第 29 卷 第 5 期 自 动 化 学 报 Vol. 29, No. 5
2003 年 9 月 ACTA AUTOMATICA SINICA Sept., 2003

The Construction of Multidimensional Morphological Wavelets Based on Lifting Scheme[1)]

HUANG Xiang-Sheng[1] YANG Xiao-Fan[2,3] WANG Yang-Sheng[1]
[1] (Institute of Automation, Chinese Academy of Sciences, Beijing 100080)
[2] (College of Computer Science, Chongqing University, Chongqing 400044)
[3] (Key Laboratory of Electro-Optical Technique and System, Chongqing University, Chongqing 400044)
(E-mail: xshuang@nlpr.ia.ac.cn)

Abstract The construction of multidimensional morphological wavelets is addressed by combining mathematical morphology with multidimensional wavelets. First, the multidimensional multi-channel lifting scheme, a general framework of multidimensional morphological wavelet construction, is presented. Then the multidimensional multi-channel Haar morphological wavelets are constructed with max (min) operator. Finally, two examples are given to reveal the advantages and applicability of morphological wavelets.

Key words Wavelet, morphology, multidimensional morphological wavelet, lifting scheme

1 Introduction

In the last decade, several constructions of one-dimensional (1-D) wavelets have originated from signal processing and mathematical analysis. In fact, we need multidimensional wavelets and filters for image processing and video coding. An obvious way to build wavelets of high dimension is through tensor products of one-dimensional constructions, which result in separable filters. However, this approach gives preferential treatment to the coordinate axes and only allows for rectangular division of the frequency spectrum. Often, symmetry axes and certain nonrectangular divisions of the frequency spectrum correspond better effects to the human visual system.

In the early 1990′s, several constructions of multidimensional filters became available[1~3]. These are typically concerned with two and three dimensions since the algebraic conditions in higher dimensions become increasingly cumbersome. The lifting scheme [4,5] provides a useful way to construct multidimensional wavelet. Its basic idea is to start with a very simple or trivial multiresolution analysis and gradually develop to a multiresolution analysis with particular properties. In [6], Kovačević and Sweldens proposed the construction of multidimensional wavelets with arbitrary vanishing moments. Its basic idea is to use low pass signal to predict each high pass signal, and then utilize all high pass signals to update low pass signal. All of the prediction operators and the update operators are Neville filter. And Neville filter can be constructed by multivariate polynomial interpolating[7,8]. Notably, all these operations are linear operations, not nonlinear operations as morphological operators.

In 2000, Henk introduced morphological operator to the framework of wavelet as prediction operator and update operator[9,10]. But Henk did not use lattices and sublattices to do signal sampling, which led to increasing difficulty in multidimensional morphological wavelet construction.

This paper addresses the construction of multidimensional morphological wavelets by combining mathematical morphology with multidimensional wavelets. Section 2 contains

1) Supported by the Visiting Scholar′s Funds of Key Laboratory of Electro-optical Technique, Chongqing University
Received November 21, 2002; in revised form March 7, 2003
收稿日期 2002-11-21; 收修改稿日期 2003-03-07

the background material including an overview of multiresolution analysis in the multivariate case. In Section 3, multidimensional multi-channel lifting scheme, a general framework of multidimensional morphological wavelet construction, is presented. Section 4 and Section 5 propose Haar morphological wavelet construction with two examples. Finally, Section 6 draws the conclusion.

2 Preliminary

Multidimensional filters, lattices, multiresolution analysis and morphological operation are briefly discussed in this section. More complete description of these notations is given in [1,2,11].

