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# Optimal $\ell_1$ Robust Control for Plants with Coprime Factor Perturbations: Continuity Properties and Adaptation[1)]

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**Abstract** This paper presents an adaptive robust control scheme for a discrete-time plant subject to both coprime factor perturbations and unknown external disturbances. Firstly, we establish nice continuity properties for $\ell_1$ optimization resulting from optimal robust control of discrete time uncertain systems with coprime factor perturbations. Then, we propose a robust parameter estimate algorithm with variant dead zone. Finally, combining the proposed parameter estimate algorithm with optimal $\ell_1$ robust control, we present a novel adaptive robust control scheme for plants with coprime factor perturbations based on the certainty equivalence principle. With the continuity properties of $\ell_1$ optimization established in this paper, the proposed adaptive scheme is shown to be uniformly stable. A posterior computable condition for the stability of the adaptive scheme is also provided in this paper.

**Key words** Coprime factor perturbations plant, continuity property, optimal $\ell_1$ robust control, adaptive control

## 1 Introduction

The $\ell_1$ design methodology, formulated in the mid-80s, is concerned with the feedback controller that reduces the effect of uncertainty on the system[1]. If this uncertainty is in the input/output, that is, disturbances are persistent bounded but unknown, $\ell_1$ controllers can be designed to minimize the amplitude of the error signal. If the uncertainty is in the plant, that is, additive or coprime factor perturbations, $\ell_1$ controllers can be designed to robustly stabilize the plant. Hence, $\ell_1$ design is viewed as a practical robustness design methodology[2].

Continuity properties of feedback design are of importance in studies of well-posedness and robustness of feedback control, especially in understanding interactions between modeling and feedback action, and in the development of the theory of adaptive control. From this point, continuity is an essential problem of $\ell_1$ design methodology. In [3] continuity properties of the problem for optimal rejection of bounded persistent disturbances were firstly studied and $\ell_1$ optimal design was proved to be as a continuous mapping from the plant to the optimal closed loop solution if the plant has no zeros on the unit circle. In [4,5], these conclusions were extended to any single block $\ell_1$ optimization problem. However, the studies of continuity properties of multi-block $\ell_1$ optimization have not been reported so far. In this paper, using properties of the graph topology, the $\ell_1$ suboptimal robust control design for the plants with coprime factor perturbations is shown to be uniformly continuous whenever the set of plants is restricted to a compact set.

On the basis of continuity properties of $\ell_1$ optimization design established in this paper, we also investigates the problem of adaptive robust control for a plant with coprime factor perturbations and unknown disturbances. This work is an extension of [3] and [6].

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The plant under consideration in [3] is subject to unknown disturbances, and the plant in [6] is subject to unknown disturbances as well as perturbations. The results about the adaptive robust control obtained in this paper is similar to that of [7], the main differences being that the plant in this paper is a discrete-time system and the characterization of uncertainty in the plant is more typical.

## 2 Preliminaries

The notations used throughout this paper are as follows:

$$\|x\|_\infty \stackrel{\text{def}}{=} \max_i |x(i)|, \ \ell_\infty \stackrel{\text{def}}{=} \{x: \|x\|_\infty < \infty\}, \ \|x\|_1 \stackrel{\text{def}}{=} \sum_{i=0}^{\infty} |x(i)|,$$

$$\ell_1 \stackrel{\text{def}}{=} \{x: \|x\|_1 < \infty\}, \ c_0 \stackrel{\text{def}}{=} \{x \in \ell_\infty \mid \lim_{k\to\infty} x(k) = 0\}; \ \hat{x} \stackrel{\text{def}}{=} \sum_{i=0}^{\infty} x(i) z^i.$$

$x$ and $\hat{x}$ can be viewed as the same where no confusion occurs.

Let $\Gamma_{TV}$ denote the space of all linear bounded and causal maps form $\ell_\infty$ to $\ell_\infty$. Given any $G \in \Gamma_{TV}$ and $x \in \ell^\infty$, we define $(Gx)(n) = \sum_{j=0}^{n} g_{n,j} x(j)$, $n \geq 0$, where $g_{n,j} \in R$. Then, the induced norm of G is given by $\|G\|_{\Gamma_{TV}} = \sup_n \sum_{j=0}^{n} |g_{n,j}|$. Denote $\Gamma_{TV}$ as the subspace of $\Gamma_{TV}$ consisting of the time invariant maps. This space is isomorphic to $\ell_1$.

**Definition 1**[8]. Operator $A \in \Gamma_{TV}$ is called slowly time-varying if there exists a constant $\gamma_A$ such that $\|A_t - A_\tau\| \leq \gamma_A$, $\forall t, \tau$. This is denoted by $A_t \in STV(\gamma_A)$.

Given operator $g \in \ell_1$, the integral time absolute error (ITAE) is defined as $ITAE(G) = \sum_{k=0}^{\infty} k |g(k)|$. Let $\hat{G}'(z) = \frac{d}{dz}\hat{G}(z)$, and it follows that $ITAE(G) = \|G'\|_1$.

## 3 Optimal robust stabilizing controller for coprime factor perturbations

Consider the class of uncertainty SISO discrete-time systems described by

$$((A + \Delta_A)y)(t) = ((B + \Delta_B)u)(t) + d(t) \tag{1}$$

where $A, B \in \Gamma_{TI}$.

