A Fast Wavelet Analysis Algorithm Based on Rotation Angle Series[1)]

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Abstract A new-style fast wavelet transform, named "rotation angle series fast wavelet transform——RAS-FWT", is put forward in this paper. The restriction on the rotation angle series is also brought forward. Instead of the traditional method based on convolution, RAS-FWT adopts iteration structure that is more easily realized on microprocessors. Digital delay is implemented by "cyclic pointers". So the code will be with higher efficiency and speed. Mallat algorithm established the one-to-one mapping between orthogonal wavelets and N-length discrete coefficient series, while RAS-FWT discovers the similar one-to-one mapping between orthogonal wavelets and N/2-length discrete angle series. So it has half operations less than FWT. In addition, new techniques based on the "rotation angle series" will inaugurate a brand-new approach for the wavelet construction theory, and will become a new branch of the academic subject.

Key words Wavelet, fast algorithm, rotation angle series, RAS-FWT, cyclic pointer

1 Introduction

In all kinds of engineering and scientific researches, wavelet analysis, an information processing technique with many outstanding performances in time-frequency analysis and multiscale feature extraction, has been widely introduced all over the world. Ultrasound signal and image processing as a significant technique in the fields of material detecting, fault diagnosis and biomedicine engineering also chooses wavelet analysis as a crucial means for the pretreatment and feature extraction. Realtime and self-adaptive performance has become the principal requirement for the relative algorithms.

In 1989, Mallet algorithm[1,2] (FWT) came out, which indicated the great span of wavelet analysis from theory to engineering. Mallet algorithm is based on the one-to-one mapping between CQMFB (conjugate quadrature mirror filter bank) and orthogonal wavelets. $h_0(n)$ and $h_1(n)$ are the "analysis filters" of CQMFB (low-pass and high-pass, respectively), with conjugate quadrature mirror relation and the same length. That is, Mallet algorithm sets up one-to-one mapping between orthogonal wavelets and N-length discrete coefficient series, and Mallet algorithm is the fast transform based on N-length discrete coefficient series.

In this paper, an N/2-length discrete angle series $\{\alpha_1, \alpha_2, \cdots, \alpha_M\}$, named "rotation angle series", is constructed. They have simple linear restriction. Accordingly, there are $M-1$ free-degrees for them. The similar one-to-one mapping exists between the N/2-length discrete angle series and orthogonal wavelets. So the corresponding algorithm, named "rotation angle series fast wavelet transform——RAS-FWT", is invented.

RAS-FWT is implemented by the iterative product of N/2 unit-stage transform matrixes, instead of the traditional convolutions. So the calculation is decreased and higher flexibility of choosing implementation schemes is achieved. An implementation scheme based on "cyclic pointers" is put forward in this paper. In addition, because of the simple linear restriction on the rotation angle series, RAS-FWT processes self-adaptive capability to choose and replace wavelet bases. All these characteristics are not possessed by Mallet algorithm.

1) Supported by the 985 Foundation of Tsinghua University (101051)
Received December 16, 2002; in revised form March 13, 2003
收稿日期 2002-12-16; 收修改稿日期 2003-03-13

2 The in-depth analysis and equivalent translation to the multiphase matrix of the orthogonal wavelet filter banks

2.1 The structure analysis of orthogonal wavelet filter banks

CQMFB, corresponding to orthogonal wavelets, satisfies the restrictions as follows[3].

$$H_1(z) = z^{-(2M-1)} H_0(-z^{-1}) \tag{1}$$

$$h_1(n) = (-1)^{N-1-n} h_0(N-1-n) \tag{2}$$

(1) is equivalent to (2) in the time domain. $N(=2M)$ is the length of the filters.

$$\sum_{n=-\infty}^{\infty} h_0(n) = \sqrt{2} \tag{3}$$

$$\sum_{n=-\infty}^{\infty} h_1(n) = 0 \tag{4}$$

According to (2), (3) and (4) are equivalent to

$$\sum_{n=0}^{M-1} h_0(2n) = \sum_{n=0}^{M-1} h_0(2n+1) = \frac{\sqrt{2}}{2} \tag{5}$$

The analysis filter bank is expressed by polyphase structure as

$$\begin{aligned} H_0(z) &= E_{00}(z^2) + z^{-1}E_{01}(z^2) \\ H_1(z) &= E_{10}(z^2) + z^{-1}E_{11}(z^2) \end{aligned} \tag{6}$$

The matrix form of (6) is

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = E(z^2)\begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} \tag{7}$$

where

$$E(z) = \begin{bmatrix} E_{00}(z) & E_{01}(z) \\ E_{10}(z) & E_{11}(z) \end{bmatrix} \tag{8}$$

$E(z)$ is called the polyphase matrix of the analysis filter bank. Thus, (5) is equivalent to

$$E(z=1) = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \tag{9}$$

The polyphase structure of the analysis filter bank is shown in Fig. 1.