2.1 Multidimensional filters

A signal x is a sequence $x=\{x_k\in R\,|\,k\in Z^d\}$ of real-valued numbers, where Z^d is the d-dimensional Euclidean integer space. The action of time invariant FIR (H) is the convolution of x with the impulse response sequence

$$(Hx)_l=\sum_{n_1}\cdots\sum_{n_d}h_{l_1-n_1,l_2-n_2\cdots l_d-n_d}x_{n_1,n_2\cdots n_d} \tag{1}$$

The z-transform of a filter H is defined as

$$H(Z)=\sum_{n_1}\cdots\sum_{n_d}h_{n_1,n_2\cdots n_d}z_1^{-n_1}\cdots z_d^{-n_d} \tag{2}$$

2.2 Lattices and sublattices

Lattice is defined as d-dimensional Euclidean integer space. Let D be a $d\times d$ nonsingular matrix with integer coefficients. A sublattice of $K=Z^d$ is DZ^d. Let $|D|$ denote the determinant of D. Further, if $|D|=M$, then there are M distinct sublattices, each of the form DZ^d+t_j with $t_j\in Z^d$ and $0\leqslant j\leqslant M-1$. Given a sublattice $L=DZ^d+t$, signal sampling can be carried out on it.

2.3 Multiresolution analysis

Let D be a $d\times d$ nonsingular matrix with integer coefficients, $|D|=M$. A d-dimensional multiresolution analysis consists of a sequence of successive approximation spaces V_j. More precisely, the closed subspaces V_j satisfy

1) $\cdots\subset V_{-1}\subset V_0\subset V_1\subset\cdots$ 2) $\mathrm{clos}_{L_2}(\bigcup_{j\in Z}V_j)=L_2(R^d)$

3) $\bigcap_{j\in Z}V_j=\{0\}$ 4) $f\in V_j\Leftrightarrow f(Dx)\in V_{j+1}$

5) Each V_j has a Riesz base, which is derived from translations and dilations of scale function $\{\varphi_{j,k}=M^{j/2}\varphi(D^jx-k)\,|\,k\in Z^d\}$.

Scale function $\varphi(x)\in L_2(R^d)$ satisfies the refinement relations $\varphi(x)=\sum_{k\in Z^d}h_k\varphi(Dx-k)$, where h_k form the impulse response of a FIR. Using a vector function notation $\Phi_j=\{\varphi_{j,k}(x)\,|\,k\in Z^d\}$ and filter operations, we can express the refinement relations further as $\Phi_j=M^{-1/2}H\Phi_{j+1}$, where H is defined as $(\downarrow D)H$. The wavelet functions satisfy refinement relations given by $\psi_{j,i}=M^{1/2}G_i\Phi_{j+1}$. Let $W_{j,i}=\mathrm{span}\{\psi_{j,i,k}\,|\,k\in Z^d\}$; then $V_{j+1}=V_j\oplus W_{j,0},\oplus\cdots\oplus W_{j,M-1}$ and $\psi_{j,i,k}$ forms an unconditional base for $L_2(R^d)$.

2.4 Morphological operation

Mathematical morphology is a very important branch of mathematics, and is often applied to image processing. The basic idea of morphology is to capture desired information from signal by using morphological operator, called detector. Many morphological operators have been constructed. Two of them, max("$\vee$") and min("$\wedge$"), are used in this paper.

3 Multidimensional multi-channel lifting scheme

3.1 Lifting scheme based on low pass signal prediction

In [6], Kovačvić and Sweldens proposed the construction of wavelet families with arbitrary vanishing moments. A multidimensional signal is divided into M components in terms of D by lazy wavelet. Then each high pass signal is predicted by low pass signal. Finally, low pass signal is updated by all high pass signals. The polyphase matrix is now $M\times M$ matrix given by

$$P=\begin{bmatrix}1 & U_1 & U_2 & \cdots & U_{M-1}\\ 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\cdots 0\\ -P_1 & 1 & 0\cdots 0\\ -P_2 & 0 & 1\cdots 0\\ \cdots & \cdots & \cdots\\ -P_{M-1} & \cdots & 1\end{bmatrix} \tag{3}$$