We assume that $\hat{A}(z)$ and $\hat{B}(z)$ are coprime in $\Gamma_{TI}$, that is, no common zeros in the disk $|z| \leq 1$; $\Delta_A$, $\Delta_B$ are unknown, possibly are time-varying or nonlinear bounded operators, and are assumed to satisfy $\|(\Delta_A \ \Delta_B)\|_1 \leq d_1$ for some $d_1 > 0$; $d(t)$ is a unknown disturbance satisfying $\sup_t |d(t)| \leq \delta_d$.

Consider the feedback system depicted in Fig. 1, where $\hat{G}_0 = \hat{B}\hat{A}^{-1}$ is the nominal model of the plant. The set of all stabilizing compensators of $\hat{G}_0$ can be parameterized in the form[9]

$$S(G_0) = \{\hat{C} = (\hat{X} + \hat{Q}\hat{A})(\hat{Y} - \hat{Q}\hat{B})^{-1} \mid \hat{Q} \in A, \ \hat{Y} - \hat{Q}\hat{B} \neq 0\} \tag{2}$$

where $XB + YA = 1$, $X, Y \in \Gamma_{TI}$.

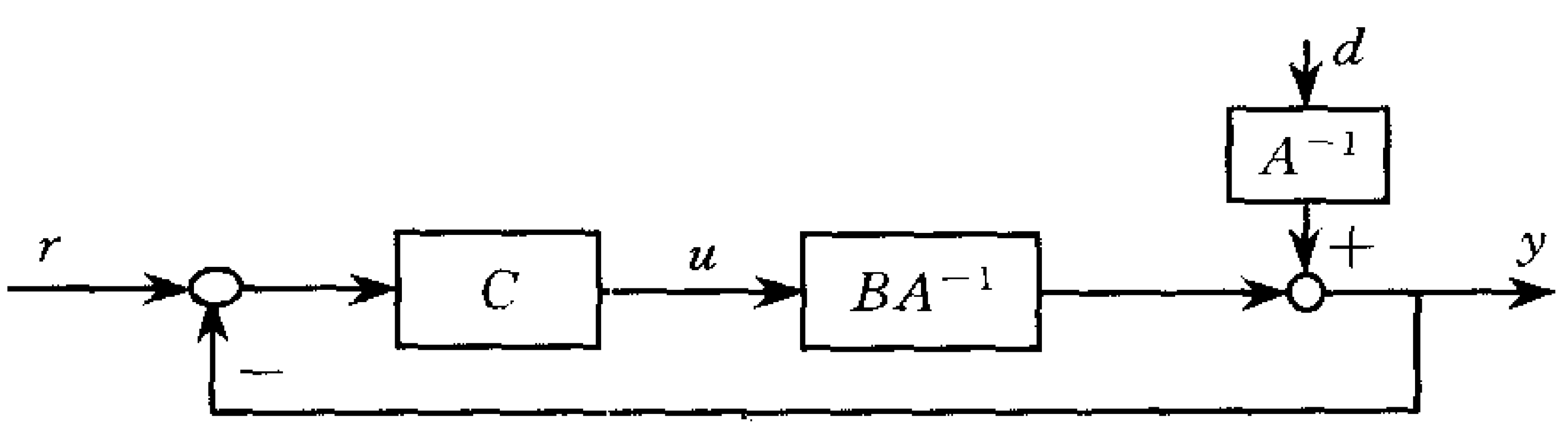


Fig. 1 The feedback system

The problem of optimal stability robustness in the presence of coprime factor perturbations can be reduced to the following tow-block $\ell_1$ optimization problem[10]:

$$\mu = \inf_{C\in S(G_0)} \left\| \begin{pmatrix} G^{yd} \\ G^{ud} \end{pmatrix} \right\|_1 = \inf_{Q\in \ell_1} \left\| \begin{pmatrix} Y-QB \\ X+QA \end{pmatrix} \right\|_1 \tag{3}$$

where $G^{yd}$ and $G^{ud}$ are the maps from the disturbance $d$ to the output $y$ and to the control $u$, respectively. Then, it follows from Theorem 3.1 in [10] that the largest stability margin defined as the maximum $\|(\Delta_A\ \Delta_B)\|$ such that the feedback system remains stable is equal to $\frac{1}{\mu}$.

The two-block $\ell_1$ optimization problem (3) can be rewritten as[10]

$$\mu = \inf_{K\in S} \left\| \begin{pmatrix} Y \\ X \end{pmatrix} - \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \right\|_1 \tag{4}$$

where $S=\left\{K\in \ell_1^{2\times 1} \,\middle|\, (A\quad B)\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}=0\right\}$.

