m}_C - \boldsymbol{m}_B) = 1$$

then $s_B = a/\sqrt{(s'_C\boldsymbol{m}_C - \boldsymbol{m}_B)^{\tau}\cdot(s'_C\boldsymbol{m}_C - \boldsymbol{m}_B)} = a/\sqrt{(s'_C)^2 - 2s'_C\cos\alpha + 1}$, and s_C, s_A are obtained by $s_C = s_B\cdot s'_C$, $s_A = s_B\cdot s'_A$.

It is clear that different set (s'_A, s'_C) must result in different set (s_A, s_B, s_C) hence in the next section, we shall show under what conditions, s'_A and s'_C in (5) and (6) can have 3 positive solutions. For notational convenience, s'_A, s'_C in (5) and (6) will be again denoted as s_A and s_C in the next section.

3 Main results

The danger cylinder is defined as a circular cylinder circumscribing control points A, B, C with axis normal to the plane A, B, C[14] (cf. Fig. 2). In [14], based on singularity analysis of the Jacobian matrix of constraints, it is concluded that if the optical center lies on the danger cylinder, the solution of the corresponding P3P is unstable. In this work, we show that this instability could stem primarily from multiple solutions.

Proposition

As shown in Fig. 2, A, B, C are the three control points, if the optical center O lies on the danger cylinder except for a few lines on it, and additionally the distance from O to the plane A, B, C is sufficiently large, then the corresponding P3P problem $\{O, (A, B, C)\}$ must have 3 positive solutions.

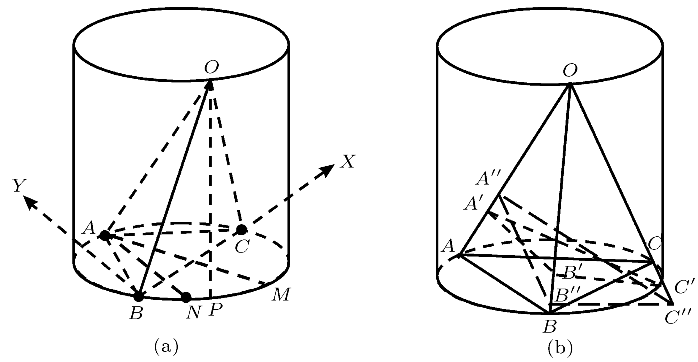

Fig. 2 (a)When the optical center O lies on the danger cylinder, and the corresponding P3P problem must have 3 positive solutions. (b) An example of 3 positive solutions: $\{O, ABC\}$, $\{O, A'B'C'\}$, $\{O, A''B''C''\}$

Before giving a formal proof of the above result, here is an outline. The proof is composed of the following two main steps: Firstly the two quadratic constraint equations of two variables (s_A, s_C) in (5) and (6) are converted into a 4^{th} degree univariate polynomial, then this polynomial is factorized into a univariate quadratic equation and two identical linear ones. Secondly, the univariate quadratic equation is shown to have two positive roots.

Proof.

Step 1. When the optical center lies on the danger cylinder, the constraints in (5) and (6) can be factorized into a univariate quadratic equation and two identical linear ones.

As shown in Fig. 2(a), P is the projection of the optical center O on the plane ABC, to be more exact, on the circumscribed circle of triangle ABC, line AM passes through the center of the circle, $AN \perp BC$, and M, N lie on the circle.

Assume $P = (x, y, 0)^\tau$; then $x^2 - ax + y^2 - \dfrac{bc\cos\angle A}{d}y = 0$, that is, P lies on the circumscribed circle.

Let $|OP| = h$; then $\boldsymbol{O} = (x, y, h)^\tau$. In addition, as shown in Section 2.2, $\boldsymbol{M}_A = (e \quad d \quad 0)^\tau$, $\boldsymbol{M}_B = (0 \quad 0 \quad 0)^\tau$, $\boldsymbol{M}_C = (a \quad 0 \quad 0)^\tau$, then

$$|OA| = \sqrt{x^2 + y^2 - 2ex - 2dy + c^2 + h^2},\ |OB| = \sqrt{x^2 + y^2 + h^2},\ |OC| = \sqrt{x^2 + y^2 - 2ax + a^2 + h^2}$$

and

$$\cos\alpha = \frac{|OB|^2 + |OC|^2 - a^2}{2|OB||OC|},\ \cos\beta = \frac{|OA|^2 + |OC|^2 - b^2}{2|OA||OC|},\ \cos\gamma = \frac{|OA|^2 + |OB|^2 - c^2}{2|OA||OB|} \tag{7}$$