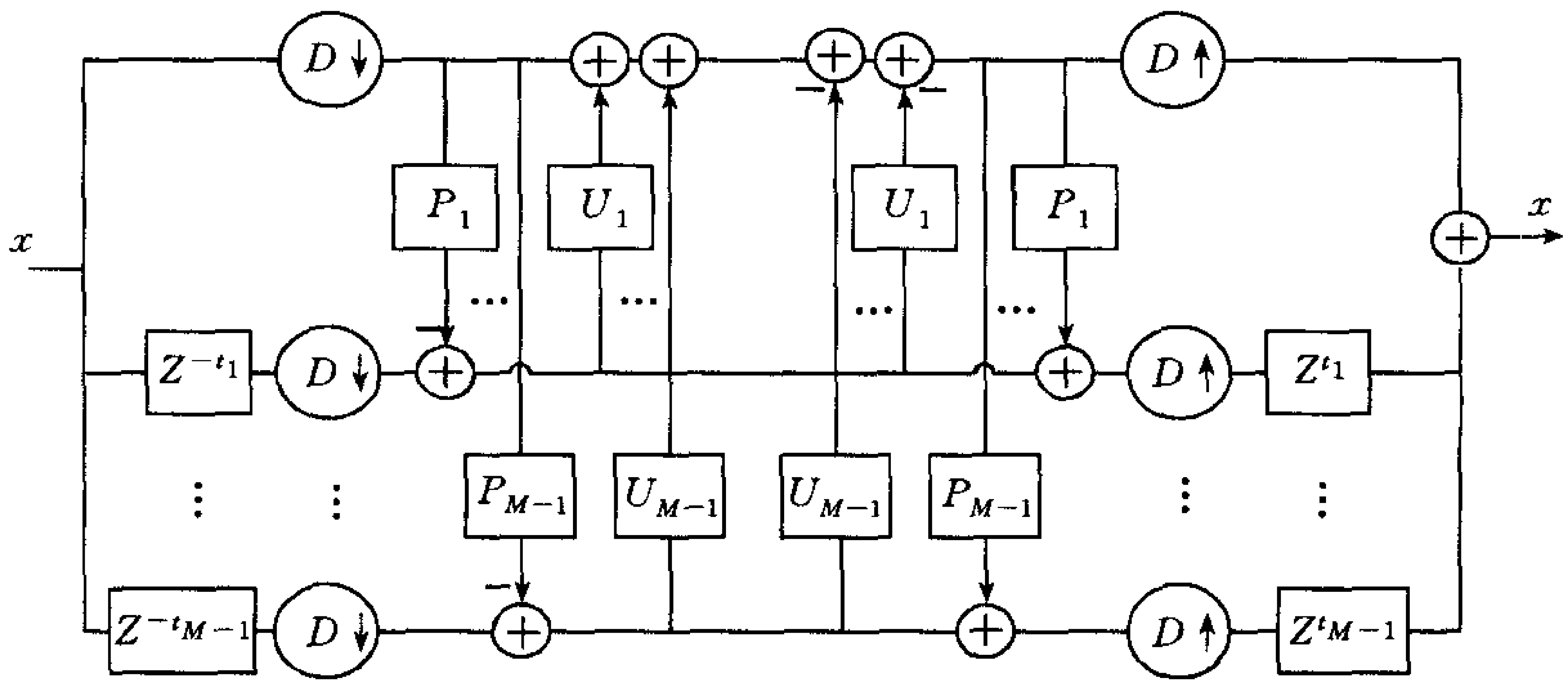


Fig. 1 Lifting scheme based on low pass signal prediction

Prediction operator P_i and update operator U_i are all Neville filters. Neville filters can be constructed by multivariate polynomial interpolating[7,8]. However, many existing wavelets, such as two-dimensional 4-channel morphological wavelet[10], could not be constructed by this scheme. We propose a novel prediction scheme based on each channel signal predicting each other.