In [1], the optimization problem(4) is shown to be equivalent to an infinite dimensional linear programming, which may not have a finite dimensional solution. This problem can be solved by FMV method. Specifically, if $\delta$ is the truncated order, then the corresponding suboptimization problem is rewritten as the following problem:

$$\mu^\delta = \min_{K\in S^\delta} \left\| \begin{pmatrix} Y \\ X \end{pmatrix} - \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \right\|_1 \tag{5}$$

where $S^\delta = \{K\in S \mid (\hat{Y}-\hat{K}_1)$ and $(\hat{X}-\hat{K}_2)$ are polynomials of degree$\leqslant\delta\}$. There always exists a sufficiently large $\delta$ such that $S^\delta$ is non-empty since $\hat{X}$ and $\hat{Y}$ are polynomials. In the sequel we always take $\delta\geqslant\max\{\deg(\hat{Y}),\deg(\hat{X})\}$. It is known from [1] that problem (5) is equivalent to a finite dimensional LP. Let $(K_1^*\quad K_2^*)^T$ be the optimal solution of (5); then the suboptimal robust stabilizing controller indexed by $\delta$ is $C^* = \frac{\hat{X}-\hat{K}_2^*}{\hat{Y}-\hat{K}_1^*} \stackrel{\text{def}}{=} \frac{\hat{N}^*}{\hat{M}^*}$.

## 4 Analysis of continuity properties of the two-block $\ell_1$ design

Consider a system model described by $G=BA^{-1}$, where $A,B\in\Gamma_{TI}$ are coprime and $A(0)\neq 0$. According to [9], the graph of $G$ is given by $G_G=\begin{bmatrix} A \\ B \end{bmatrix}$. Let $CF$ denote the set of the graphs of all plants $G=BA^{-1}$ and be equipped with the standard graph topology[9] induced over $\ell_\infty$. Let $G_i=B_iA_i^{-1}$, $C_i^*=N_i^*(M_i^*)^{-1}$, $i=0,1,\cdots$ denote a sequence of plants and their corresponding two-block $\ell_1$ optimization design solutions, respectively. We may state the continuity problem as: does $G_{C_i^*}$ approach $G_{C_0^*}$ as $G_{P_i}\to G_{P_0}$ in $CF$? We will study this problem in the followings.

**Lemma 1**[9, p.238]. $G_{G_i}\to G_{G_0}$ in $CF$ if and only if $A_i\to A$, and $B_i\to B$ in $\ell_1$.

**Lemma 2**[9, p.239]. Suppose $\{A_i\}\subset\ell_1$ and $\{B_i\}\subset\ell_1$. If $A_i\to A$ and $B_i\to B$ in $\ell_1$, then there exist $X_i$, $Y_i$, $X$, $Y\in\ell_1$ such that $X_iB_i+Y_iA_i=1$ and $XB+YA=1$, respectively, and $X_i\to X$, $Y_i\to Y$ in $\ell_1$ space.

**Lemma 3**[11]. Let $M$ be a subspace in a real normed linear space $X$. Let $x^*\in X^*$, and let $d$ denote its distance from $M^\perp$. Then

$$d = \min_{m^*\in M^\perp} \|x^*-m^*\| = \sup_{x\in BM} \langle x,x^*\rangle,$$

where the minimum is achieved for some $m_0^*\in M^\perp$.

**Lemma 4.** Suppose $\mu_i$ and $\mu$ are the corresponding optimal values of problem (4) for $G_i$ and $G_0$, respectively. If $G_{G_i}\to G_{G_0}$ in $CF$, then $\mu_i\to\mu$.

The proof is omitted due to the limitation of space.

As mentioned above, the general two-block $\ell_1$ optimal design problem may not have a finite dimensional solution. This might lead to the discontinuity of the optimal solution on system parameters. For this reason, we show alternatively that for a suitable truncated or-

*der* $\delta$, the corresponding suboptimal value $\mu^\delta$ is continuous.

**Lemma 5.** Given truncated degree $\delta-1$, if $G_{G_i}\to G_{G_0}$ in $CF$, then $\mu_i^{\delta-1}\to\mu^{\delta-1}$.

The proof is omitted due to the limitation of space.

**Theorem 1.** Suppose that for given $\delta$, suboptimal problem (5) has a unique solution for all $i$, and denote the corresponding suboptimal controllers by $\{C_i^*\}$ and $C_0^*$. If $G_{G_i}\to G_{G_0}$, then $G_{C_i^*}\to G_{C_0^*}$ in $CF$.

**Proof.** Let $G_{C_i^*}=(M_i^* \quad N_i^*)^{\mathrm T}$, $G_{C_0^*}=(M_0^* \quad N_0^*)^{\mathrm T}$. From Lemma 1, to prove the above theorem we need only show $\|M_i^*-M_0^*\|_1\to 0$ and $\|N_i^*-N_0^*\|_1\to 0$. Since $\hat M_i^*$ and $\hat N_i^*$ are polynomials whose degrees not higher than $\delta$, and since the bounded closed set in finite dimensional space is compact, it follows that the sequence $\{(M_i^* \quad N_i^*)^{\mathrm T}\}$ has a convergent subsequence, denoted by $\{(M_{i_k}^* \quad N_{i_k}^*)^{\mathrm T}\}$. Let $\lim\limits_{k\to\infty}(M_{i_k}^* \quad N_{i_k}^*)^{\mathrm T}=(\widetilde M_0^* \quad \widetilde N_0^*)^{\mathrm T}$. Then it follows from the constraint set of problem(5) that $(A_{i_k} \quad B_{i_k})\times\begin{pmatrix}M_{i_k}^*\\ N_{i_k}^*\end{pmatrix}=1$, therefore, $(A \quad B)\times\begin{pmatrix}\widetilde M_0^*\\ \widetilde N_0^*\end{pmatrix}=\lim\limits_{k\to\infty}(A_{i_k} \quad B_{i_k})\times\begin{pmatrix}M_{i_k}^*\\ N_{i_k}^*\end{pmatrix}=1$. This means that $(\widetilde M_0^* \quad \widetilde N_0^*)^{\mathrm T}$ is a feasible solution of (5). Also, from Lemma 5 we have $\lim\limits_{k\to\infty}\|(M_{i_k}^* \quad N_{i_k}^*)^{\mathrm T}\|_1=\|(\widetilde M_0^* \quad \widetilde N_0^*)^{\mathrm T}\|_1=\mu^\delta$; then $(\widetilde M_0^* \quad \widetilde N_0^*)^{\mathrm T}$ is a solution to problem (5). It results from the uniqueness assumption that $(\widetilde M_0^* \quad \widetilde N_0^*)^{\mathrm T}=(M_0^* \quad N_0^*)^{\mathrm T}$. This claims that all the cluster points of $\{(M_i^* \quad N_i^*)^{\mathrm T}\}$ are the same and equal to $(M_0^* \quad N_0^*)^{\mathrm T}$, hence, $\|(M_i^* \quad N_i^*)^{\mathrm T}-(M_0^* \quad N_0^*)^{\mathrm T}\|_1\to 0$, that is, $\|M_i^*-M_0^*\|_1\to 0$ and $\|N_i^*-N_0^*\|_1\to 0$. This completes the proof. □