1) When $\cos\beta s_C - \cos\gamma = 0$

By some simple calculations, we can prove that when $\cos\beta s_C - \cos\gamma = 0$, P must coincide with point A, B, or N. In addition, we can prove that:

1.1) If P coincides with A, (5) and (6) must have three positive solutions;

1.2) If P coincides with M or N but $b \neq c$, (5) and (6) still have three positive solutions;

1.3) If P coincides with M or N but $b = c$, M and N will coincide each other, thus (5) and (6) have only two positive solutions.

Since the above three cases are isolated ones and can be proved by simply substituting the related conditions into (5) and (6), the detailed steps will be omitted.

2) When $\cos\beta s_C - \cos\gamma \neq 0$

By (5), we have

$$s_A = (es_C^2 - (2e-a)\cos\alpha s_C - g)/a(\cos\beta s_C - \cos\gamma)$$

Substituting s_A into (6) gives

$$(\cos^2\beta - \cos^2\angle B)c^2 \cdot s_C^4 + 2c\left[\frac{1}{a}(c^2-b^2)\cos\angle B\cos\alpha - b\cos\angle A\cos\gamma\cos\beta - c\cos^2\beta\cos\alpha\right]\cdot s_C^3 +$$
$$\left[2(b^2+c^2)\cos\alpha\cos\beta\cos\gamma + (b^2-a^2)\cos^2\gamma - (\frac{b^2-c^2}{a})^2\cos^2\alpha + (c^2-a^2)\cos^2\beta + 2bc\cos\angle B\cos\angle C\right]\cdot s_C^2 +$$
$$2b\left[\frac{1}{a}(b^2-c^2)\cos\angle C\cos\alpha - c\cos\angle A\cos\gamma\cos\beta - b\cos^2\gamma\cos\alpha\right]\cdot s_C + (\cos^2\gamma - \cos^2\angle C)b^2 = 0 \quad (8)$$

Substituting (7) into (8), *i.e.*, when the optical center is limited to the danger cylinder, (8) can be factorized into

$$\left(s_C - \frac{|OC|}{|OC|}\right)\cdot(s_2 \cdot s_C^2 + s_1 \cdot s_C + s_0) = 0 \quad (9)$$

where

$$s_2 = [C_1 \cdot h^4 + (C_2x + C_3y + C_4)\cdot h^2 + (C_5xy + C_6x + C_7y^2 + C_8y)]|OC|$$
$$s_1 = [C_9 \cdot h^4 + (C_{10}x + C_{11}y + C_{12})\cdot h^2 + (C_{13}xy + C_{14}y^2 + C_{15}y)]|OB|$$
$$s_0 = [C_{16} \cdot h^4 + (C_{17}x + C_{18}y + C_{19})\cdot h^2 + (C_{20}xy + C_{21}x + C_{22}y^2 + C_{23}y)]|OC|$$

$C_1 = 16al^3$, $C_9 = -32al^3$, $C_{16} = 16al^3$, and $l = \frac{1}{2}ac\sin\angle B$ (other coefficients C_i are irrelevant to our proof and will not be listed here).

From (9), we have

$$s_C = \frac{|OC|}{|OB|} \quad (10)$$

or

$$s_2 \cdot s_C^2 + s_1 \cdot s_C + s_0 = 0 \quad (11)$$

Step 2. Three different positive solutions from (10) and (11).

Step 2.1. Except some isolated cases, S_C in (10) is not a root of (11)

When $s_C = |OC|/|OB|$, we have $s_A = |OA|/|OB|$. This is one group of positive solutions of the P3P problem. Additionally, it is necessary to show under what conditions $s_C = |OC|/|OB|$ is not a root of (11). Substituting $s_C = |OC|/|OB|$ into the left side of (11), we have

$$|OC|/|OB|^2(f(a,b,c,x,y)h^2 + g(a,b,c,x,y)) \quad (12)$$

where both $f(a,b,c,x,y)$ and $g(a,b,c,x,y)$ are functions of $\{a,b,c,x,y\}$. Hence if $f(a,b,c,x,y)$ and $g(a,b,c,x,y)$ are not equal to 0 simultaneously, then (12) cannot equal 0, that is, $s_C = |OC|/|OB|$ cannot also be a root of (11). The following is an enumeration of all the possible cases when $f(a,b,c,x,y)$ and $g(a,b,c,x,y)$ are equal to 0 simultaneously.