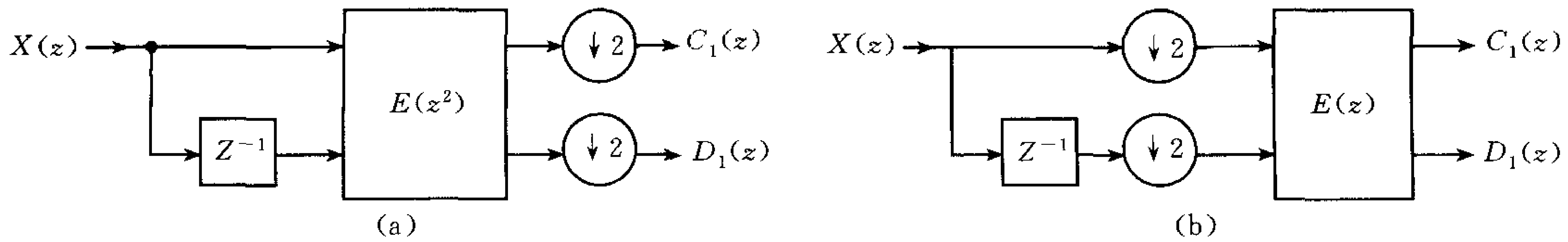

Fig. 1 The polyphase structure of the analysis filter bank

2.2 The paraunitary structure of the polyphase matrix depicted by rotation angle series, and its equivalent translation

Above all, we review Section 2.1: (1) and (9) become the "necessary and sufficient condition" for the orthogonal wavelet analysis filter bank.

For the lattice structure shown in Fig. 2, assume that $\alpha_i \neq 0$. Then

$$Q_i(z) = z^{-(2i-1)} P_i(-z^{-1}) \tag{10}$$

is achieved by induction.

Thus it can be seen that every stage of this structure satisfies (1). So, as long as (9) is satisfied, this structure can be used to construct the orthogonal wavelet analysis filter bank.

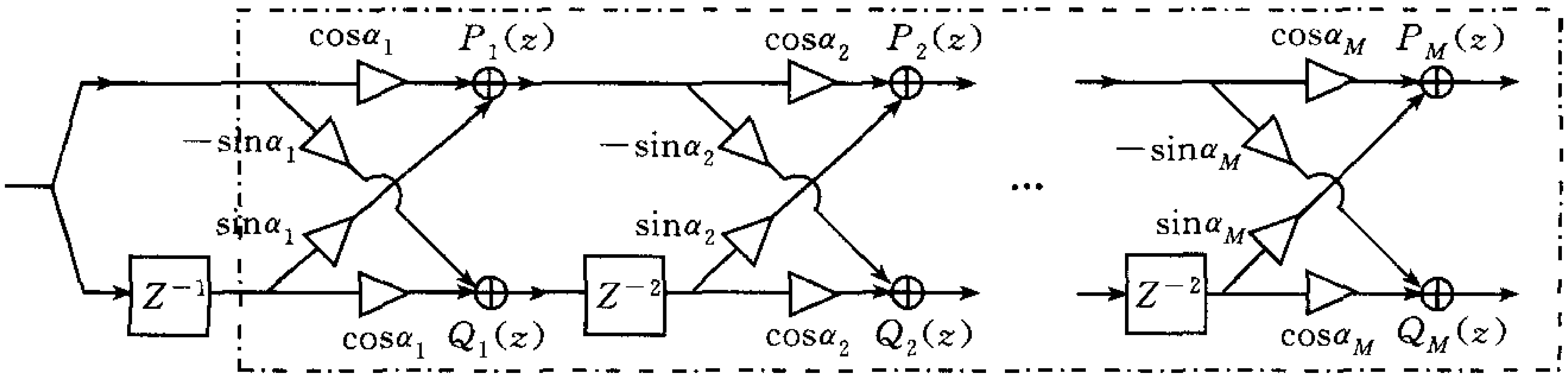

Fig. 2 The lattice structure of the analysis filter bank

Contrasting Fig. 1 with Fig. 2, we find two significant equivalence relations shown in Fig. 3.