**Remark 1.** The condition that problem (5) has a unique solution is generic. It is well known that (5) is equivalent to a LP. Geometrically, the nonuniqueness of the solution of LP is due to the fact that an active surface of the boundary of the feasible region lies on the hyperplane defined by the objective function. It is clear that this situation is nongeneric since an arbitrarily small perturbation in the plant parameters will remove this situation.

Let $\rho=\mu^\delta-\mu$ denote the deviation of suboptimal value from the optimal value. It should be pointed out that the FMV method gives no information about the relation between $\rho$ and $\delta$. The following lemma shows that for any given $\rho>0$, there is a uniform bound on the degree of suboptimal solutions of any convergent sequence of plants such that their deviation of suboptimal value from the optimal one is bounded by $\rho$.

**Lemma 6.** If in $CF$, $G_{G_i}\to G_{G_0}$, then for any given $\rho>0$, there exits a truncation $\delta^*$ such that $|\mu_i^{\delta^*}-\mu_i|\leqslant\rho$ for all $i$.

The proof is omitted due to the limitation of space.

Furthermore, in order to apply $\ell_1$ design methodology to adaptive control, it is desirable to prove that the two-block $\ell_1$ optimization design is uniformly continuous with respect to certain restricted set of plants. To do so, we define a parameter set of plants, and denote it by $PS$. Assume that the set $PS$ is a compact and convex set such that plants whose parameters in $PS$ have no common zeros in the disk $|z|\leqslant 1$, and that the corresponding FMV formulation has a unique solution for a suitable given truncation.

**Theorem 2.** Denote by $\hat C_\delta(G)$ the suboptimal controller corresponding to plant $G$ and $\mu^\delta(G)$ the corresponding suboptimal value; denote by $F_{PS}$ the set of all plants with parameters in the set $PS$. Then the set of all suboptimal controllers $\hat C_\delta(G)$ for all $G\in F_{PS}$ has the following properties: 1) given any $\rho>0$, there exists $\delta^*$ such that $|\mu^{\delta^*}(G)-\mu(G)|\leqslant\rho$ for all $G\in F_{PS}$; 2) $\hat C_{\delta^*}(G)$ is uniformly continuous on $F_{PS}$.

**Proof.** It is clear from Lemma 6 that given $\rho>0$ and $G_0\in F_{PS}$, there exist $\delta$ and $\varepsilon>0$

such that $|\mu^{\delta}(G)-\mu(G)|\leqslant\rho$ for any $G\in B(G_0,\varepsilon)$, where $B(G_0,\varepsilon)$ denotes the open ball in $F_{PS}$ with center at $G_0$ and radius of $\varepsilon$. Such balls provide an open covering of $F_{PS}$. By compactness of $F_{PS}$, there exists a subcover of $F_{PS}$. By taking $\delta^*$ to be the largest of the $\delta$ corresponding to this finite subcover, $|\mu^{\delta^*}(G)-\mu(G)|\leqslant\rho$ for all $G\in F_{PS}$. Property 1) follows.

By Theorem 1, for $\delta^*$ given above, $\hat{C}_{\delta^*}(G)$ is continuous at $G\in F_{PS}$. It results from compactness of $F_{PS}$ that $\hat{C}_{\delta^*}(G)$ is uniformly continuous on $F_{PS}$. This completes the proof. □

## 5 Adaptive robust control scheme

In this section, we demonstrate an application of the results developed in previous sections to adaptive robust control. We will consider the system described by (1) whose parameters are unknown, and make the following assumptions:

**Assumption 1.** The degree of the model $n=\max\{n_1,n_2\}$ is known a priori.

**Assumption 2.** The plant parameter of the model lies in the compact convex set $PS$ (introduced in section 4) which is known a priori.

**Assumption 3.** $\max_t|d(t)|\leqslant\delta_d$, $\|\Delta_A\|_1\leqslant\delta_\Delta$, where $d$ and $d_1$ are known.