3.2 Lifting scheme based on each other channel signal predicting (novel prediction)

The polyphase matrix is an $M\times M$ matrix given by

$$P=\begin{bmatrix}1 & U_1 & U_2 & \cdots & U_{M-1}\\ 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0 & \cdots & 0\\ -P_{1,0} & 1 & -P_{1,2} & -P_{1,3} & \cdots & -P_{1,M-1}\\ -P_{2,0} & -P_{2,1} & 1 & -P_{2,3} & \cdots & -P_{2,M-1}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ -P_{M-1,0} & -P_{M-1,1} & -P_{M-1,2} & -P_{M-1,3} & \cdots & 1\end{bmatrix} \tag{4}$$

where $P_{j,i}$ is a prediction operator, which is used by ith channel signal to predict jth channel signal. U_i is an update operator used by ith channel to update low pass signal. Many rich filters can be constructed by this approach.

4 One dimensional Haar morphological wavelet construction

In the following sections, x is the original signal. s and d denote low pass signal and high pass signal, respectively.

4.1 One-dimensional 2-channel Haar morphological wavelets

In this case, matrix D degenerates to an integer M in one-dimension case. M is the number of channels. In [10], Henk provided one-dimensional 2-channel Haar morphological wavelet construction. The analysis operation can be expressed as $s_n=x_{2n}\wedge x_{2n+1}$, $d_n=x_{2n}-x_{2n+1}$. An alternative expression is the lifting scheme. $s_n=x_{2n}$, $d_n=x_{2n+1}$; $d_n^{(1)}=s_n-d_n$, $s_n^{(1)}=s_n\wedge(s_n-d_n^{(1)})$. Its synthesis is defined as $s_n=s_n^{(1)}+(d_n^{(1)}\vee 0)$, $d_n=s_n-d_n^{(1)}$.

4.2 One-dimensional multi-channel Haar morphological wavelets

In $M \geqslant 3$ case, the analysis of one-dimensional multi-channel Haar morphological wavelets can be defined as

$$s_n = x_{Mn}, \quad d_{i,n} = x_{Mn+i}, \quad 1 \leqslant i \leqslant M-1,$$

$$d_{i,n}^{(1)} = s_n - d_{i,n}, \quad s_n^{(1)} = s_n \wedge (s_n - \max(d_{1,n}^{(1)}, d_{2,n}^{(1)}, \cdots, d_{M-1,n}^{(1)}, 0))$$

Its synthesis is defined as $s_n = s_n^{(1)} + \max(d_{1,n}^{(1)}, d_{2,n}^{(1)}, \cdots, d_{M-1,n}^{(1)}, 0), d_{i,n} = s_n - d_{i,n}^{(1)}$.

If we use min operator instead of max operator, the analysis can be expressed as

$$s_n = x_{Mn}, \quad d_{i,n} = x_{Mn+i}, \quad 1 \leqslant i \leqslant M-1$$

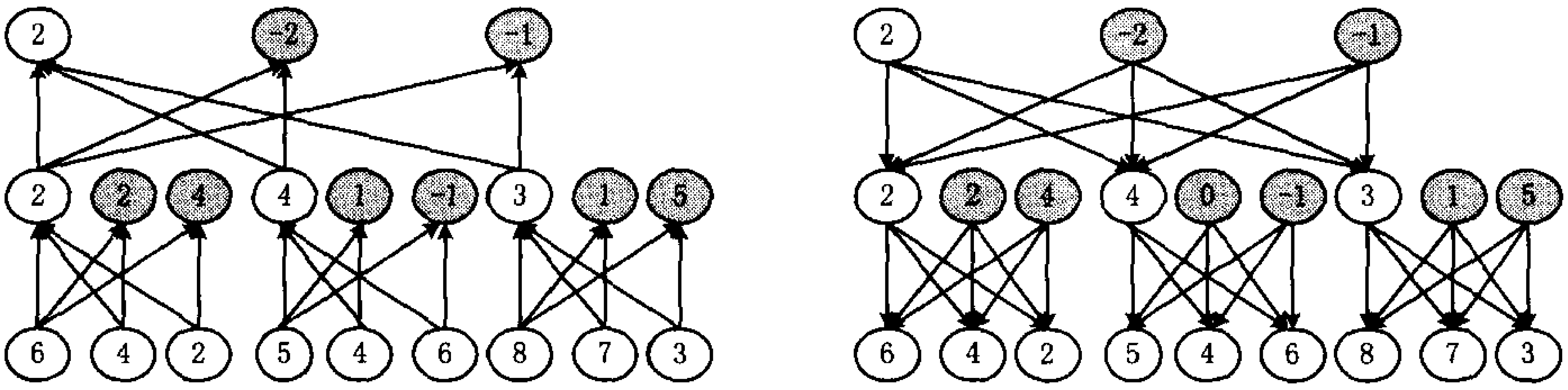
$$d_{i,n}^{(1)} = s_n - d_{i,n}, \quad s_n^{(1)} = s_n \vee (s_n - \min(d_{1,n}^{(1)}, d_{2,n}^{(1)}, \cdots, d_{M-1,n}^{(1)}, 0))$$