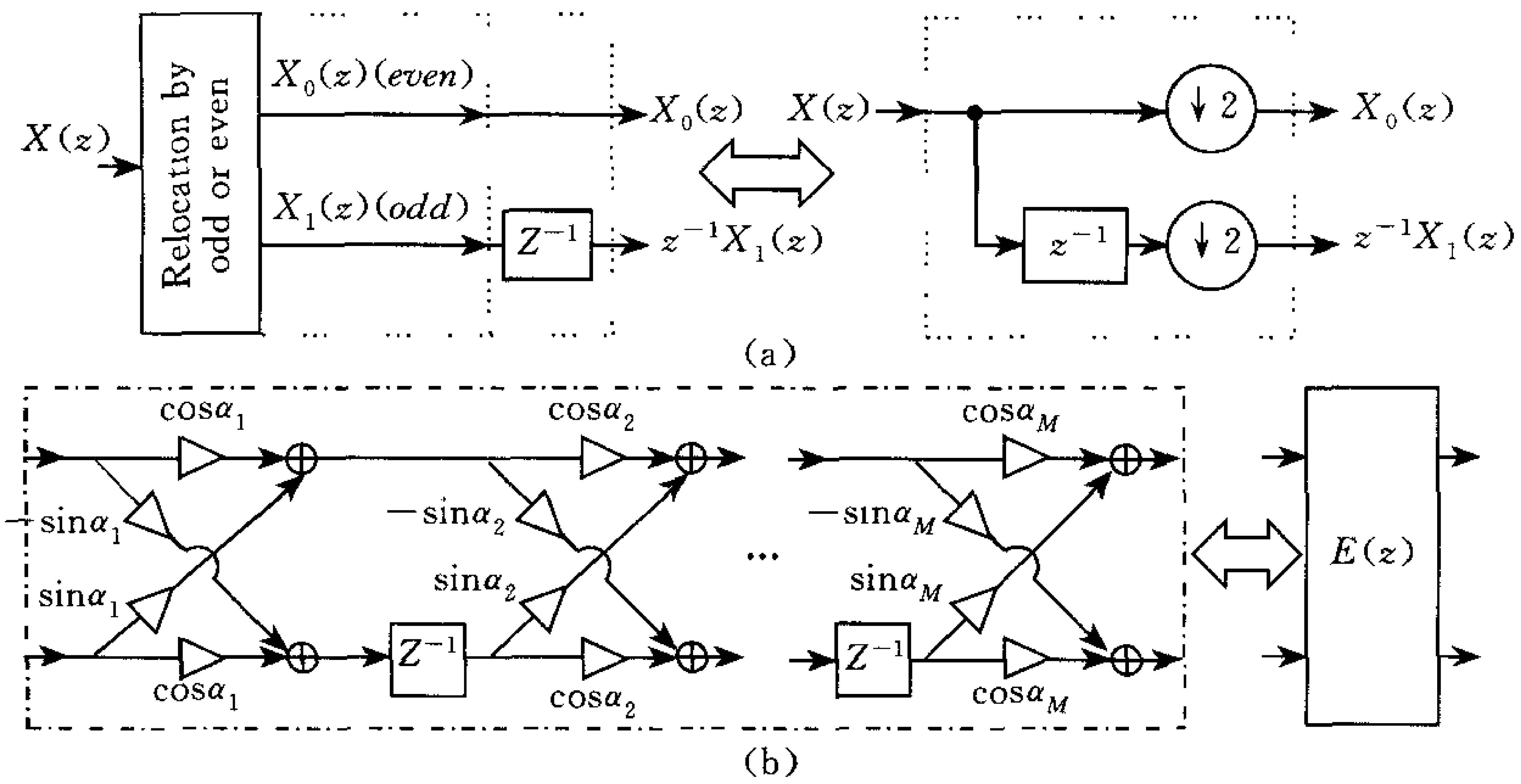

Fig. 3 The equivalent translation

In Fig. 3,

$$X(z) = X_0(z^2) + z^{-1}X_1(z^2) \tag{11}$$

Fig. 3(b) is expressed in matrix form as

$$E(z) = \prod_{i=M}^{2}\left\{\begin{bmatrix} \cos\alpha_i & \sin\alpha_i \\ -\sin\alpha_i & \cos\alpha_i \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}\right\} \times \begin{bmatrix} \cos\alpha_1 & \sin\alpha_1 \\ -\sin\alpha_1 & \cos\alpha_1 \end{bmatrix} \tag{12}$$

where $\begin{bmatrix} \cos\alpha_i & \sin\alpha_i \\ -\sin\alpha_i & \cos\alpha_i \end{bmatrix}$ is the "rotation matrix" as we know in algebra. Thus α_i is called "rotation angle".