Equation (1) can be rewritten as:

$$y(t)=\phi(t-1)^{\mathrm{T}}\theta_0+(-\Delta_A y)(t)+(\Delta_B u)(t)+d(t) \tag{8}$$

where $\theta_0\in PS$, $\theta_0^{\mathrm{T}}=(-a(1),\cdots,-a(n_1),b(1),\cdots,b(n_2))$, $\phi(t-1)^{\mathrm{T}}=(y(t-1),\cdots,y(t-n_1),u(t-1),\cdots,u(t-n_2))$. Let

$$d_\Delta(t)=\|(\Delta_A\quad\Delta_B)\|_1\max_{0\leqslant\tau\leqslant t}(|y(\tau)|,|u(\tau)|) \tag{9}$$

### 5.1 The adaptive scheme

It should be noted in (8) that coprime factor perturbations will cause uncertainty in addition to disturbance $d(t)$. In the followings we will present a robust projection algorithm with dead zone.

$$\theta(t)=\text{projection of } p(t) \text{ on } PS \tag{10}$$

$$p(t)=\theta(t-1)+\frac{\eta(t-1)\phi(t-1)}{c+\phi(t-1)^{\mathrm{T}}\phi(t-1)}[y(t)-\phi(t-1)^{\mathrm{T}}\theta(t-1)] \tag{11}$$

where $c>0$. Denote by $e(t)=y(t)-\phi(t-1)^{\mathrm{T}}\theta(t-1)$ the estimate error, and take

$$\eta(t-1)=\begin{cases}\dfrac{|e(t)|-2(d_\Delta(t)+d)}{|e(t)|}, & |e(t)|>2(d_\Delta(t)+d)\\ 0, & \text{Otherwise}\end{cases} \tag{12}$$

**Lemma 7.** For the algorithm (10)~(12) when (8) and Assumptions 1~3, it follows that:

1) $\|\theta(t)-\theta_0\|\leqslant\|\theta(t-1)-\theta_0\|\leqslant\|\theta(0)-\theta_0\|$;

2) $\lim\limits_{N\to\infty}\sum\limits_{t=1}^{N}\dfrac{[|e(t)|-2(d_\Delta(t)+d)]^2}{c+\phi(t-1)^{\mathrm{T}}\phi(t-1)}<\infty$, which implies

a) $\lim\limits_{t\to\infty}\dfrac{[|e(t)|-2(d_\Delta(t)+d)]^2}{c+\phi(t-1)^{\mathrm{T}}\phi(t-1)}=0$, and

b) $\lim\limits_{t\to\infty}\|\theta(t)-\theta(t-1)\|=0$.

**Proof.** It results from [13] that the convergent properties of the algorithm will be improved by constraining $\theta(t)$ to remain within the region of $PS$. Thus, we need only to prove the properties in the non-constrained case. Taking $\theta(t)=p(t)$ and subtracting $\theta_0$ from both side of (11) give $\tilde{\theta}(t)=\tilde{\theta}(t-1)+\dfrac{\eta(t-1)\phi(t-1)}{c+\phi(t-1)^{\mathrm{T}}\phi(t-1)}e(t)$ with $\tilde{\theta}(t)=\theta(t)-\theta_0$. Letting $w(t)=(-\Delta_A y)(t)+(\Delta_B u)(t)+d(t)$, and using (12), we have $\|\tilde{\theta}(t)\|^2\leqslant$

$\|\tilde{\theta}(t-1)\|^2+\frac{2\eta(t-1)e(t)w(t)-2\eta(t-1)e(t)^2+\eta(t-1)^2e(t)^2}{c+\phi(t-1)^{\mathrm{T}}\phi(t-1)}$. When $|e(t)|>2(d_\Delta(t)+d)$, by noting that $0\leqslant\eta(t-1)<1$, we get

$$\|\tilde{\theta}(t)\|^2\leqslant\|\tilde{\theta}(t-1)\|^2+\frac{[|e(t)|-2(d_\Delta(t)+d)][2|w(t)|-|e(t)|]}{c+\phi(t-1)^{\mathrm{T}}\phi(t-1)}\leqslant$$

$$\|\tilde{\theta}(t-1)\|^2-\frac{[|e(t)|-2(d_\Delta(t)+d)]^2}{c+\phi(t-1)^{\mathrm{T}}\phi(t-1)} \tag{13}$$

When $|e(t)|\leqslant 2(d_\Delta(t)+d)$, it follows that $\|\tilde{\theta}(t)\|=\|\tilde{\theta}(t-1)\|$. It is clear that $\{\|\tilde{\theta}(t)\|\}$ is a non-increasing sequence. This establishes 1).

Summing both sides of (11) with respect to $t$, we obtain 2). Then a) follows since $\left\{\frac{[|e(t)|-2(d_\Delta(t)+d)]^2}{c+\phi(t-1)^{\mathrm{T}}\phi(t-1)}\right\}$ is a non-negative sequence. To establish b), let $\theta(t)=p(t)$. It follows form(11) that $\theta(t)-\theta(t-1)=\frac{\eta(t-1)\phi(t-1)}{c+\phi(t-1)^{\mathrm{T}}\phi(t-1)}e(t)$, hence, $\|\theta(t)-\theta(t-1)\|^2=\frac{\eta(t-1)^2\phi(t-1)^{\mathrm{T}}\phi(t-1)}{[c+\phi(t-1)^{\mathrm{T}}\phi(t-1)]^2}e(t)^2\leqslant\frac{[|e(t)|-2(d_\Delta(t)+d)]^2}{c+\phi(t-1)^{\mathrm{T}}\phi(t-1)}$. Then b) follows from a). This completes the proof. □

Suppose $G_{t-1}=B_{t-1}A_{t-1}^{-1}$ is the system model generated by an estimate procedure described above at time $t$, and the input/output sequences are related by

$$A_{t-1}y(t)=B_{t-1}u(t)+e(t) \tag{14}$$

where $e(t)$ is the estimate error.