Its synthesis is defined as

$$s_n = s_n^{(1)} + \min(d_{1,n}^{(1)}, d_{2,n}^{(1)}, \cdots, d_{M-1,n}^{(1)}, 0), \quad d_{i,n} = s_n - d_{i,n}^{(1)}$$

4.3 Example

A simple two level transform of one-dimensional 3-channel morphological wavelets is depicted in Fig. 2.



(a) The analysis of morphological wavelet (b) The synthesis of morphological wavelet

Fig. 2 Two level transform of one-dimensional 3-channel Haar morphological wavelet

5 Multidimensional multi-channel Haar morphological wavelets

5.1 Multidimensional Haar morphological wavelet based on low pass signal prediction

In this section, n is an integer vector. The analysis of multidimensional Haar morphological wavelet is defined as

$$s_n = x_{Dn}, \quad d_{i,n} = x_{Dn+t_i} \quad 1 \leqslant i \leqslant M-1,$$

$$d_{i,n}^{(1)} = s_n - d_{i,n}, \quad s_n^{(1)} = s_n \wedge (s_n - \max(d_{1,n}^{(1)}, d_{2,n}^{(1)}, \cdots, d_{M-1,n}^{(1)}, 0))$$

Its synthesis is defined as

$$s_n = s_n^{(1)} + \max(d_{1,n}^{(1)}, d_{2,n}^{(1)}, \cdots, d_{M-1,n}^{(1)}, 0), d_{i,n} = s_n - d_{i,n}^{(1)}$$

5.2 Multidimensional Haar morphological wavelets based on novel prediction

The novel prediction scheme is more complex than low pass signal prediction scheme. The construction procedure of multidimensional Haar morphological wavelets is discussed for two cases, i. e., M is even or odd.

5.2.1 M is even

The novel prediction scheme is to predict each channel using all other channels. Its analysis is defined as

$$s_n = x_{Dn}, \quad d_{i,n} = x_{Dn+t_i}, \quad 1 \leqslant i \leqslant M-1, \quad d_{i,n}^{(1)} = 2\left(s_n + \sum_{j=1}^{M-1} (-1)^{j+i+1} d_{i,n}\right)/M$$

$$s_n^{(1)} = s_n \wedge (s_n - \max(d_{1,n}^{(1)} + d_{2,n}^{(1)}, d_{2,n}^{(1)} + d_{3,n}^{(1)}, \cdots, d_{M-1,n}^{(1)} + d_{1,n}^{(1)}, 0) \cdot M/4)$$

Its synthesis is defined as

$$s_n = s_n^{(1)} + \max(d_{1,n}^{(1)} + d_{2,n}^{(1)}, d_{2,n}^{(1)} + d_{3,n}^{(1)}, \cdots, d_{M-1,n}^{(1)} + d_{1,n}^{(1)}, 0) \cdot M/4$$

$$d_{i,n} = s_n - (d_{i,n}^{(1)} + d_{j,n}^{(1)}) \cdot M/4, \quad 1 \leqslant i < M-1, j = i+1, \; i = M-1, j = 1$$

The two-dimensional 4-channel Haar morphological wavelet, proposed by Henk[10], is a particular example of these wavelet families.