Apparently, $E(z)$ is a paraunitary matrix, namely $\widetilde{E}(z)E(z) = I$, where $\widetilde{E}(z) = E^{\mathrm{T}}(z^{-1})$.

That is to say, the polyphase matrix of the orthogonal wavelet analysis filter bank has the paraunitary structure in (12) and in Fig. 3(b).

2.3 The restriction on the rotation angle series

From (9) and (12), $\alpha_i (i=1,\cdots,M)$ should satisfy

$$E(z=1) = \prod_{j=M}^{1}\begin{bmatrix} \cos\alpha_i & \sin\alpha_i \\ -\sin\alpha_i & \cos\alpha_i \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \tag{13}$$

namely,

$$E(z=1) = \begin{bmatrix} \cos\left(\sum_{i=1}^{M}\alpha_i\right) & \sin\left(\sum_{i=1}^{M}\alpha_i\right) \\ -\sin\left(\sum_{i=1}^{M}\alpha_i\right) & \cos\left(\sum_{i=1}^{M}\alpha_i\right) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \tag{14}$$

So,

$$\sum_{i=1}^{M} \alpha_i = \frac{\pi}{4} + 2k\pi \tag{15}$$

3 RAS-FWT algorithm

To reduce the amount of calculation in the algorithm implementation, the cosine element of each stage is taken out. Note that

$$a_i = \mathrm{tg}\alpha_i \tag{16}$$

$$\beta = \prod_{i=1}^{M} \cos\alpha_i < 1; \quad \beta = \prod_{i=1}^{M} \frac{1}{\sqrt{(1+a_i^2)}} \tag{17}$$

From the equivalence relations shown in Fig. 3, a novel analyzing and synthesizing cascaded wavelet transform scheme is invented as shown in Fig. 4.