Suppose the controller is given by

$$M_tu(t)=N_t(r-y)(t) \tag{15}$$

Combining parameter estimation and $\ell_1$ optimal control design gives the following indirect adaptive scheme:

1) Estimate $A_t$, $B_t$ using algorithm (10)~(12);

2) Select a suitable truncated order and compute $M_t$ and $N_t$ via solving suboptimizaton (5);

3) Compute and implement control

$$M_tu(t)=N_t(r-y)(t),\quad e(t)=y(t)-\phi(t-1)^{\mathrm{T}}\theta(t-1);$$

4) Back to 1).

### 5.2 Analysis of global stability and robustness of the adaptive system

**Lemma 8.** Suppose sequences $\{d_\Delta(t)\}$, $\{e(t)\}$ and $\{\phi(t-1)\}$ satisfy the following conditions:

1) $\lim\limits_{t\to\infty}\frac{[|e(t)|-2(d_\Delta(t)+d)]^2}{c+\phi(t-1)^{\mathrm{T}}\phi(t-1)}=0$;

2) There exist constants $0\leqslant c_1<\infty$, and $0<c_2<\infty$ such that $\|\varphi(t)\|\leqslant c_1+c_2\max\limits_{0\leqslant\tau\leqslant t}|e(\tau)|$;

3) There exist constants $0\leqslant r_1<\infty$, and $0<r_2<\frac{1}{2}$, such that $d_\Delta(t)\leqslant r_1+r_2\max\limits_{0\leqslant\tau\leqslant t}|e(\tau)|$.

Then, sequences $\{e(t)\}$ and $\{\phi(t-1)\}$ are bounded.

**Proof.** If sequence $\{e(t)\}$ is bounded, then condition 2) implies that sequence $\{\phi(t-1)\}$ is bounded. Now assume that $\{e(t)\}$ is unbounded; there exists a subsequence $\{e(t_k)\}$ such that $\lim\limits_{k\to\infty}|e(t_k)|=\infty$, and $|e(t)|\leqslant|e(t_k)|$ as $t<t_k$. Then it follows from conditions 2) and 3) that there is an integer $N$ such that $\|\phi(t_k)\|\leqslant c_1+c_2|e(t_k)|$, $d_\Delta(t_k)\leqslant r_1+r_2|e(t_k)|$ whenever $k>N$. Thus, $|e(t_k)|-2(d_\Delta(t_k)+d)\geqslant(1-2r_2)|e(t_k)|-2(d+r_1)$. Since $\{e(t_k)\}$ is unbounded and $0<r_2<\frac{1}{2}$, $|e(t_k)|-2(d_\Delta(t_k)+d)\geqslant(1-2r_2)$

$|e(t_k)|-2(d+r_1)>0$ when $N$ is large enough. Therefore, we have

$$\frac{[|e(t_k)|-2(d_\Delta(t_k)+d)]^2}{c+\phi(t_k-1)^{\mathrm T}\phi(t_k-1)}\geqslant\frac{[(1-2r_2)|e(t_k)|-2(d+r_1)]^2}{c+\phi(t_k-1)^{\mathrm T}\phi(t_k-1)}\geqslant\frac{[(1-2r_2)|e(t_k)|-2(d+r_1)]^2}{c+[c_1+c_2|e(t_k)|]^2}.$$

It follows from $\lim_{k\to\infty}|e(t_k)|=\infty$ that

$$\lim_{k\to\infty}\frac{[|e(t_k)|-2(d_\Delta(t_k)+d)]^2}{c+\phi(t_k-1)^{\mathrm T}\phi(t_k-1)}\geqslant\frac{(1-2r_2)^2}{c_2^2}>0,$$

which contradicts condition 1) in the lemma. The lemma follows. □

**Theorem 3.** Suppose the plant described by (1) satisfies Assumptions 1~3, and that the adaptive system depicted by Fig. 2 satisfies $\sup_t\left\|\begin{pmatrix}G_t^{ye}\\ G_t^{ue}\end{pmatrix}\right\|_1\leqslant\frac{r_2}{d_1}$, where, $0<r_2<\frac{1}{2}$, $G_t^{ue}$ and $G_t^{ye}$ are transfer functions from $e$ to $u$ and $y$ at time $t$, respectively. Then sequences $\{e(t)\}$, $\{u(t)\}$ and $\{y(t)\}$ are bounded.

To prove this theorem we need the following theorem.