Case A. If $b = c$, then when $P = N$, or $P = \left(\frac{a}{2} + \frac{\sqrt{3}b}{4\sin\angle B}, \frac{b + 2b\cos\angle A}{4\sin\angle B}, 0\right)$ or $P = \left(\frac{a}{2} - \frac{\sqrt{3}b}{4\sin\angle B}, \frac{b + 2b\cos\angle A}{4\sin\angle B}, 0\right)$, both $f(a,b,c,x,y)$ and $g(a,b,c,x,y)$ are equal to 0. In these cases, there are only two positive roots of (5) and (6).

Case B. When $P = N$ but $b \neq c$, both $f(a,b,c,x,y)$ and $g(a,b,c,x,y)$ are equal to 0. However, there are still three different positive roots of (5) and (6).

Case C. When $P = P_0\left(\frac{a}{2} + \frac{b\cos\theta}{2\sin\angle B}, \frac{bc\cos\angle A}{2d} + \frac{b\sin\theta}{2\sin\angle B}, 0\right)^{\tau}$ but $b \neq c$, both $f(a,b,c,x,y)$ and $g(a,b,c,x,y)$ are equal to 0, there are only two positive roots of (5) and (6), here $\sin\theta$ and $\cos\theta$ satisfy (13) and (14).

$$8a^2bc\sin^3\theta - 6a^2bc\sin\theta + a^2c^2 + a^2b^2 - b^4 + 2b^2c^2 - c^4 = 0 \quad (13)$$

$$\cos\theta = \frac{4a^2bc\sin^2\theta + (a^2c^2 + a62b^2 - b^4 + 2b^2c^2 - c^4)\sin\theta - 2a^2bc}{2ac(b^2 - c^2)\sin\angle B} \tag{14}$$

In summary, if P is not one of the above points in Cases A, B and Case C, $s_C = |OC|/|OB|$ cannot be a root of (11).

Step 2.2. (11) has two different positive roots

At first, if the optical center is enough far from the plane ABC, then the signs of s_2, s_1 and s_0 will be the same to those of coefficients C_1, C_9, and C_{16}. Hence $s_2 > 0$, $s_1 < 0$, $s_0 > 0$.

In addition, the discriminant of (11) is

$$\begin{aligned}\Delta =&32C_{24}C_{25}[(C_{26}x + C_{27}y + C_{28})\cdot h^6 + (C_{29}xy + C_{39}x + C_{31}y^2 + C_{32}y + C_{33})\cdot h^4 + \\ &(C_{34}x + C_{35}xy^3 + C_{36}xy^2 + C_{37}xy + C_{38}y^4 + C_{39}y^3 + C_{40}y^2 + C_{41}y)\cdot h^2 + \\ &(C_{42}xy^4 + C_{43}xy^3 + C_{44}xy^2 + C_{45}y^5 + C_{46}y^4 + C_{47}y^3)]\end{aligned}$$

where $C_{24} = l^2$, $C_{25} = 1/d^2$, $C_{26} = -128(b-c)(b+c)al^6$, $C_{28} = 32a^4b^2c^2l^4\cos^2\angle C$, $C_{27} = 32(a^2b^2 + a^2c^2 - b^4 + 2bc^2 - c^4)al^5$ (other coefficients C_i are irrelevant to our proof and will not be listed here).

Denote the coefficient of h^6 in Δ by δ. Then

$$\delta = 1024al^6/d^2(4(c^2-b^2)l^2\cdot x + (a^2b^2 + a^2c^2 - b^4 + 2b^2c^2 - b^4 + 2b^2c^2 - c^4)l\cdot y + a^3b^2c^2\cos^2\angle C)$$

when h is large, in order for $\Delta > 0$, it suffices $\delta > 0$. In the following, we will prove indeed.

Because the projection of the optical center on the plane ABC $P = (x, y, 0)^\tau$ satisfies the circle equation, x can be expressed as a function of variable y, consequently, δ can also be expressed as a function of y, say $\delta(y)$. It is evident that $\delta(y)$ is continuous and differentiable, so $\delta(y)$ can reach its maximum and minimum within the following closed interval:

$$\left[\frac{(a+b-c)(b-a-c)}{4b}, \frac{(a+b+c)(b+c-a)}{4d}\right]$$

at points at which its first derivative is 0 or at the two endpoints of the interval.