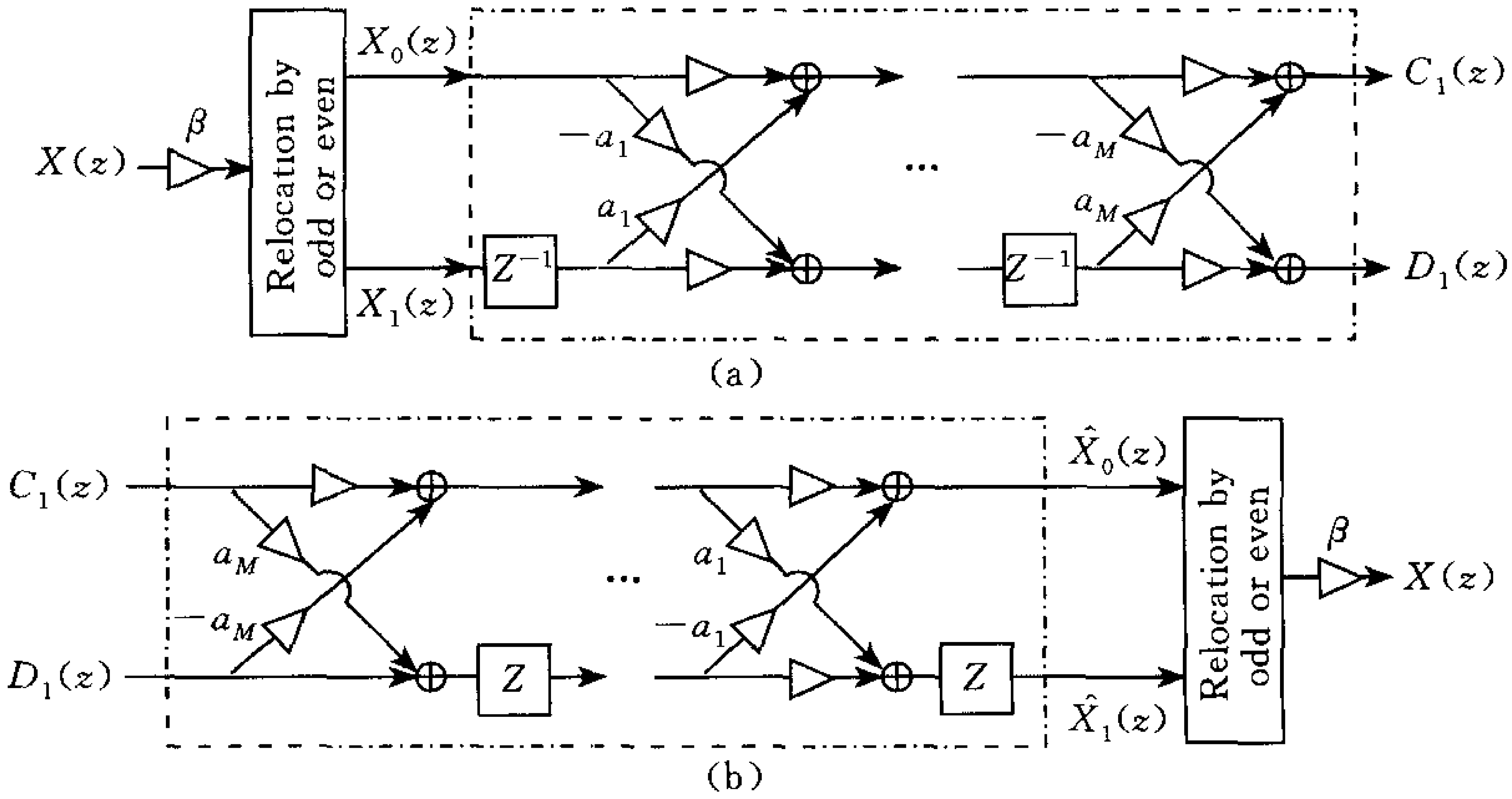

Fig. 4 The analyzing and synthesizing cascaded wavelet transform scheme

The forward transform unit-stage matrix (analyzing)

$$T(\alpha_i) = \begin{bmatrix} \cos\alpha_i & \sin\alpha_i \\ -\sin\alpha_i & \cos\alpha_i \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \tag{18}$$

The backward transform unit-stage matrix (synthesizing)

$$\hat{T}(\alpha_i) = T^{-1}(\alpha_i) = \widetilde{T}(\alpha_i) = \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} \times \begin{bmatrix} \cos\alpha_i & -\sin\alpha_i \\ \sin\alpha_i & \cos\alpha_i \end{bmatrix} \tag{19}$$

Perfect reconstruction

$$[\hat{T}(\alpha_1) \times \cdots \times \hat{T}(\alpha_M)] \times [T(\alpha_M) \times \cdots \times T(\alpha_1)] = I \tag{20}$$

The algorithm description for the signal with periodic extension at the edges

Let $R_1 = \begin{bmatrix} 0 & & & 1 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}_{n\times n}$, $R_2 = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{bmatrix}_{n\times n} = R_1^{\mathrm{T}}$.

R_1 and R_2 are both cyclic matrixes. The signal is periodically extended at the edges, and the forward and backward transform unit-stage matrixes are expressed as orthogonal matrixes in real number domain:

$$T(\alpha_i) = \begin{bmatrix} \cos\alpha_i I & \sin\alpha_i I \\ -\sin\alpha_i I & \cos\alpha_i I \end{bmatrix} \times \begin{bmatrix} I & 0 \\ 0 & R_1 \end{bmatrix} \tag{21}$$

$$\hat{T}(\alpha_i) = \begin{bmatrix} I & 0 \\ 0 & R_2 \end{bmatrix} \times \begin{bmatrix} \cos\alpha_i I & -\sin\alpha_i I \\ \sin\alpha_i I & \cos\alpha_i I \end{bmatrix} \tag{22}$$

$$\hat{T}(\alpha_i) = T^{-1}(\alpha_i) = T^{\mathrm{T}}(\alpha_i) \tag{23}$$

Here, (21) and (22) are expressed as the product of sub-matrixes with particular meaning. You will find that no operation is performed on the signal by the two sub-matrixes which include R_1 and R_2. The cyclic matrixes, R_1 and R_2, just cyclically shift the odd serial signal (namely, so-called digital delay). So no time will be expended on algorithm by "cyclic pointers". The rest is simple algebraic operations. Table 1 and 2 show the amount of calculation of RAS-FWT.