**Theorem 4**[8]. Given a time-varying closed loop system composed of $C\in\Gamma_{TV}$ and $G\in\Gamma_{TV}$, denote by $\hat G_t=\hat N_t\hat M_t^{-1}$ the estimate model, $\hat C_t=\hat B_t\hat A_t^{-1}$ the controller designed on the basis of the estimate model, $\hat T_t=\hat M_t\hat A_t+\hat N_t\hat B_t$ the corresponding closed loop polynomial for each time instant $t$, and assume the following

1) There exist positive $\gamma_A$ and $\gamma_B$, such that $A_t\in STV(\gamma_A)$, and $B_t\in STV(\gamma_B)$;

2) There exist positive $\gamma_M$ and $\gamma_N$, such that $M_t\in STV(\gamma_M)$, and $N_t\in STV(\gamma_N)$;

3) The $\ell_1$ norm of $A_t$, $B_t$, $M_t$, $N_t$, $N_t$ and ITAE $e_{ITA}$ are uniformly bounded in $t$, and from assumption (1) this also implies that there exists $\gamma_T$ such that $T_t\in STV(\gamma_T)$;

4) The inverse $T_t^{-1}$ and ITAE $e_{ITA}$ are uniformly bounded.

Then, there exists a nonzero constant $\gamma$ such that if $\gamma_A$, $\gamma_B$, $\gamma_M$, $\gamma_N\leqslant\gamma$, then the time-varying system is internally stable.

**The proof of theorem 3.** First, we show that the closed loop system in Fig. 2 is internally stable. To do so, it is enough to verify the assumptions in Theorem 4. Since $\|A_t-A_{t-1}\|\leqslant\|\theta(t)-\theta(t-1)\|$, $\|B_t-B_{t-1}\|\leqslant\|\theta(t)-\theta(t-1)\|$, and from property b) in Lemma 7, it is clear that Assumption 1 is satisfied.

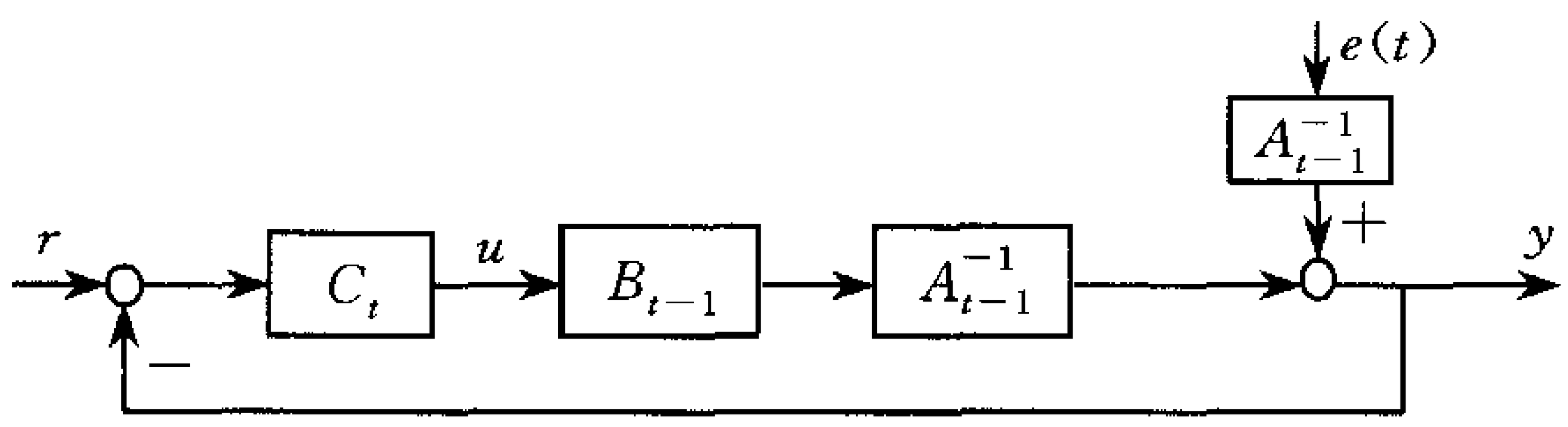


Fig. 2 The adaptive system

The $\ell_1$ optimization design procedure induces the following map $D:\ell^1\otimes\ell^1\to\ell^1\otimes\ell^1$, $D(B,A)=(N,M)$. From Assumption 2, $PS$ is compact. It follows from Theorem 2 that the function $D$ is uniformly continuous on $PS$, that is, given $\varepsilon>0$, there exists $\delta>0$, and if $\|A_t-A_{t-1}\|<\delta$ and $\|B_t-B_{t-1}\|<\delta$ then $\|M_t-M_{t-1}\|<\varepsilon$ and $\|N_t-N_{t-1}\|<\varepsilon$. From property b) in Lemma 7, we have $\|A_t-A_{t-1}\|<\delta$ and $\|B_t-B_{t-1}\|<\delta$, as $t$ is large enough. This implies that Assumption 2 is satisfied.

From property 1) in Lemma 7 and Assumption 1, $\|\theta(t)\|_1$ is bounded; furthermore, since $PS$ is compact, $\|\theta(t)\|_1$ is uniformly bounded in $t$, that is, $\|A_t\|_1$ and $\|B_t\|_1$ are uniformly bounded. Since the number of the degrees of $\theta(t)$ is not larger than $2n+1$, $ITAE(A_t)$ and $ITAE(B_t)$ are uniformly bounded in $t$. According to Theorem 2, it follows from

Assumption 2 that the controller is uniformly continuous with respect to the plant model, implying that $\|M_t\|_1$ and $\|N_t\|_1$ is uniformly bounded in $t$. From Theorem 2, for a given $\rho$, the number of the degrees of suboptimal controller is uniformly bounded on $PS$. Combining the above, it follows that $ITAE(M_t)$ and $ITAE(N_t)$ are uniformly bounded in $t$, that is, Assumption 3 is satisfied.