Denote the first derivative of δ with respect to y by δ'_y, the second derivative by δ''_y. Since (x, y) lies on the circle:

$$x^2 - ax + y^2 - \frac{bc\cos\angle A}{d}y = 0$$

when $y \in \left(\frac{(a+b-c)(b-a-c)}{4d}, \frac{(a+b+c)(b+c-a)}{4d}\right)$, by some simple calculations, we can prove that when $x = b\cos\angle C$, $y = -\frac{b\cos\angle B\cos\angle C}{\sin\angle B}$, $\delta'_y = 0$ and $\delta''_y > 0$, so δ reaches its minimum of 0. However in this case, P is the point M. As we said before, P is assumed different from points M, N, we can conclude that when $y \in \left(\frac{(a+b-c)(b-a-c)}{4d}, \frac{(a+b+c)(b+c-a)}{4d}\right)$, $\delta > 0$ holds.

In addition, when $y = \frac{(a+b-c)(b-a-c)}{4d}$ or $\frac{(a+b+c)(b+c-a)}{4d}$, there is $x = \frac{a}{2}$, and we have

$$\delta = \delta_1 = \frac{256a^2bcl^6}{d^2}(b-c)^2(a+b+c)(b+c-a)$$

or

$$\delta = \delta_2 = \frac{256a^2bcl^6}{d^2}(b+c)^2(a+b-c)(a+c-b)$$

It is clear that $\delta_2 > 0$ always holds. However, for δ, if $b \neq c$, $\delta_1 > 0$; if $b = c$, $\delta_1 = 0$. But as we said before, when $b = c$, P coincides with point M, and this case is excluded in our assumption. Hence at these two endpoints, $\delta > 0$ holds also.

Combining these two cases, we know $\delta > 0$ holds when y is within the closed interval, *i.e.*, $y \in \left[\frac{(a+b-c)(b-a-c)}{4d}, \frac{(a+b+c)(b+c-a)}{4d}\right]$.

In the following, we investigate under what condition s_A is positive when s_C is a positive root of (11). From, $s_A = (e\cdot s_C^2 - (2e-a)\cos\alpha\cdot s_C - g)/a(\cos\beta\cdot s_C - \cos\gamma)$, $s_A > 0$ is equivalent to the following inequality: $p_A = (e\cdot s_C^2 - (2e-a)\cos\alpha\cdot s_C - g)\cdot(\cos\beta\cdot s_C - \cos\gamma) > 0$.

From (11), there is

$$s_C = -(s_2\cdot s_C^2 + s_0)/s_1 \tag{15}$$

Substituting (15) into p_A, we obtain

$$p_A = ((C_{48}s_C^4 + C_{49}s_C^2 + C_{50})\cdot h^{12} + \lambda_{10}\cdot h^{10} + \lambda_8\cdot h^8 + \lambda_6\cdot h^6 + \lambda_4\cdot h^4 + \lambda_2\cdot h^2 + \lambda_0)/4s_1^2\cdot|OA|\cdot|OB|^3$$

where $C_{48} = 1024a^3l^6$, $C_{49} = -2048a^3l^6$, $C_{50} = 1024a^3l^6$ (the other parameters λ_i are irrelevant to our proof and will not be listed here)

The coefficient of the highest degree in h of the polynomial p_A is

$$\lambda_{12} = C_{48}s_C^4 + C_{49}s_C^2 + C_{50} = 1024a^3l^6(s_C-1)^2\cdot(s_C+1)^2 \tag{16}$$

It is clear that $\lambda_{12} \geqslant 0$. In addition, since when $s_C > 0$, $\lambda_{12} = 0$ if and only if $s_C = 1$. The following is to prove $s_C = 1$ cannot be a root of (11).

Case 1 If $|OB| \neq |OC|$, then by substituting $s_C = 1$ into the left side of (11), it can be shown that $s_C = 1$ is not a root of (11) when h is large enough.