Table 1 The algorithm description for the signal with periodic extension at the edges

Wavelet analysis algorithm	Wavelet synthesis algorithm
$X_0 = X$(original signal), $i=1$; relocate X_0 by odd or even	$i=M$
Step1. $X_i = T(\alpha_i)X_{i-1}$	Step1. $X_{i-1} = T^{\mathrm{T}}(\alpha_i)X_i$
Step2. $i=i+1$, if $i>M$ stop; else goto Step1	Step2. $i=i-1$, if $i<1$ stop; else goto Step1
Analysis completed	Relocate X_0 by odd or even; synthesis completed

Table 2 The amount of calculation contrast between RAS-FWT and FWT

Algorithm name	FWT	RAS-FWT
Multiplication	$4Mn$	$2n\times M$
Addition	$4(M-1)n$	$2n\times M$
Total	$8Mn-4n=2NL-2L$	$4Mn=NL$
Total /signal length	$2N-2$	N

Note: M(the length of rotation angle series); N(the length of filters, $N=2M$); L(the length of signals, $L=2n$)

It is obvious that only when $N=2M=2$, corresponding to Haar wavelet, will the two algorithms have the same amount of calculation. For higher wavelet order, RAS-FWT will have only half amount of FWT's calculation.

4 Further discussion on RAS-FWT

Adjust the composite relation in (12):

$$E(z) = \prod_{i=M}^{2}\left\{\begin{bmatrix}\cos\alpha_i & \sin\alpha_i\\ -\sin\alpha_i & \cos\alpha_i\end{bmatrix}\times\begin{bmatrix}1 & 0\\ 0 & z^{-1}\end{bmatrix}\right\}\times\begin{bmatrix}\cos\alpha_1 & \sin\alpha_1\\ -\sin\alpha_1 & \cos\alpha_1\end{bmatrix}=$$

$$\begin{bmatrix}\cos\alpha_M & \sin\alpha_M\\ -\sin\alpha_M & \cos\alpha_M\end{bmatrix}\times\prod_{i=M-1}^{1}\left\{\begin{bmatrix}1 & 0\\ 0 & z^{-1}\end{bmatrix}\times\begin{bmatrix}\cos\alpha_i & \sin\alpha_i\\ -\sin\alpha_i & \cos\alpha_i\end{bmatrix}\right\} \tag{24}$$

Let

$$\begin{cases} p(i) = \{h^{(i)}(0), h^{(i)}(2), \cdots, h^{(i)}(2i-2)\} \\ q(i) = \{h^{(i)}(1), h^{(i)}(3), \cdots, h^{(i)}(2i-1)\} \end{cases} \tag{25}$$

The boundary conditions are

$$\begin{cases} p(1) = \{\cos\alpha_M\} \\ q(1) = \{\sin\alpha_M\} \end{cases} \tag{26}$$

$$\begin{cases} p(M) = \{h_0(0), h_0(2), \cdots, h_0(2M-2)\} = e_{00}(n) \\ q(M) = \{h_0(1), h_0(3), \cdots, h_0(2M-1)\} = e_{01}(n) \end{cases} \tag{27}$$

where $e_{00}(n)$ is the even serial of $h_0(n)$, and $e_{01}(n)$is the odd serial of $h_0(n)$.

The recurrence formula from the rotation angle series to filter coefficients:

From (24), we have the recurrence formula

$$\begin{cases} p(i+1) = \{\cos\alpha_{M-i}(p(i),0) - \sin\alpha_{M-i}(0,q(i))\} \\ q(i+1) = \{\sin\alpha_{M-i}(p(i),0) + \cos\alpha_{M-i}(0,q(i))\} \end{cases} \quad i = 1,\cdots,M-1 \tag{28}$$

with (26) being the initial condition.

The recurrence formula from filter coefficients to the rotation angle series:

From (28), the reverse recurrence formula is

$$\begin{cases} (p(i),0) = \{\cos\alpha_{M-i}p(i+1) + \sin\alpha_{M-i}q(i+1)\} \\ (0,q(i)) = \{-\sin\alpha_{M-i}p(i+1) + \cos\alpha_{M-i}q(i+1)\} \end{cases} \quad i = M-1,\cdots,1 \tag{29}$$

The upper formula is divided by the lower one in (28). Taking out first element of the series, we get

$$a_{M-i} = tg\alpha_{M-i} = h^{(i+1)}(1)/h^{(i+1)}(0), i = M-1,\cdots,1 \tag{30}$$

$$a_1 = tg\alpha_1 = h^{(M)}(1)/h^{(M)}(0) = h_0(1)/h_0(0) \tag{31}$$

Thus, from (31), (29), (30), we can get the total rotation angle series successively.