From the controller $\hat{C}_t=\frac{\hat{N}_t}{\hat{M}_t}=\frac{\hat{X}_t-\hat{K}_{t2}^*}{\hat{Y}_t-\hat{K}_{t1}^*}$ obtained at time instant $t$, the closed loop polynomial $\hat{T}_t=\hat{M}_t\hat{A}_t+\hat{N}_t\hat{B}_t=1$, which implies that Assumption 4 is satisfied.

Secondly, we show that the estimate error $e(t)$ is bounded for all time instants $t$'s. Since, the closed loop system is internally stable by Theorem 4, it follows that there exist constants $k_1\geqslant 0$ and $k_2>0$ such that $|y(t)|\leqslant k_1+k_2\max\limits_{0\leqslant\tau\leqslant t}|e(\tau)|$ and $|u(t)|\leqslant k_1+k_2\max\limits_{0\leqslant\tau\leqslant t}|e(\tau)|$. This implies that there exist constants $0\leqslant c_1<\infty$ and $0<c_2<\infty$ such that $\|\varphi(t)\|\leqslant c_1+c_2\max\limits_{0\leqslant\tau\leqslant t}|e(\tau)|$. It follows from (9) that

$$d_\Delta(t)\leqslant d_1\max_{0\leqslant\tau\leqslant t}\{|y(\tau)|,\ |u(\tau)|\}\leqslant d_1\sup_\tau\{\max_{0\leqslant\tau\leqslant t}\{\|G_\tau^{ye}\|_1,\ \|G_\tau^{ue}\|_1\}\}\max_{0\leqslant\tau\leqslant t}|e(\tau)|\leqslant$$
$$d_1\sup_\tau\left\|\begin{bmatrix}G_\tau^{ye}\\ G_\tau^{ue}\end{bmatrix}\right\|_1\max_{0\leqslant\tau\leqslant t}|e(\tau)|\leqslant r_2\max_{0\leqslant\tau\leqslant t}|e(\tau)|.$$

By property a) in Lemma 7, we obtain $\lim\limits_{t\to\infty}\frac{[|e(t)|-2(d_\Delta(t)+d)]^2}{c+\phi(t-1)^{\mathrm{T}}\phi(t-1)}=0$. According to Lemma 8, it follows that $\{e(t)\}$ and $\{\phi(t-1)\}$ are bounded sequences. This completes the proof. □

In the following, we discuss the robustness of the adaptive scheme. From Theorem 3, *if the plant satisfies Assumptions* 1～3, *then the unmodeled dynamics permitted by the* adaptive system are $d_1\leqslant r_2\left[\sup\limits_t\left\|\begin{pmatrix}G_t^{ye}\\ G_t^{ue}\end{pmatrix}\right\|_1\right]^{-1}$. In the adaptive scheme, the controller is designed on the basis of two-block $\ell_1$ optimization, which implies that $\sup\limits_t\left\|\begin{pmatrix}G_t^{ye}\\ G_t^{ue}\end{pmatrix}\right\|_1$ is minimal. Therefore, under Assumptions 1～3, the $\ell_1$ optimization is the best one to be applied to design controller for adaptive system. Combining (10) and Lemma 8, it is clear that $r_2$ becomes larger as the estimate error gets smaller. This claims that the robustness of adaptive control system can be improved by decreasing the estimate error.

## 6 Conclusions

This paper has studied the continuity properties of the optimal design of BIBO stability robustness for discrete time systems in the presence of coprime factor perturbations. It has been shown that the $\ell_1$ suboptimal design for the plants with coprime factor perturbations is continuous in terms of a map from the plant model to the suboptimal controller parameter. Furthermore, the $\ell_1$ suboptimal design is shown to be uniformly continuous as long as the set of plants is restricted to a compact set. On the basis of the continuity of $\ell_1$ optimization, an adaptive robust control scheme is proposed when a discrete-time plant is subject to both coprime factor perturbations and unknown external disturbances. The scheme is shown to have the optimal robust stability under certain conditions. A posterior computable condition for the stability of the adaptive scheme is also provided.

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# 互质因子摄动系统最优 $\ell_1$ 鲁棒控制：连续性与自适应控制

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**摘　要**　针对具有互质因子摄动和未知干扰的离散时间系统研究了一种自适应鲁棒控制策略. 本文的主要工作包括三个方面. 首先建立了互质因子摄动系统最优 $\ell_1$ 鲁棒控制设计的连续性. 然后，提出了一种带变死区的参数鲁棒估计投影算法. 最后，结合所提出的参数估计算法和最优 $\ell_1$ 鲁棒控制，利用确定性等价原理提出了互质因子摄动系统的一种新的自适应鲁棒控制方法. 基于本文建立的 $\ell_1$ 优化设计的连续性，证明了自适应鲁棒控制的全局稳定性，给出了自适应控制系统稳定性的后验可计算条件.

**关键词**　互质因子摄动系统，连续性，最优 $\ell_1$ 鲁棒控制，自适应控制

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