Case 2 If $|OB| = |OC|$, *i.e.*, if $P = P_1 = \left(\frac{a}{2}, \frac{b(\cos\angle BAC + 1)}{2\sin\angle ABC}, 0\right)$ or $P = P_2 = \left(\frac{a}{2}, \frac{b(\cos\angle BAC - 1)}{2\sin\angle ABC}, 0\right)$ then by substituting into the left side of (11), we have

$$L_1 = \frac{a^2b(b+c)^2(a+b+c)(a-b-c)(a+b-c)^2(a-b+c)^2}{8\sin\angle ABC}|OB| \text{ when } P = P_1$$

$$L_2 = \frac{a^2b(b-c)^2(a+b-c)(a-b+c)(a+b+c)^2(a-b-c)^2}{8\sin\angle ABC}|OB| \text{ when } P = P_2$$

It is evident that $L_1 \neq 0$ and when $b \neq c$, $L_2 \neq 0$. This indicates that in these cases, $s_C = 1$ is not a root of (11). However when $b = c$, $L_2 = 0$. But in this last case, P_2 coincides with point M, and it is excluded in our assumption.

Combining Case 1 and Case 2, $s_C = 1$ is not a root of (11), *i.e.*, $s_C \neq 1$ holds, then $\lambda_{12} > 0$ holds. Thus, if the optical center is enough far from the plane of control points, we have $p_A > 0$, hence $s_A > 0$.

Therefore, we can conclude that when the optical center lies on the danger cylinder except for a few lines on it, and is placed enough far from the plane of control points, the corresponding P3P problem must have three positive solutions. □

At this stage, we can give an explicit specifications of those invalided "a few lines" in the proposition. In fact, these lines are those defined in Case A and Case C in Step 2.1.

4 Simulations

All simulations are performed by Maple 7. The simulation steps are as follows: Firstly, choose three control points A, B, C at random to form a triangle, then construct its danger cylinder like that in Fig. 2, and arbitrarily select a point as the optical center on the danger cylinder, then compute three sides a, b, c and three angles α, β, γ, finally, substitute a, b, c and α, β, γ into Equations in (1), and solve them. For convenience, we setup a coordinate frame such that the X-axis coincides with AC, Y positive axis passes through B, then $\boldsymbol{M}_A = (-c\cos\angle BAC, 0, 0)^\tau$, $\boldsymbol{M}_B = (0, c\sin\angle BAC, 0)^\tau$, $\boldsymbol{M}_C = (a\cos\angle ACB, 0, 0)^\tau$.

We will report an example only for the obtuse triangle due to the space limit.

For an obtuse triangle, we set $a = 78$, $b = 36$, $c = 47$. The optical center is at $\left(\frac{3875}{72} + \frac{\sqrt{-277426451116549 + 1209135756780\sqrt{4800215}}}{960043}, 25, 456\right)^\tau$ so $\cos\alpha = 0.9860329534$, $\cos\beta = 0.9969727446$, $\cos\gamma = 0.9949725637$. The three groups of positive solutions are:

$$\begin{cases}|OA| = 461.8272676\\ |OB| = 467.8626706\\ |OC| = 457.8559471\end{cases}, \quad \begin{cases}|OA| = 462.9167228\\ |OB| = 468.3185048\\ |OC| = 462.2441451\end{cases}, \quad \begin{cases}|OA| = 445.1718116\\ |OB| = 428.0557374\\ |OC| = 453.7209075\end{cases}$$

Remarks

1) In this paper, we show that if the optical center is enough far from the supporting plane of control points, the corresponding P3P problem must have 3 positive solutions. However, an explicit specification of "farness" is unavailable at this stage, it depends on the specific relation with respect to the triangle ABC. From the simulations, it seems that the distance does not need to be too large.

2) It is shown that the highest degree of h in the discriminant of (11) Δ is 6, while the highest degrees of h in both s_1 and s_2 of (11) are 4, so when the optical center is enough far from the plane ABC, *i.e.* h is very large, then (11) will have two very close solutions (c.f. the example for the obtuse triangle in the simulations). This accords with the instability explanation in [13].

5 Conclusions

A general sufficient condition for the P3P problem to have 3 positive solutions is obtained. We think this occurrence of multiple solutions when the optical center lies on the danger cylinder is the primary source of the observed instability in [14]. Our results are purely geometric and instructive enough for practitioners to properly arrange control points in real applications.

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ZHANG Cai-Xia Lectuer at North China University of Technology. Her research interests include computer vision and multiple view geometry.

HU Zhan-Yi Professor at Institute of Automation, Chinese Academy of Sciences. His research interests include computer vision, geometric primitive extraction, vision guided robot navigation, and pattern recognition.