5.2.2 M is odd

In this case, the analysis of multidimensional Multi-channel Haar morphological wavelet can be defined as

$$s_n = x_{Dn}, d_{i,n} = x_{Dn+t_i}, 1 \leqslant i \leqslant M-1, \quad d_{i,n}^{(1)} = 2\left(s_n - d_{j,n} + \sum_{k=1}^{M-1}(-1)^{k+i+1} d_{i,n}\right)/(M-1)$$

where $i=1$, $j=M-1$; $1<i\leqslant M-1$, $j=i-1$

$$s_n^{(1)} = s_n \wedge \Big\{s_n - \Big[\max\Big(\Big((M-2)\cdot d_{1,n}^{(1)} - \sum_{i=1}^{M-1}(-1)^{i+1}\cdot d_{i,n}^{(1)}\Big), \Big((M-2)\cdot d_{2,n}^{(1)} - \sum_{i=1}^{M-1}(-1)^{i+2}\cdot d_{i,n}^{(1)}\Big), \cdots, \Big((M-2)\cdot d_{M-1,n}^{(1)} - \sum_{i=1}^{M-1}(-1)^{i+(M-1)}\cdot d_{i,n}^{(1)}\Big), 0\Big)\Big]\cdot (M-1)/[2\cdot(M-2)]\Big\}$$

Its synthesis is defined as

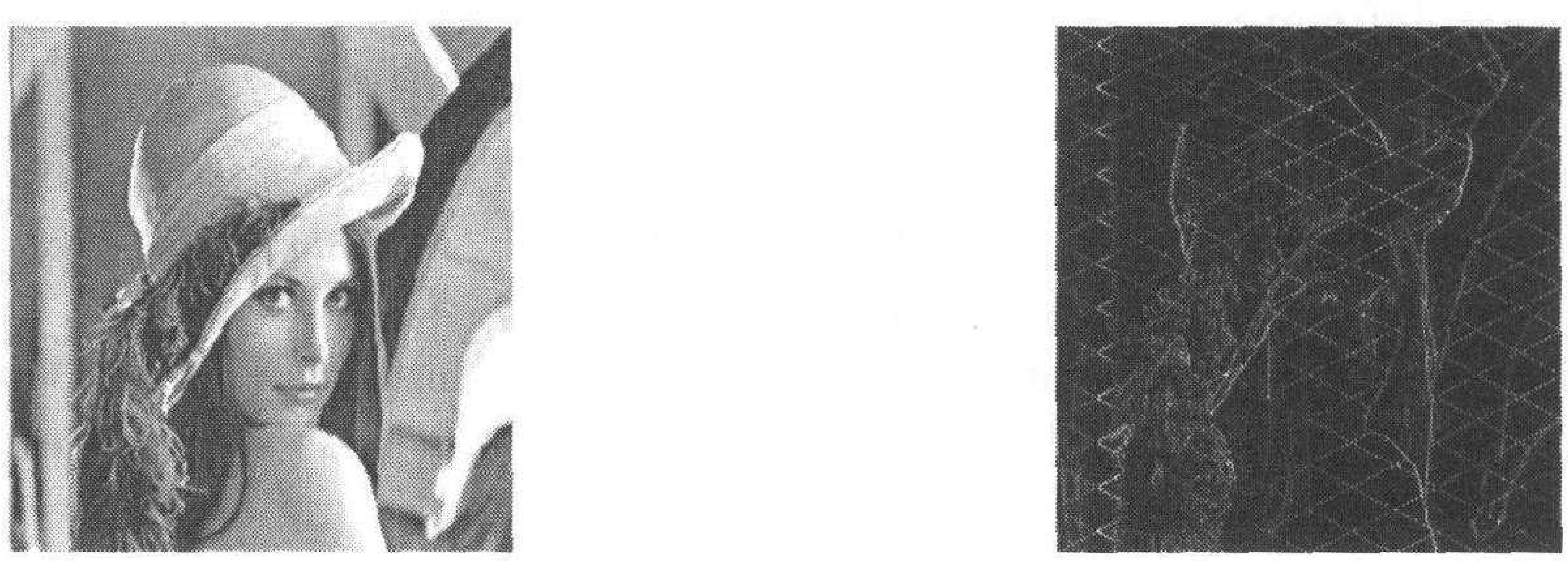
$$s_n = s_n^{(1)} + \Big\{\Big[\max\Big(\Big((M-2)\cdot d_{1,n}^{(1)} - \sum_{i=1}^{M-1}(-1)^{i+1}\cdot d_{i,n}^{(1)}\Big), \Big((M-2)\cdot d_{2,n}^{(1)} - \sum_{i=1}^{M-1}(-1)^{i+2}\cdot d_{i,n}^{(1)}\Big), \cdots, \Big((M-2)\cdot d_{M-1,n}^{(1)} - \sum_{i=1}^{M-1}(-1)^{i+(M-1)}\cdot d_{i,n}^{(1)}\Big), 0\Big)\Big]\cdot (M-1)/[2\cdot(M-2)]\Big\}$$