Examples for the rotation angle series

Table 3 specializes the rotation angle series for Daubechies orthogonal wavelets[2,4].

Table 3 The rotation angle series of Daubechies orthogonal wavelets

Wavelet	Vanishing moments	Filter coefficients		Rotation angle series	Sum of rotation angles
Haar	1	0.7071068,	0.7071068	$\pi/4$	$\pi/4$
Db2	2	0.4829629,	0.8365163	$\pi/3$	$\pi/4$
		0.2241439,	−0.1294095	$-\pi/12$	
Db3	3	0.3326706,	0.8068915	1.1797431	0.7853981
		0.4598775,	−0.1350110	−0.4998413	
		−0.0854413,	0.0352262	0.1054963	
Db4	4	0.2303778,	0.7148466	1.2590303	0.7853982
		0.6308808,	−0.0279838	−0.6813709	
		−0.1870348,	0.0308414	0.2537065	
		0.0328830,	−0.0105974	−0.0459677	
Db5	5	0.1601024,	0.6038293	1.3116150	0.7853982
		0.7243085,	0.1384281	−0.8163694	
		−0.2422949,	−0.0322449	0.4017260	
		0.0775715,	−0.0062415	−0.1324053	
		−0.0125808,	0.0033357	0.0208319	
Db6	6	0.1115407,	0.4946239	1.3490001	0.7853981
		0.7511339,	0.3152504	−0.9181872	
		−0.2262647,	−0.1297669	0.5327508	
		0.0975016,	0.0275229	−0.2383934	
		−0.0315820,	0.0005538	0.0698859	
		0.0047773,	−0.0010773	−0.0096581	

Note: Limited by the space, the rotation angle series for higher order wavelets are not listed.

From Table 3, the rotation angle series of Daubechies orthogonal wavelets have the apparent structure characteristic of "fast oscillatory attenuation". This brings significant elicitation to wavelet construction theory. That is, as long as we have construct a set of rotation angle series with specific rule which satisfies (15), we can achieve a new orthogonal wavelet with special performance. In addition, RAS-FWT completely casts away fussy filter designs, using the rotation angle series with much less parameters as the replacer. These characteristics establish a firm theoretical basis for self-adaptive selection of wavelet bases.

The in-depth study is being taken on the relations between the rotation angle series and the frequency selectivity, vanishing moments, and regularity of orthogonal wavelets. Applied research findings are expected to construct special performance wavelets.

5 Conclusion

1) RAS-FWT has only half amount of FWT calculation in higher order wavelet analysis. That is two times faster in speed.

2) RAS-FWT adopts iteration structure that is more easily realized on microprocessors. The relocation by odd or even and digital delay are implemented by pointer operations, instead of real data shift. So its code has higher efficiency and speed, and its data structure is more compact.

3) The simple linear restriction and the apparent structure characteristic of rotation

angle series offer the capability for self-adaptively searching the best wavelet bases, and will simplify wavelet construction greatly.

Based on the above characteristics, RAS-FWT possesses higher technique advantage and theoretical value. It will bring great effect in fields of numerous realtime applications, such as speech and image processing, biomedicine ultrasound imaging and industry fault diagnosis.

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旋转角序列小波分析快速算法

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摘　要　提出了一种新型小波分析快速算法——"旋转角序列快速小波变换(RAS-FWT)",并给出了正交小波旋转角序列的约束关系.该算法将传统的基于卷积的小波变换快速实现方法,转化为微处理器更易实现的迭代结构,并采用"循环指针"实现数字延迟,代码更加高效简洁.Mallat算法将正交小波与 N 长度离散系数序列建立起了一一映射关系;而 RAS-FWT 建立起正交小波与 $N/2$ 长度离散角度序列的一一映射关系,故计算量降低为 Mallet 算法(FWT)的一半.另外,基于"旋转角序列"的特征构造,这一技术将为正交小波构造理论开辟一条崭新的技术路线,成为这一学科的新分支.

关键词　小波,快速算法,旋转角序列,RAS-FWT,循环指针

中图分类号　TP391