$$d_{i,n} = s_n - \Big\{\Big[(M-2)\cdot d_{j,n}^{(1)} - \sum_{k=1}^{M-1}(-1)^{k+j} d_{k,n}^{(1)}\Big]\cdot (M-1)/[2\cdot(M-2)]\Big\}$$

where $i=M-1$, $j=1$; $1\leqslant i<M-1$, $j=i+1$.

5.3 Image transform using Haar morphological wavelets

In this section, the two-dimensional Haar morphological wavelet is used to transform image. One of the possible sampling matrices in the hexagonal is given by $D = \begin{bmatrix} 20 & 20 \\ 10 & -10 \end{bmatrix}$. Fast in-place calculation [6] is used in this experiment.



(a) Lena (240×240) (b) Two-dimensional Haar morphological wavelet

Fig. 3 Image transformation using two-dimensional Haar morphological wavelet

The result of two-dimensional Haar morphological wavelet transform is shown in Fig. 3. Broken line is used to explain the role of hexagonal matrix in Fig. 3.

The main difference between Haar morphological wavelet and linear Haar wavelet is the morphological operator. From Section 4.3 and Section 5.3, we can learn that there are many advantages using morphological wavelet.

1) The morphological Haar wavelet decomposition scheme may do a better job in preserving edges in x_0, as compared to the linear case. This is expected, since the signal analysis filters in the linear Haar wavelet decomposition scheme are linear low pass filters and smooth-out edges as such.

2) The signal analysis of morphological wavelet guarantees that the range of values of

the scale signal is the same as that of the original signal x_0. Furthermore, it guarantees that if the original signal x_0 is discrete-valued, the scaled signals will be discrete-valued as well, a desirable property in lossless coding application.

The basic idea of mathematical morphology is to use morphological operator, *detector*, to capture desired information from signal. In Section 4.3, the *min* operator is used to detect the minimum information of a signal. And in Section 5.3, the two-dimensional detector is used to capture the minimum information from image.

6 Conclusion

Classical wavelet transform is based on linear operation. The lifting scheme, recently introduced by Sweldens, has provided a useful way to construct nonlinear wavelet decompositions. Mathematical morphology and wavelet are combined by Henk[10] to construct one-dimensional Haar morphological wavelet. In this paper, one-dimensional morphological wavelets are generalized to multidimensional morphological wavelet by combining mathematical morphology with multidimensional wavelets, introduced by Kovačvić and Sweldens[6]. All kinds of Haar morphological wavelet analysis and synthesis are given in this paper. One example is given to show one-dimensional Haar morphological wavelet computing procedure. Furthermore, two-dimensional Haar morphological wavelet is applied to image transform. All these results indicate that morphological is not only feasible but also necessary for many applications, because it has many advantages, such as capturing desired information.

However, there are still many problems in this field. For example, the relations between morphological operator and vanishing moments are worth researching to construct good wavelet. Furthermore, not only *min* operator and *max* operator, but also many other good morphological operators need to be introduced into the framework of multidimensional wavelet.

References

1 Kalker T A C M, Shah I A. A group theoretic approach to multidimensional filter banks: Theory and application. *IEEE Transactions on Signal Processing*, 1996, **44**(6): 1397～1405

2 Viscito E, Allebach J P. The analysis and design of multidimensional FIR perfect reconstruction filter banks for arbitrary sampling lattices. *IEEE Transactions on Circuits and Systems*, 1991, **38**(1):29～41

3 Chen T, vaidyanathan P P. Multidimensional multirate filters and filter banks derived from one-dimensional filters. *IEEE Transactions on Signal Processing*, 1993, **41**(5): 1749～1765

4 Sweldens W. The lifting scheme: A custom-design construction of biorthogonal wavelets. *Applied and Computational Harmonic Analysis*, 1996, **3**:186～200

5 Sweldens W. The lifting scheme: A construction of second generation wavelets. *SIAM Journal of Mathematical Analysis*, 1998, **29**:511～546

6 Kovačvić J, Sweldens W. Wavelet families of increasing order in arbitrary dimensions. *IEEE Transactions on Image Processing*, 2000, **9**(3):480～496

7 De Boor C, Ron A. Computational aspects of polynomial interpolation in several variables. *Mathematics Computation*, 1992,**58**:705～727

8 De Boor C, Ron A. On multivariate polynomial interpolation. *Constructive Approximation*, 1990,**6**:287～302

9 Goutsias J, Heijmans H J A M. Nonlinear multiresolution signal decomposition schemes—Part I: Morphological pyramids. *IEEE Transactions on Image Processing*, 2000, **9**(11): 1862～1876

10 Heijmans H J A M, Goutsias J. Nonlinear multiresolution signal decomposition schemes—Part II: Morphological wavelets. *IEEE Transactions on Image Processing*, 2000, **9**(11): 1897～1913

11 Van der B, Waerden. Algebra I & II. vol. 12 and 23, Heidelberger Taschenbucher, 8th ed. Berlin: Springer-Verlag, 1971

HUANG Xiang-Sheng Received his master degree from Chongqing University in 2002. Now, he is a Ph. D. candidate. His research interests include theory and application of wavelet, video coding and computer vision.

YANG Xiao-Fan Received his master and Ph. D. degrees from Chongqing University in 1988 and 1994, respectively. He is a professor of Chongqing University of P. R. China. His research interests include wavelet and its application, graph theory.

WANG Yang-Sheng Received his master and Ph. D. degrees from Huazhong University of Science and Technology in 1984 and 1989, respectively. In October 1989, he joined the University of Edinburgh and did his postdoctor research. He is a professor of Institute of Automation of Chinese Academy of Sciences of P. R. China. His research interests include biometrics, robotics, pattern recognition and computer vision.

基于提升方案的高维形态小波构造

黄向生[1] 杨小帆[2,3] 王阳生[1]

[1](中国科学院自动化研究所 北京 100080)

[2](重庆大学计算机学院 重庆 400044)

[3](重庆大学光电技术及系统重点验室 重庆 400044)

(E-mail: xshuang@nlpr. ia. ac. cn)

摘 要 研究基于提升方案的高维形态小波构造方法. 结合数学形态学和高维小波分析来构造高维形态小波. 提出高维多通道形态小波构造的一般性框架——高维多通道提升方案. 采用最大值(最小值)形态算子构造一维多通道 Haar 形态小波和高维多通道 Haar 形态小波. 最后通过两个例子说明形态小波变换的优点和可行性.

关键词 小波,形态学,高维形态小波,提升方案

中图分类号 TP391